

High-fidelity error-resilient composite phase gates

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We present a method to construct high-fidelity quantum phase gates, which are insensitive to errors in various experimental parameters. The phase gates consist of a pair of two sequential broadband composite pulses, with a phase difference $\pi + \alpha/2$ between them, where α is the desired gate phase. By using composite pulses which compensate systematic errors in the pulse area, the frequency detuning, or both the area and the detuning, we thereby construct composite phase gates which compensate errors in the same parameters. Particularly interesting are phase gates which use the recently discovered universal composite pulses, which compensate systematic errors in *any* parameter of the driving field, which keep the evolution Hermitian (e.g., pulse amplitude and duration, pulse shape, frequency detuning, Stark shifts, residual frequency chirps, etc.).

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I. INTRODUCTION

Quantum computers exploit the coherent superposition nature of quantum states and involve numerous phase-sensitive manipulations [1]. Insofar as quantum algorithms, such as Shor's factorization [2] and Grover's search [3], involve a great number of phase gates, the accuracy of the latter is of crucial importance for high-fidelity quantum information processing.

As far as a phase shift of π is concerned the simplest approach is to use a resonant 2π pulse that couples one of the qubit states with an ancilla state. A variable phase shift α , however, requires a field with a suitable detuning, intensity, and duration; such a variable phase shift is required, for example, for the construction of conditional quantum gates and the quantum Fourier transform [1].

There are two major types of phase gates: dynamic [4] and geometric [5]. The dynamic phase gate benefits from the simplicity of implementation because it requires just a single far-off-resonant pulsed field, which determines its widespread use. The geometric phase gate has certain advantages in terms of robustness against parameter fluctuations, which come at the cost of more demanding implementations. An alternative phase gate uses adiabatic passage and relative laser phases [6].

In the present paper we propose a different approach to construct a phase gate with an arbitrary phase. Our method is based on the use of composite pulses (CPs) [7–10], which are a powerful method for quantum state control. CPs combine high accuracy of manipulation with robustness to variations of the interaction parameters. A CP is a sequence of pulses with suitably chosen relative phases. These phases are used as control parameters to correct the errors which emerge in the interaction between a single pulse and a qubit. The vast majority of CPs are designed to produce complete or partial population transfer in two-state or multistate quantum systems. Here we show how CPs can be used to produce well-defined phase shifts of the two states of a qubit.

The paper is organized as follows. First, we start with a brief overview of the theory of CPs and we show how CPs can be derived. Then we explain how CPs can be used to produce broadband (BB), adiabatic, and universal robust phase gates, which are insensitive to various types of experimental errors. Then we test the performance of the proposed gates in terms of fidelity and robustness.

II. COMPOSITE PULSES

Let us consider a two-state quantum system (a qubit), in a general state $|\Psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$, interacting with an external coherent field. Our goal is to create a phase gate, which is defined as an operation which changes the phase difference between c_1 and c_2 by some predefined amount α . In a matrix form it can be written as

$$\Phi = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}. \quad (1)$$

In order to achieve this, we are going to use the powerful method of CPs. To explain the idea of CPs, we first note that the evolution of our qubit is described by the Schrödinger equation,

$$i\hbar\partial_t\mathbf{c}(t) = \mathbf{H}(t)\mathbf{c}(t), \quad (2)$$

where $\mathbf{c}(t) = [c_1(t), c_2(t)]^T$ is a column vector with the probability amplitudes of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$. The Hamiltonian after the rotating wave approximation [11] reads

$$\mathbf{H}(t) = (\hbar/2)\Omega(t)e^{-iD(t)}|\psi_1\rangle\langle\psi_2| + \text{H.c.}, \quad (3)$$

with $D = \int_0^t \Delta(t')dt'$, where $\Delta = \omega_0 - \omega$ is the detuning between the field frequency ω and the Bohr transition frequency ω_0 . The Rabi frequency $\Omega(t)$ is a measure of the field-system interaction: for laser-driven electric-dipole atomic transitions, $\Omega(t) = -\mathbf{d} \cdot \mathbf{E}(t)/\hbar$, where $\mathbf{E}(t)$ is the laser electric-field envelope and \mathbf{d} is the transition dipole moment of the atom. It is convenient to describe the evolution of the quantum system by means of the propagator $\mathbf{U}(t, t_i)$, which connects the probability amplitudes at any time t to their initial values at time t_i : $\mathbf{c}(t) = \mathbf{U}(t, t_i)\mathbf{c}(t_i)$. A general 2×2 unitary propagator is parametrized by the Cayley-Klein parameters a and b as

$$\mathbf{U} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}. \quad (4)$$

A constant phase shift ϕ in the driving field, $\Omega(t) \rightarrow \Omega(t)e^{i\phi}$, is mapped onto the propagator as

$$\mathbf{U}(\phi) = \begin{bmatrix} a & be^{i\phi} \\ -b^*e^{-i\phi} & a^* \end{bmatrix}. \quad (5)$$

If we have a sequence of n identical pulses, each with a phase ϕ_k , we obtain a CP whose effect upon the quantum system is described by the propagator

$$\mathbf{U}^{(n)} = \mathbf{U}(\phi_n) \cdots \mathbf{U}(\phi_2) \mathbf{U}(\phi_1). \quad (6)$$

If now the phases ϕ_k are chosen appropriately, the propagator $\mathbf{U}^{(n)}$ can be made much more robust to variations in the experimental parameters than the single-pulse propagator \mathbf{U} . This is the basic idea behind CPs and in such a way one can produce a huge variety of broadband [12,13], narrow-band [14], and passband [15] CPs with respect to variations in essentially any experimental parameter.

III. COMPOSITE PHASE GATES: GENERAL

In this section we show how the phase gate (1) can be constructed as a sequence of two CPs, with a fixed phase difference between them. In order to explain the idea, we first consider a simple sequence of two single pulses, wherein the second one has a relative phase χ with respect to the first one. By using the single-pulse propagators (4) and (5) we find that for the sequence of two pulses, the total propagator is

$$\begin{aligned} \mathbf{U}_{\text{tot}} &= \mathbf{U}(\chi) \mathbf{U}(0) \\ &= \begin{bmatrix} a^2 - |b|^2 e^{i\chi} & ab + a^* b e^{i\chi} \\ -a^* b^* - ab^* e^{-i\chi} & a^{*2} - |b|^2 e^{-i\chi} \end{bmatrix}. \end{aligned} \quad (7)$$

This simple expression reveals the idea of the phase gates proposed here: if each of the two single pulses is such that it causes complete population transfer ($a = 0$ and hence $|b| = 1$) and if we choose the phase χ of the second pulse to be equal to $\pi + \alpha/2$, we obtain for the total propagator $\mathbf{U}_{\text{tot}} = \mathbf{U}(\pi + \alpha/2) \mathbf{U}(0) = \Phi$, which is exactly the phase gate defined by Eq. (1). This can be achieved by a sequence of two resonant pulses, for which the Cayley-Klein parameters are $a = \cos A/2$, $b = -i \sin A/2$, where A is the pulse area. If we choose the area A to be equal to π , we obtain $a = 0$ and the total propagator is exactly the phase gate of Eq. (1).

This scheme, despite being quite simple and natural, has the same drawbacks as the single π pulses, regarding the lack of robustness against variations of the interaction parameters. However, it is possible to overcome these shortcomings by replacing the two single π pulses with two identical broadband CPs; then the Cayley-Klein parameters a and b are determined by each of these CPs. Each of the two CPs is robust against variations in one or more experimental parameters and hence we obtain a robust phase gate. Explicitly, the total propagator reads

$$\mathbf{U}_{\text{tot}} = \mathbf{U}_{\text{CP}_2} \mathbf{U}_{\text{CP}_1}, \quad (8)$$

where

$$\mathbf{U}_{\text{CP}_1} = \mathbf{U}(\phi_n) \cdots \mathbf{U}(\phi_2) \mathbf{U}(\phi_1), \quad (9a)$$

$$\mathbf{U}_{\text{CP}_2} = \mathbf{U}(\chi_n) \cdots \mathbf{U}(\chi_2) \mathbf{U}(\chi_1), \quad (9b)$$

and

$$\chi_k = \phi_k + \pi + \alpha/2. \quad (10)$$

Since the phases ϕ_k and χ_k are connected with this simple relation, from now on we will only refer to ϕ_k . The phases ϕ_k are chosen in such a way that they produce robust propagators U_{CP_1} and U_{CP_2} , which are insensitive to variations in different

experimental parameters. It is straightforward to verify that if each of the two CPs of Eqs. (8) and (9) produces complete population inversion up to the order ϵ^m , i.e., $a = O(\epsilon^m)$, where ϵ is the deviation from the desired value of some interaction parameter, then the composite phase gate of Eq. (7) will produce the desired target phase gate of Eq. (1) with the same accuracy $O(\epsilon^m)$. In other words, the composite phase gate is accurate to the same order as the two composite π pulses that it is composed of.

Depending on which type of CPs we use, we can produce different types of phase gates. For example, a phase gate that is insensitive to errors in the pulse area can be produced by using BB CPs, which are robust against variations in the pulse area too. In a similar way we can produce phase gates that are robust against any single parameter, or even such phase gates that are robust against *all* experimental parameters. In the next section, we will study these cases more closely.

In order to test the performance of our phase gates, we use the infidelity (failure) measure F , which we define as the Frobenius norm of the distance between the actual composite gate Φ' and the desired phase gate Φ of Eq. (1),

$$F = \sqrt{\sum_{jk} |\Phi'_{jk} - \Phi_{jk}|^2}. \quad (11)$$

IV. COMPOSITE PHASE GATES: EXAMPLES

A. Broadband composite phase gates

A phase gate, which is robust against errors in the pulse area, is produced by using a sequence of two area-compensating CPs. Numerous such CPs have been proposed and demonstrated in the literature [9]. We use here the symmetric resonant CPs that we have derived recently [12]; their phases are given by the analytic formula [16]

$$\phi_k = k(k-1) \frac{\pi}{n} \quad (k = 1, 2, \dots, n), \quad (12)$$

where n is the number of pulses, used in the CP. Explicitly, the phases of the first few CPs are (modulo 2π)

$$(0, \frac{2}{3}, 0)\pi, \quad (13a)$$

$$(0, \frac{2}{5}, \frac{6}{5}, \frac{2}{5}, 0)\pi, \quad (13b)$$

$$(0, \frac{2}{7}, \frac{6}{7}, \frac{12}{7}, \frac{6}{7}, \frac{2}{7}, 0)\pi, \quad (13c)$$

$$(0, \frac{2}{9}, \frac{2}{3}, \frac{4}{3}, \frac{2}{9}, \frac{4}{3}, \frac{2}{9}, 0)\pi. \quad (13d)$$

The phases of the second CP are found from here and Eq. (10). These phases allow one to suppress the error in the transition probability up to order $O(\epsilon^{2n})$, where ϵ is the deviation of the pulse area A from π , $A = \pi(1 + \epsilon)$. Obviously, the total number of pulses needed for the composite phase gate is $2n$. As an example, for $n = 3$, the total number of pulses is $2n = 6$ and the relative phases are

$$(0, \frac{2}{3}\pi, 0, \pi + \frac{1}{2}\alpha, \frac{5}{3}\pi + \frac{1}{2}\alpha, \pi + \frac{1}{2}\alpha). \quad (14)$$

Figure 1 shows the infidelity of our composite phase gate as a function of the pulse area of the pulses in the sequence for target phases $\alpha = \pi/2$ and $\alpha = \pi/4$. These phase gates are the most widely used ones in quantum information processing [1]. We see that by increasing the number of pulses in the two

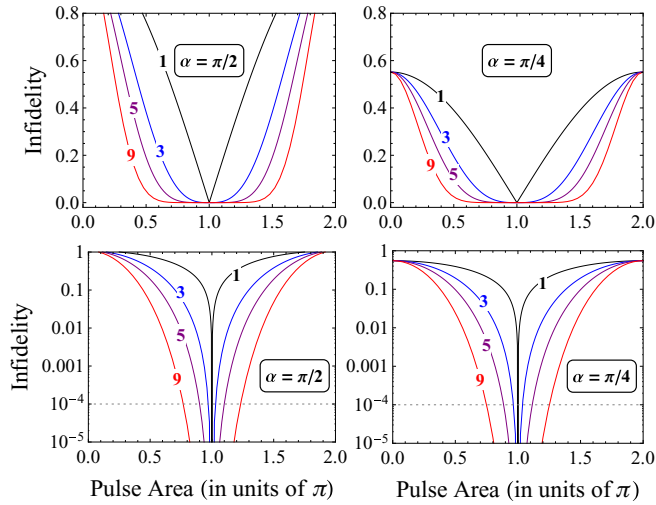


FIG. 1. (Color online) Infidelity of the BB phase gate as a function of pulse area, for $n = 1$, $n = 3$, $n = 5$, and $n = 9$. The target phase is $\alpha = \pi/2$ (left frames) and $\alpha = \pi/4$ (right frames). The lower frames show the same infidelities as the upper frames, but in a logarithmic scale.

CPs, which compose the phase gates, the latter get increasingly robust to variations in the pulse area.

B. Adiabatic composite phase gates

It was demonstrated in [13] that CPs can be used to improve the adiabatic passage by using a sequence of phase-shifted

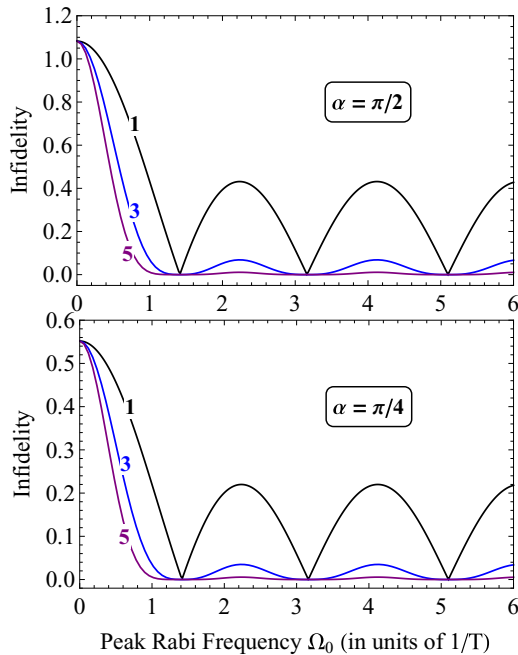


FIG. 2. (Color online) Infidelity of the adiabatic phase gate for $n = 3$ and $n = 5$, compared with the infidelity, produced by a sequence of two single (noncomposite) chirped pulses. We use hyperbolic-secant pulses, $\Omega(t) = \Omega_0 \operatorname{sech}(t/T)$, and hyperbolic-tangent detuning, $\Delta(t) = B \tanh(t/T)$, where Ω_0 is the peak Rabi frequency, B is the chirp rate, and T is the pulse duration. We have used chirp rate $B = 1/T$. The target phase is $\alpha = \pi/2$ (top frame) and $\alpha = \pi/4$ (bottom frame).

chirped pulses. It was shown that the composite phases do not depend on the particular shape of the pulse and the chirp, but only demand symmetric Rabi frequency and antisymmetric detuning. The analytic formula for the phases is the same as the one for the BB pulses, Eq. (12). In Fig. 2 we compare the infidelities of the composite adiabatic phase gates for $n = 3$ and $n = 5$ with the single-pulse adiabatic phase gate ($n = 1$). We conclude that the composite adiabatic phase gate is extremely robust against variations in the peak Rabi frequency, which is due to the high fidelity and robustness of the composite adiabatic passage [13].

C. Detuning-compensated composite phase gates

Another major type of CPs are the ones which stabilize the excitation profile with respect to the frequency detuning from exact resonance. These can be used to construct detuning-compensated composite phase gates. We have derived earlier [12] several detuning-compensated CPs,

$$(0, \frac{1}{3}\pi, 0), \quad (15a)$$

$$(0, 0.747, 0.424, 0.747, 0)\pi, \quad (15b)$$

$$(0, 1.308, 1.153, 1.251, 0.562, 1.251, 1.153, 1.308, 0)\pi. \quad (15c)$$

The first, second, and third CPs of these produce a unit transition probability around single pulse areas of π , $3\pi/5$, and $4\pi/9$, respectively.

The detuning-compensated composite phase gates are obtained by using Eqs. (10) and (15). For example, the

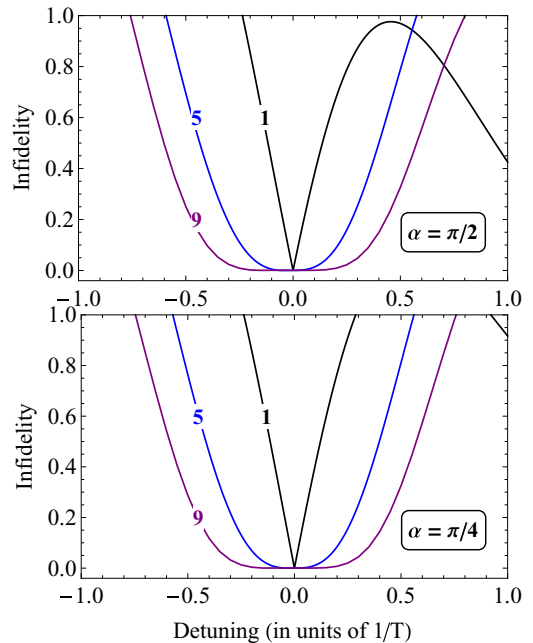


FIG. 3. (Color online) Infidelity of the detuning-compensated phase gate for a sequence of two single hyperbolic secant pulses ($n = 1$) and for a sequence of two CPs with $n = 5$ and $n = 9$. The composite phases are given by Eq. (15). The target phase is $\alpha = \pi/2$ (top frame) and $\alpha = \pi/4$ (bottom frame).

TABLE I. Universal CPs for complete population inversion, which compensate errors in arbitrary field parameters [17].

CP	$(\phi_1, \phi_2, \dots, \phi_n)$
U3	$(0, \frac{1}{2}, 0)\pi$
U5a	$(0, \frac{5}{6}, \frac{1}{3}, \frac{5}{6}, 0)\pi$
U5b	$(0, \frac{11}{6}, \frac{1}{3}, \frac{11}{6}, 0)\pi$
U7a	$(0, \frac{11}{12}, \frac{5}{6}, \frac{17}{12}, \frac{5}{6}, \frac{11}{12}, 0)\pi$
U7b	$(0, \frac{23}{12}, \frac{5}{6}, \frac{5}{12}, \frac{23}{12}, 0)\pi$
U13a	$(0, \frac{3}{8}, \frac{42}{24}, \frac{11}{24}, \frac{8}{24}, \frac{37}{24}, \frac{2}{24}, \frac{37}{24}, \frac{8}{24}, \frac{11}{24}, \frac{42}{24}, \frac{3}{8}, 0)\pi$
U13b	$(0, \frac{33}{24}, \frac{42}{24}, \frac{35}{24}, \frac{8}{24}, \frac{13}{24}, \frac{2}{24}, \frac{13}{24}, \frac{8}{24}, \frac{35}{24}, \frac{42}{24}, \frac{33}{24}, 0)\pi$

six-pulse phase gate reads

$$(0, \frac{1}{3}\pi, 0, \pi + \frac{1}{2}\alpha, \frac{4}{3}\pi + \frac{1}{2}\alpha, \pi + \frac{1}{2}\alpha). \quad (16)$$

In Fig. 3 we plot the simulated infidelity of such gates. It is evident from the figure that the phase gates are robust against variations in the detuning, which makes the method very useful for applications in situations, when exact resonance is not possible (for instance, in the presence of inhomogeneous broadening or Doppler shift).

If we want to achieve compensation of errors in more than one parameter, we need to use CPs, which are robust against variations in several parameters, for instance pulse area and detuning [12]. In the next subsection we examine the cases when a more general compensation of errors is possible.

D. Universal composite phase gates

Recently, Genov *et al.* [17] derived CPs, which compensate errors in any parameter of the driving field. This is done by using the most general parametrization (4) of the propagator, without any assumptions for the properties of the constituent pulses and their parameters, which justifies the term “universal” CPs. The phases of the lowest-order universal CPs are given in Table I. By using these universal CPs, we can build a universal phase gate, which is insensitive, up to a certain order, to any experimental error of the driving field. For example, the six- and ten-pulse composite phase gates read

$$\text{UPh6} = (0, \frac{1}{2}\pi, 0, \pi + \frac{1}{2}\alpha, \frac{3}{2}\pi + \frac{1}{2}\alpha, \pi + \frac{1}{2}\alpha), \quad (17a)$$

$$\begin{aligned} \text{UPh10a} = & (0, \frac{5}{6}\pi, \frac{1}{3}\pi, \frac{5}{6}\pi, 0, \pi + \frac{1}{2}\alpha, \frac{11}{6}\pi + \frac{1}{2}\alpha, \frac{4}{3}\pi \\ & + \frac{1}{2}\alpha, \frac{11}{6}\pi + \frac{1}{2}\alpha, \pi + \frac{1}{2}\alpha), \end{aligned} \quad (17b)$$

$$\begin{aligned} \text{UPh10b} = & (0, \frac{11}{6}\pi, \frac{1}{3}\pi, \frac{11}{6}\pi, 0, \pi + \frac{1}{2}\alpha, \frac{5}{6}\pi + \frac{1}{2}\alpha, \frac{4}{3}\pi \\ & + \frac{1}{2}\alpha, \frac{5}{6}\pi + \frac{1}{2}\alpha, \pi + \frac{1}{2}\alpha), \end{aligned} \quad (17c)$$

where a and b refer to different universal CP solutions [17].

In Fig. 4, we test the performance of the universal phase gates by plotting the infidelity as a function of the pulse duration and the detuning. We see that by using the single-pulses approach, it is almost impossible to achieve high fidelity, while universal CPs deliver quite large high-fidelity areas of $F < 0.01$, and even areas of ultrahigh fidelity $F < 10^{-4}$.

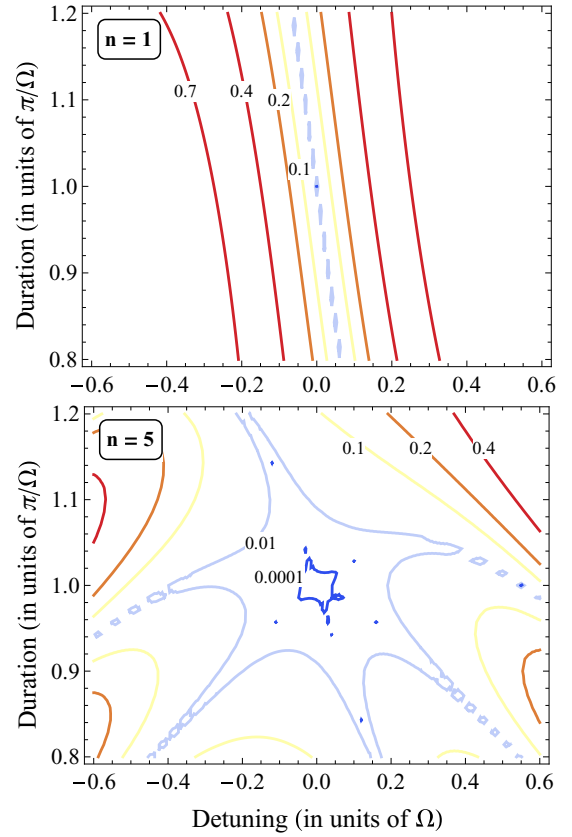


FIG. 4. (Color online) Infidelity of the universal phase gate for (top frame) a sequence of two rectangular pulses and (bottom frame) a sequence of two CPs with $n = 5$. The composite phases are $\phi_k = (0, 11, 2, 11, 0)\pi/6$. The target phase is $\alpha = \pi/4$.

V. CONCLUSIONS

We have presented an approach to construct high-fidelity error-resistant composite phase gates for quantum information processing. These phase gates are formed by two identical composite π pulses, the second of which is shifted by phase $\pi + \alpha/2$ with respect to the first. The properties of these CPs are directly transferred to the composite phase gate. For example, a composite π pulse, which compensates errors in the pulse area to a certain order ϵ^m , produces a composite phase gate, which compensates errors in the pulse area to the same order ϵ^m . Of special interest are the universal phase gates, which compensate errors in all field parameters (pulse duration, pulse amplitude, detuning, unwanted chirp, Stark shift, etc.). We note that these universal CPs have the huge practical advantage that they simultaneously compensate all kinds of errors, even unknown ones, as far as the evolution is coherent and hence unitary (no losses to other states). However, if we wish robustness against a single parameter, e.g., pulse area or detuning, then better performance is provided by the dedicated CP, which compensates this particular error.

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