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Universality and scaling in the N-body sector of Efimov physics

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Universal behavior has been found inside the window of Efimov physics for systems with $N = 4,5,6$ particles. Efimov physics refers to the emergence of a number of three-body states in systems of identical bosons interacting via a short-range interaction becoming infinite at the verge of binding two particles. These Efimov states display a discrete scale invariance symmetry, with the scaling factor independent of the microscopic interaction. Their energies in the limit of zero-range interaction can be parametrized, as a function of the scattering length, by a universal function. We have found, using the form of finite-range scaling introduced by A. Kievsky and M. Gattobigio [Phys. Rev A **87**, 052719 (2013)], that the same universal function can be used to parametrize the ground and excited energy of $N \leqslant 6$ systems inside the Efimov-physics window. Moreover, we show that the same finite-scale analysis reconciles experimental measurements of three-body binding energies with the universal theory.

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I. INTRODUCTION

Universality is one of the concepts that have attracted physicists along the years. Different systems, even having different energy scales, share common behaviors. The most celebrated example of universality comes from the investigation of critical phenomena [1,2]: At the critical point, materials that are governed by different microscopic interactions share the same macroscopic laws, for instance, the same critical exponents. The theoretical framework to understand universality has been provided by the renormalization group (RG); the critical point is mapped onto a fixed point of a dynamical system, the RG flow, whose phase space is represented by Hamiltonians. At the critical point the systems have scale-invariant (SI) symmetry, forcing all of the observables to be exponential functions of the control parameter. A consequence of SI symmetry is the scaling of the observables: For different materials, in the same class of universality, a selected observable can be represented as a function of the control parameter and, provided that both the observable and the control parameter are scaled by some material-dependent factor, all representations collapse onto a single universal curve [3].

More recently, a new kind of universality has captured the interest of physicists, namely the Efimov effect [4,5]. A system of three identical bosons interacting via two-body short-range interaction whose strength is tuned, by scientists or by nature, to the verge of binding the two-particle subsystem, exhibits the appearance of an infinite tower of three-particle bound states whose energies accumulate to zero. Moreover, the ratio between the energies of two consecutive states is constant and independent of the very nature of the interaction; this last property points out to the emergence of a discrete-scale invariance (DSI) symmetry (for a complete review, see Ref. [6]).

Even this example of universality has found in the RG its theoretical framework. Systems sharing the Efimov effect are mapped onto a limit cycle of the RG flow, where they manifest the emergence of DSI. In turn, DSI implies that all of the observables are log-periodic functions of the control parameter [7], and this property is what characterizes Efimov physics, of which the Efimov effect is an example. The limit cycle implies the emergence of a new dimensional quantity, which in the case of Efimov physics is known as the three-body parameter. Strictly speaking, the DSI is an exact symmetry for systems with zero-range interaction, or equivalently in the scaling limit; for real systems, which possess an interaction with finite range r_0, there are deviations from DSI called finite-range effects.

Atomic physics, and more precisely experiments using ultracold-alkali atoms, has recently (re)sparked the interest in Efimov physics [8]. At present, several different experimental groups have observed the Efimov effect in alkali systems [9–12], where the key point has been the scientists' ability to change the two-body scattering length a by means of Fano-Feshbach resonances. In fact, the theory predicts how observables change as a function of the control parameter, $\kappa_* a$, which is proportional to the scattering length, making the tuning of a crucial to test the theory's predictions. In particular, the Efimov equation for the three-body binding energies E_3^n can be expressed in a parametric form as follows [6]:

$$E_3^n/(\hbar^2/ma^2) = \tan^2\xi, \quad \kappa_* a = e^{(n-n^*)\pi/s_0}\,\frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}\,, \quad (1)$$

with $\Delta(\xi)$ a universal function whose parametrization can be found in Ref. [6], $s_0 = 1.00624$, and κ_* is the emergent three-body parameter which gives the energy $\hbar^2\kappa_*^2/m$ for $n = n^*$ at the unitary limit $1/a = 0$.

The ability of tuning a has allowed the different experimental groups to measure the value of the scattering length a_- at which the three-body bound state disappears into the continuum ($\xi \to -\pi$). From Eq. (1) we see that measuring $a_- = -e^{-\Delta(\pi)/2s_0}/\kappa_* \approx -1.56/\kappa_*$ [13,14] is a way to measure the three-body parameter κ_*, which in principle should be different for different systems. However, it has been experimentally found [9–12], and theoretically justified [15,16], that in the class of alkali atoms $a_-/\ell \approx -9.5$, with

ℓ being the van der Waals length, a universality inside universality. Recently the same behavior has been seen in a gas of ^{4}He atoms [17].

II. FINITE-RANGE CORRECTIONS

Equation (1), as well as the parametrization of $\Delta(\xi)$, have been derived in the scaling limit, where the DSI is exact. Experiments and calculations made for real systems deal with finite-range interactions, $r_0 \neq 0$, and for this reason finite-range corrections have to be considered [18–21]. In a recent paper [22], the authors have observed in which manner finite-range corrections manifest in numerical calculations using potential models:

(i) There are corrections coming from the two-body sector which can be taken into account by substituting a_B for a, defined by $E_2 = \hbar^2/ma_B^2$, with E_2 being the two-body binding energy if $a > 0$, or the two-body virtual-state energy in the opposite case, $a < 0$ [23]. One simple way to obtain the virtual-state energy is looking for the poles of the two-body S matrix using a Padè approximation, as shown in Ref. [24]. It should be noticed that in the zero-range limit $a_B \to a$.

(ii) The finite-range corrections enter as a shift in the control parameter $\kappa_* a_B$. The value of the shift depends on the observable under investigation.

For instance, in the case of three-body binding energies, (i) and (ii) applied to Eq. (1) give

$$E_3^n/E_2 = \tan^2 \xi, \quad \kappa_n^3 a_B + \Gamma_n^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}, \tag{2}$$

where we have defined $\kappa_n^3 = \kappa_* e^{(n-n^*)\pi/s_0}$ and introduced the shifts Γ_n^3. In Ref. [22] the authors have shown that this type of correction appears in the energy spectrum, in atom-dimer scattering length and in the effective range function of three boson atoms. Moreover, in Ref. [25] it has been shown that the shift appears in recombination rate of three atoms close to threshold too.

Our finite-range analysis can be applied to describe experimental data. In Fig. 1 we report the experimental three-body binding energies measured in ^{7}Li [10] and for reference the corresponding magnetic field [26]. Using Eq. (2) with the values of the two three-body parameters $\Gamma_1^3 = 4.95 \times 10^{-2}$ and $\kappa_1^3 = 1.61 \times 10^{-4}\, a_0$ the experimental points collapse on the universal curve (solid line).

To explain the origin of this form of finite-range correction, we refer to the original derivation [5] of Eq. (1) and to the parametrization of the universal phase $\Delta(\xi)$, for instance, in Ref. [6]. In the zero-range limit $r_0 = 0$ the adiabatic approximation is exact, and the three-body problem is equivalent to a single Schrödinger equation in a scale-invariant $1/R^2$ potential, where $R^2 \propto r_{12}^2 + r_{13}^2 + r_{23}^2$ is the hyperradius; Eq. (1) has been derived by matching the scale-invariant phase shift $\Delta(\xi)$ originating from the long-range physics to the scaling-violating phase shift originating from the short-range physics (see Eq. (193) of Ref. [6]). The short-range physics can be encoded in a scale-violating momentum Λ_0, see Eq. (147) of Ref. [6], and the parametrization, for a zero-range theory, of $\Delta(\xi)$ is such that $\Lambda_0 = \kappa_*$. Now, when we consider a finite-range system, $r_0 \neq 0$, the lowest adiabatic potential is coupled to the other adiabatic potentials: For instance, it has been demonstrated by Efimov [18] that the coupling can be taken into account by a correction $\sim r_0/R^3$ on the lowest potential. This means that, keeping the same parametrization of $\Delta(\xi)$, the relation between Λ_0 and κ_* is modified, and at the first order we can expect

$$\Lambda_0 \simeq \kappa_* \left(1 + \mathcal{A}\,\frac{r_0}{a_B}\right), \tag{3}$$

which gives the shift $\Gamma_{n^*}^3 = \mathcal{A}\,\kappa_* r_0$. The constant $\mathcal{A}$ is expected to take natural values.

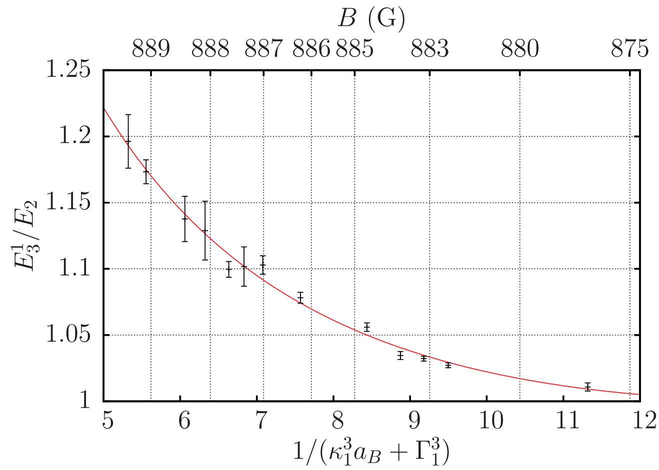


FIG. 1. (Color online) The experimental data on ^{7}Li [10], in the form of ratio between the three-body binding energy E_3^1 and the two-body binding energy E_2, as a function of $1/(\kappa_1^3 a_B + \Gamma_1^3)$ with the values of the three-body parameters given in the text. The solid curve represents the prediction of the universal function, Eq. (2). In the upper abscissa the magnetic field from Ref. [26] is given.

III. SCALING AND COLLAPSE

In this work we extend the application of the modifications to the zero-range theory in order to analyze the ground- and excited-binding energy of N-body systems obtained by numerical calculations inside the window of Efimov physics. The Efimov effect is strictly related to the $N = 3$ system, but one can try to investigate if and how Efimov physics affects $N > 3$ sectors. Some seminal-theoretical studies [27–29], and subsequent experimental investigation [30], have demonstrated that for each trimer belonging to the Efimov tower there are only two attached four-body states. This property has also been observed in $N = 5,6$ [31–33]: There are only two attached five-body states to the four-body ground state and there are only two attached six-body states to the five-body ground state. These states have been characterized by measuring ratios between energies close to the unitary limit, and these ratios have been found to be universal. Moreover, their stability has been analyzed along the Efimov plane in wide region of the angle ξ [34].

We want to make a step forward showing that the three-body equation, Eq. (2), can be modified to predict N-body ground- and excited-state energies E_N^0 and E_N^1. Even though our calculations have been done up to $N = 6$, a clear indication of validity for generic N can be inferred. We have solved the

Schrödinger equation using two different potentials: The first is an attractive two-body Gaussian (TBG) potential

$$V(r) = V_0 \, e^{-r^2/r_0^2} , \tag{4}$$

where r_0 is the range of the potential and $V_0 < 0$ is the strength that can be modified in order to tune the scattering length inside the Efimov window. This kind of potential has been previous used to investigate clusters of ^{4}He [22,25,33,35], and some numerical results, used in this work, have been previous given in Ref. [33]. The second is a Pöschl-Teller (PT) potential [36]

$$V(r) = -\frac{\hbar^2}{mb^2} \frac{2(1+C)}{\cosh^2(r/b)} , \tag{5}$$

where the dimensionless parameter C can be varied to change the scattering length. The solution of the N-body Schrödinger equation has been found using the nonsymmetrized hyperspherical harmonic (NSHH) expansion method with the technique recently developed by the authors in Refs. [31,37–39]

In Fig. 2(a) we show selected results for the $N = 3,4,5,6$ ground-state binding energies calculated with both potentials. The N-body ground-state binding energies E_N^0 are divided by E_2, which is the two-body binding energy for $a > 0$ or the virtual-state energy for $a < 0$. These ratios are given as a function of the inverse of the control parameter $\kappa_0^N a_B$. The parameter κ_0^N is fixed by the N-body ground-state binding energy $E_N^0 = \hbar^2(\kappa_0^N)^2/m$ calculated at the unitary limit $1/a = 0$. The corresponding values and some relevant ratios are given in Tables I and II. The solid curve represents the result of the $N = 3$ zero-range theory given in Eq. (1) for $n = n^*$. In the figure only a subset of the numerical data is shown in order to better appreciate the trend.

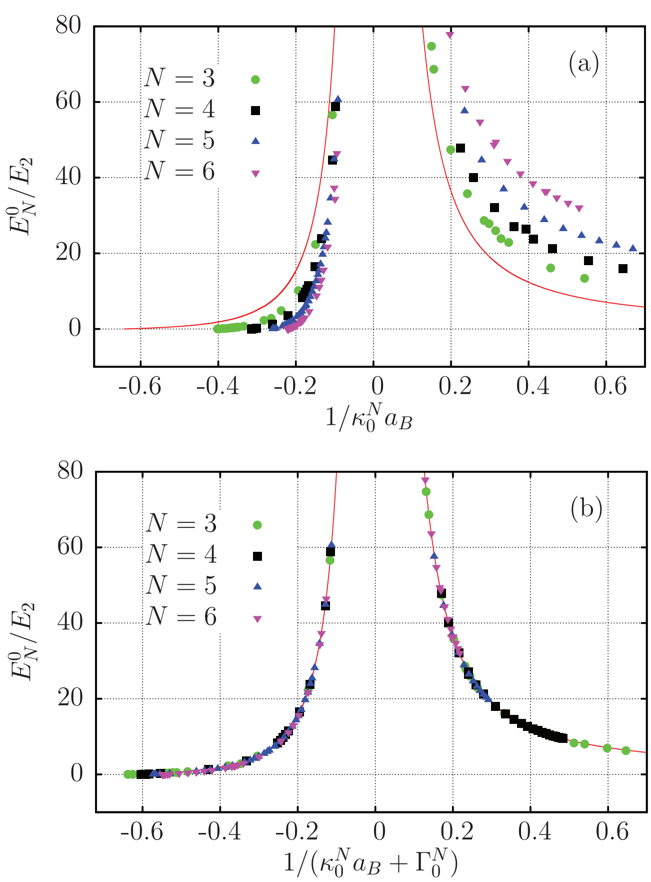


FIG. 2. (Color online) The TBG and PT ground-state N-body binding-energy E_N^0 in units of E_2 as a function of (a) $1/\kappa_0^N a_B$, and (b) $1/(\kappa_0^N a_B + \Gamma_0^N)$. E_2 is the two-body binding energy for $a > 0$, and the two-body virtual state energy for $a < 0$. In panel (b) the ground-state energies in terms of the scaled variable collapse onto the zero-range curve (solid line).

TABLE I. The parameters κ_n^N, in unit of r_0, and selected ratios for the TBG potential. When available, we report the ratios at the scaling limit between parenthesis.

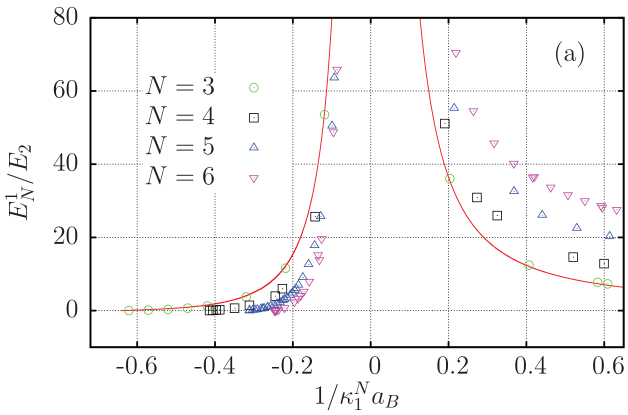
	$N = 3$	$N = 4$	$N = 5$	$N = 6$
$\kappa_0^N r_0$	4.88×10^{-1}	1.18	1.96	2.77
$\kappa_1^N r_0$	2.12×10^{-2}	5.11×10^{-1}	1.24	2.07
κ_0^N/κ_1^N	23.0 (22.7 [4])	2.31	1.58	1.34
$\kappa_0^N/\kappa_0^{N-1}$		2.42 (2.147 [29])	1.66	1.41
$\kappa_1^N/\kappa_0^{N-1}$		1.05 (1.001 [29])	1.05	1.06
$\kappa_1^N/\kappa_1^{N-1}$		24.1	2.43	1.67

In Fig. 2(b) the same data are shown, but this time the control parameter, $\kappa_0^N a_B$, has been shifted by a quantity Γ_0^N, different for each potential and each particle sector. As a remarkable result, the different sets of data collapse on the three-body zero-range universal curve. This is very reminiscent of the scaling property in critical phenomena [3]. In our case we have a N-dependent parameter, κ_0^N, that fixes the scale of the system and, in this respect, we refer here to it as a scaling parameter. Furthermore an N-dependent parameter, Γ_0^N, appears to take into account finite-range corrections. In this respect we refer to it as a finite-range scaling parameter.

It should be noticed that the values of κ_0^N has been obtained from our data, and in doing so we have included some range corrections into these quantities. As is well known, the lower energy states, as those considered here, have some dependence on the form of the potential. This dependence decreases in higher level states [29]. However, we want to emphasize that κ_0^N are not new N-body parameters in the same sense as the emergent three-body parameter κ_*. In the present treatment, where we only use two-body interaction (eventually, we could have also added three-body interactions [33]), there are not such things as four-, five-, and six-body parameters; in the

TABLE II. The parameters κ_n^N, in units of b, Γ_n^N, and selected ratios for the Pöschl-Teller potential.

	$N = 3$	$N = 4$	$N = 5$	$N = 6$
$\kappa_0^N b$	0.3668	0.9088	1.521	2.175
$\kappa_1^N b$		0.3934	0.971	1.633
κ_0^N/κ_1^N		2.31	1.58	1.33
$\kappa_0^N/\kappa_0^{N-1}$		2.48	1.68	1.43
$\kappa_1^N/\kappa_0^{N-1}$		1.07	1.07	1.07
$\kappa_1^N/\kappa_1^{N-1}$			2.46	1.68



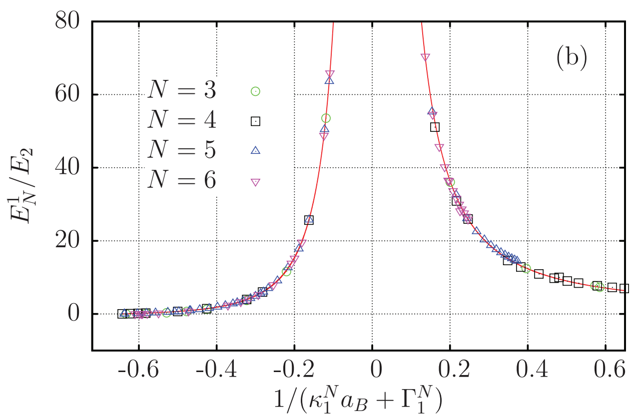


FIG. 3. (Color online) The TBG and PT excited-state N-body binding-energy E_N^1 in units of E_2 as a function of (a) $1/\kappa_1^N a_B$ and (b) $1/(\kappa_1^N a_B + \Gamma_1^N)$. E_2 is the two-body binding energy for $a > 0$ and the two-body virtual state for $a < 0$. In panel (b) the excited-state energies in terms of the scaled variable collapse onto the zero-range curve (solid line).

scaling limit all their values are fixed by the three-body parameter κ_*, as discussed below (for a different scenario see Ref. [40]).

In Fig. 3(a) we show our calculations for the N-body excited states E_N^1 using the TBG and PT potentials. We report the ratios E_N^1/E_2, where E_2 is still either the two-body binding energy for $a > 0$ or the virtual-state energy for $a < 0$, as a function of the inverse of the control parameter $\kappa_1^N a_B$. As for the ground states, the parameters κ_1^N are fixed by the excited-binding energy $E_N^1 = \hbar^2(\kappa_1^N)^2/m$ at the unitary limit. The solid line shows the universal function. Again, we stress that they are not new N-body parameters, but they are fixed by the value of κ_*. As before, κ_1^N has some range corrections which can be estimated in the case of $N = 3$ from Tables I and II. The zero-range theory imposes $\kappa_0^3/\kappa_1^3 \approx 22.7$ whereas we found $\approx$23.0 and $\approx$22.4 using TBG and PT potentials respectively.

In Fig. 3(b) our data sets are shown with a shift in the control variable $\kappa_1^N a_B$ by a N-dependent quantity Γ_1^N. As for the ground states, the excited states collapse on the universal curve too, pointing out to the emergence of a common universal behavior in the N-boson system.

Our numerical findings can be summarized in a modified version of Eq. (1). We propose for the N-body bound-state

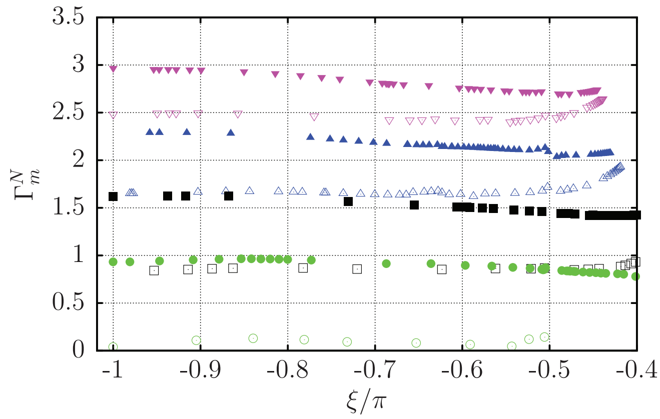


FIG. 4. (Color online) The TBG finite-range scaling parameters Γ_m^N as a function of the angle ξ derived from Eq. (6). The symbols are the same as in Figs. 2 and 3.

energies

$$E_N^m/E_2 = \tan^2\xi, \quad \kappa_m^N a_B + \Gamma_m^N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}, \tag{6}$$

where the function $\Delta(\xi)$ is universal and it is determined by the three-body physics. The above equation shows the same universal character of the three-boson system with, due to the DSI, the same universal function $\Delta(\xi)$. It is valid for a general N as long as the N-body bound-state energies do not leave the Efimov window. We stress the fact that the index m can take two values indicating the ground-sate ($m = 0$) and the excited state ($m = 1$) of the N-boson system. This form a tree structure with two levels attached to the ground state of the three-boson system. The parameter κ_m^N appears as a scale parameter and the shift Γ_m^N is a finite-range scale parameter introduced to take into account finite-range corrections. The introduction of the shifts Γ_m^N is probably a first-order correction of finite-range effects. In fact, we can use Eq. (6) to see that a small dependence on the parameter ξ still remains. This is illustrated in Fig. 4 in which the TBG Γ_m^N parameters are obtained by subtracting to the universal term $e^{-\Delta(\xi)/2s_0}/\cos\xi$ the computed value $\kappa_m^N a_B$ at the corresponding values of the angle ξ.

In Fig. 5 we report all our numerical calculations, for both ground- and excites-state energies, in the full range of scattering length, where the collapse of data is most evident.

IV. SCALING PARAMETERS

In this section we comment on the scaling parameters κ_m^N. In the zero-range limit, their values are fixed by the three-body parameter κ_*. For instance, by identifying $\kappa_0^3 = \kappa_*$ as the ground-state energy wave number, an accurate study in the four-body sector gives $\kappa_0^4 = 2.147\kappa_0^3$ [29]. In Table I, together with the values for TBG potential, we report, when available, the corresponding zero-range-limit values, whereas in Table II the same values are reported for the PT potential.

From the tables we can deduce a linear relation between the ground states that can be approximated as

$$\frac{\kappa_0^N}{\kappa_0^3} = 1 + (N-3)\left(\frac{\kappa_0^4}{\kappa_0^3} - 1\right). \tag{7}$$

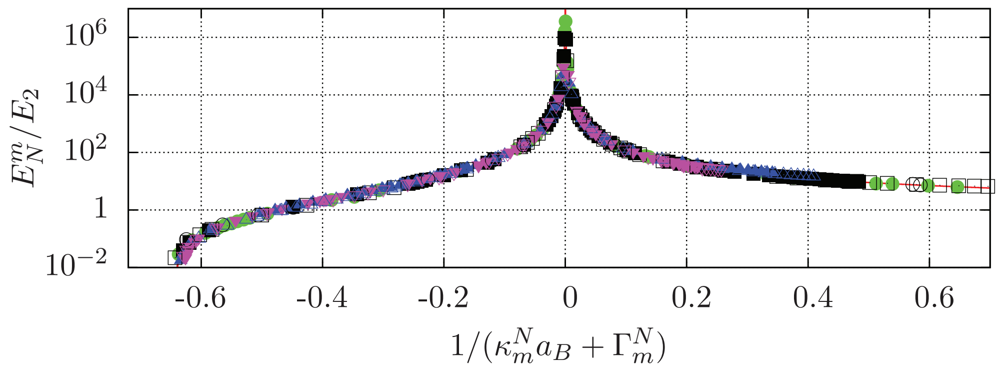


FIG. 5. (Color online) The TBG and PT ground- and excited-state N-body binding energies E_N^m in units of E_2 as a function of $1/(\kappa_m^N a_B + \Gamma_m^N)$. E_2 is the two-body binding energy for $a > 0$, and the two-body virtual state for $a < 0$. The symbols are the same as in Figs. 2 and 3 of the paper. All the data collapse on the three-body universal curve (solid line) calculated in the scaling limit.

In the scaling limit, using the universal value of κ_0^4/κ_0^3, this relation reduces to $\kappa_0^N/\kappa_0^3 = 1 + 1.147(N-3)$. The linear relation with N can also been seen in Refs. [41,42].

V. CONCLUSIONS

In the first part of this work we have studied finite-range corrections in a three-boson system having a large two-body scattering length. Following a previous finding (see Ref. [22]) we have modified the zero-range Efimov equations, Eq. (1), introducing a finite-range parameter. Its origin has been discussed in Eq. (3); in the case of a finite-range interaction the scale-violating momentum Λ_0 cannot be directly identified with the three-body parameter κ_*. The relation of these two parameters can be expressed in terms of an expansion in the small parameter of the theory r_0/a_B. By limiting the expansion up to first order Eq. (2) is directly obtained. From one side Eq. (2) is able to describe results obtained using different interactions and here we have shown two examples, the TBG and PT potentials. On the other side we have shown that our finite-range analysis reconciles experimental measurements of trimer-binding energies on ^{7}Li [10] with the universal theory, showing the collapse of the data on the universal curve.

In the second part of the work we have used our finite-range analysis to investigate N-body bound-state energies, up to $N = 6$, obtained by solving the Schrödinger equation with two different potential, the TGB and PT potentials. This analysis has allowed us to extend Eq. (2) to describe the ground and excited states of an N-boson systems. Varying the two-body scattering length, we have demonstrated that scaled ground- and excited-state energies of systems up to (at least) $N = 6$ collapse over the same universal curve, which is parametrized by the universal function appearing in Eq. (6); the only parameters are the N-body scaling parameters κ_m^N and the finite-range scaling parameters Γ_m^N. In the Appendix we further show that our scaling hypothesis applies also to calculations of Ref. [41] up to $N = 8$ particles.

In the final part of the work we have proposed a linear relation with N of the scaling parameters κ_0^N valid in the zero-range limit. This relation has been extracted from our numerical results given in Tables I and II, and points to the fact that, in the scaling limit, all the parameters are governed by the sole emergent three-body parameter.

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We grateful acknowledge Prof. Lev Khaykovich for providing us with the experimental data of Refs. [10,26].

APPENDIX: ANALYSIS OF DATA OF REF. [41]

In this appendix we show that our theory can be applied to calculations of Ref. [41], where the ground-state energies are calculated up to $N = 8$ and where different potentials have used. In that paper the author performed few-body calculations using two- plus three-body Gaussian potentials for $N = 6$, and a two-body square-well potential plus three-body hard-wall potential for $N = 7,8$. In that paper the author gives a four-parameter parametrization of

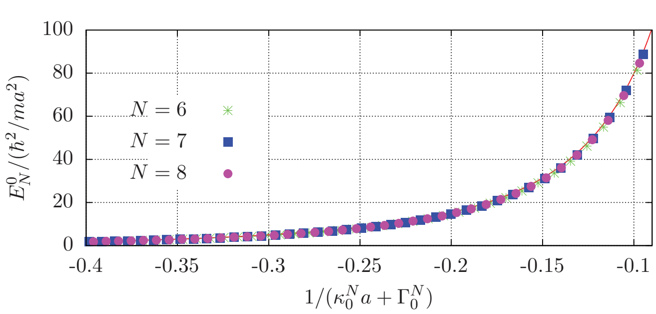

FIG. 6. (Color online) The ground-state N-body binding energy E_N^0 from Ref. [41] in units of $\hbar^2/ma^2$ as a function of $1/(\kappa_0^N a + \Gamma_0^N)$. The solid curve represent the universal zero-range curve.

TABLE III. The parameters κ_0^N, in units of κ_0^3, and Γ_0^N, used to analyze the data of Ref. [41].

	$N = 6$	$N = 7$	$N = 8$
κ_0^N/κ_0^3	4.29	5.22	6.18
Γ_0^N	−0.393	−0.414	−0.382

the ground-state energies for $N = 6,7,8$, and we have used that parametrization in order to reconstruct the calculated data.

In Fig. 6 we show that the extracted data, once analyzed with Eq. (6) of our paper, collapse onto the universal zero-range curve. As a side effect, we see that only two parameters are needed to describe data, and that the collapse does not depend on the Gaussian form of the potential. In Table III we report the parameters we used to make the collapse of the data onto the universal curve.

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