Using a biased quantum random walk as a quantum lumped element router

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Quantum random walks have received much attention for their intrinsic interest and many possible uses and have been experimentally demonstrated. In this work we look at the possibility of using a biased one-dimensional (1D) quantum walk as an element within a larger quantum device. We ask whether one can use a quantum walk to act as a router with one bias setting engineering the quantum walk to route probability flow one direction while another bias setting routes flow in the opposite direction. Appealing to electrical circuit terminology, we consider a biased quantum walk over a large spatial lattice to act as a single "lumped element" whose routing action depends on the coin bias. We discover that the lumped-element current, when summed over the quantum walk lattice, reaches a steady state and for specific initial states we derive an analytic form for this steady-state lumped-element current. We show that we can control the magnitude and the direction (routing) of the steady-state current. Curiously the control phase and steady-state total current exhibits a sinusoidal current-phase relationship indicating that the lumped element may be similar to that found in Josephson junctions. Finally we illustrate that conservative 1D Hamiltonian systems can also exhibit steady-state dynamics similar to the quantum walk.

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I. INTRODUCTION

Discrete quantum systems can be used to study novel quantum phenomena such as quantum walks (QWs). Quantum walks have received much attention over the past two decades, initially motivated as being the quantum analog of the well-known classical random walk (RW). Initial interest in QWs focused on the behavior of the spatial evolution of the walks. In the classical case the spatial probability distribution of the walker after a time τ is Gaussian with a width $\sigma(\tau) \sim \tau^{1/2}$ while the typical coined quantum walk has a bimodal spatial distribution with a width that scales linearly with time, $\sigma(\tau) \sim \tau$ [1–3].

Beyond fundamental interest, researchers have also shown that quantum walks can be the basis for constructing quantum algorithms, such as graph searching [4–6] and graph isomorphism testing [7,8], and for moving towards full-blown quantum computing [9], which, in part, is due to the presence of quantum entanglement in quantum walks [9,10]. A number of interesting physical implementations of quantum walks have been proposed, including trapped ions [11], Bose-Einstein condensates [12], linear optics [13], neutral atoms in optical lattices [14], and circuit quantum electrodynamics [15], and a number of experimental demonstrations include nuclear magnetic resonance (NMR) [16] and photonic systems [17]. Finally, an emerging direction of study for quantum walks is their use in quantum simulation, for example, the simulation of bosonic or fermionic quantum walks using integrated photonics [18].

In this work we look at the possibility of using a biased one-dimensional (1D) quantum walk as an element within a larger quantum device to route quantum information either in one direction or another by appropriately biasing the quantum walk. By considering the entire quantum walk as a single "lumped element" whose routing action depends on the coin bias (see Fig. 1), we investigate this idea further and find that we must generalize the concept of a quantum probability current density, which is typically defined for a continuous time and space setup, to the case of a coined quantum walk with discrete time and space. Although there are many ways to perform such a generalization, we argue that to be physically relevant the generalized discrete probability current density must satisfy a discretized continuity equation which essentially encodes the microscopic detailed balance of the probability density with time. We show that the lumped-element current, obtained by summing over the entire quantum walk lattice, reaches a steady state and for specific initial states we derive an analytic form for this steady-state current. We show that by altering a phase factor within the biased coin we can engineer the magnitude and the direction (routing) of the steady-state current. The control phase and steady-state total current exhibits a sinusoidal current-phase relationship indicating some similarity to the behavior of a Josephson junction. Finally we illustrate that conservative 1D Hamiltonian systems can also exhibit steady-state dynamics similar to the lumped-element quantum walk router.

As mentioned above, central to our study is the existence of a probability current density which satisfies an appropriately physical continuity relation. We note that although quantum walks have received much attention and that their evolution can be studied analytically we are aware of only two previous studies that make use of a QW probability current density [19,20]. Moreover, in both cases, neither of their proposed currents satisfies a continuity equation and thus are physically of little relevance. Below we derive a form for the local probability current density which manifestly satisfies the local discrete continuity equation for probability. We derive an analytical expression for the *total current* when summed over the spatial lattice of the QW—this corresponds to the current through the entire lumped element of the QW router—and we show that this total current reaches a steady state whose value can be controlled by considering a two-parameter family of $SU(2)$ biased coined QWs. We find that the total stationary current is an oscillatory function of this one parameter and we discuss the similarity in behavior to the current-phase relation of Josephson junctions. We then show that one can

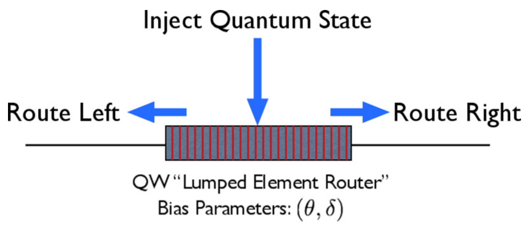

FIG. 1. (Color online) Using the "lumped-element" biased quantum walk as a router. We examine whether one can control the routing of an initial quantum state injected at the middle of a quantum walk to either end of the lumped element. We consider a 1D biased quantum walk with a large number of lattice sites as a lumped element, which can be used to redirect quantum probability to either end and into attached quantum wires (not described here). The bias parameters (θ,δ) parametrize the type of coin used in the quantum walk.

find a more symmetric expression for the local probability current density through a central difference approach to the continuity equation and finally we explore what types of continuous conservative dynamical systems possess similar stationary currents.

II. DISCRETE-TIME COINED QUANTUM WALK

We quickly review the model of the typical coined quantum walk. For a detailed overview of quantum walks, see [2]. In this model, a particle with an internal chirality (or two-state coin) resides on a discrete 1D lattice and is displaced conditioned on the state of the quantum coin, where the latter periodically experiences an operation similar to coin tossing in a classical random walk. Let the position eigenkets of the particle at position n be $|n\rangle \in \mathcal{H}_p$ and with an internal chirality state $|c\rangle \in \mathcal{H}_c$, where $\mathcal{H}_p$ and $\mathcal{H}_c$ are the position and coin Hilbert space, respectively: $\mathcal{H}_p = \{|n\rangle : n \in \mathbb{Z}\}, \mathcal{H}_c = \{|\uparrow\rangle, |\downarrow\rangle\}$, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are the coin states that specifies the direction of motion. The combined Hilbert space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$ and the coin-flip operation is performed via a unitary coin operator, $\hat{C}$, which operates only on the coin Hilbert space. The most general form of this operator is

$$\hat{C}(\xi,\theta,\eta) = \begin{pmatrix} e^{i\xi}\cos\theta & e^{i\eta}\sin\theta \\ e^{-i\eta}\sin\theta & -e^{-i\xi}\cos\theta \end{pmatrix}, \tag{1}$$

which is a three-parameter unitary operator. The unitary conditional translation (or step) operator $\hat{S}$ is

$$\hat{S} = |\uparrow\rangle\langle\uparrow| \otimes |n+1\rangle\langle n| + |\downarrow\rangle\langle\downarrow| \otimes |n-1\rangle\langle n|, \tag{2}$$

which implies that, if the particle is at position n with an internal coin state $|\uparrow\rangle$, $\hat{S}$ shifts the particle to position $n+1$, whereas if its chirality state is $|\downarrow\rangle$, $\hat{S}$ shifts it to position $n-1$. If we put $\theta = \frac{\pi}{4}$, $\xi = \eta = 0$, we realize the Hadamard coin that has been used extensively to study quantum walks [1,21].

In our study of the transport properties of the quantum walk we use the two-parameter unitary coin operator $\hat{C}(\theta,\delta)$:

$$\hat{C}(\theta,\delta) = \begin{pmatrix} \cos\theta & e^{i\delta}\sin\theta \\ e^{-i\delta}\sin\theta & -\cos\theta \end{pmatrix}, \tag{3}$$

which we find sufficient to study the properties of quantum transport in our model, namely the spread of the quantum walk through θ and also any bias of the walk in the 'lumped element' through δ. Moreover, later in the paper we show that any other phase factor in a three-parameter coin appears only in combination with the phase factors of the initial coin state of the walker, a fact that has been noted too by previous authors [22]. Hence, we mainly use a two-parameter coin operator.

We represent the state of the quantum particle at any time t by a state vector

$$|\psi(t)\rangle = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \alpha_{n,t} \\ \beta_{n,t} \end{pmatrix} \otimes |n\rangle = \sum_{n=-\infty}^{\infty} \alpha_{n,t}|\uparrow,n\rangle + \beta_{n,t}|\downarrow,n\rangle, \tag{4}$$

where $\alpha_{n,t}$ and $\beta_{n,t}$ are the amplitudes associated with the walker which has a given chirality. We have chosen the representation $|\uparrow\rangle = \binom{1}{0}$ and $|\downarrow\rangle = \binom{0}{1}$. The update rule for our quantum walk is given by

$$|\psi(t+1)\rangle = \hat{S}(\hat{C}\otimes \mathbb{1})|\psi(t)\rangle, \tag{5}$$

from which we obtain the (recurrence) relations satisfied by the amplitudes $\alpha_{n,t}$ and $\beta_{n,t}$,

$$\begin{aligned} \alpha_{n,t} &= \cos\theta\alpha_{n-1,t-1} + e^{i\delta}\sin\theta\beta_{n-1,t-1}, \\ \beta_{n,t} &= e^{-i\delta}\sin\theta\alpha_{n+1,t-1} - \cos\theta\beta_{n+1,t-1}. \end{aligned} \tag{6}$$

These are the basic recurrence relations defining the evolution of the quantum walk on the 1D lattice which we use to derive the probability current density. Denoting the probability $\rho(n,t)$ for the particle to be found at position n at time t as $\rho(n,t) = |\alpha_{n,t}|^2 + |\beta_{n,t}|^2$, we have the conservation of probability with time, i.e., $\sum_{n=-\infty}^{n=\infty}\rho(n,t) = 1,\ \forall t$.

III. PROBABILITY CURRENT DENSITY OF THE COINED QUANTUM WALK

Some authors have previously found an expression for the probability current density for the QW of the form [19,20]

$$j(n,t) = |\alpha_{n,t}|^2 - |\beta_{n,t}|^2, \tag{7}$$

which, however, does not satisfy the continuity relation and hence is not suitable for our purpose. We now outline how to derive an expression for the current density in our quantum walk from the recurrence relations (6) starting from the continuity equation. The continuity equation (in its continuous form) can be written as $\partial_x j = -\partial_t \rho$ and implies that the net flow of probability, $\partial_x j$, into or out of a region is equal to the rate of change of the overall probability, $-\partial_t\rho$, in that region. We consider the discrete version of the continuity equation to be

$$-\triangle_t\ \rho(n,t) = \triangle_n\ j(n,t), \tag{8}$$

where $\triangle_{n,t}$ is a forward difference operator in space and time given as $\triangle_t\ \rho(n,t) \equiv \rho(n,t+1) - \rho(n,t)$, and $\triangle_n\ j(n,t) \equiv j(n+1,t) - j(n,t)$.

The left-hand side of Eq. (8) can be computed easily if we consider that the local probability $\rho(n,t) = |\alpha_{n,t}|^2 + |\beta_{n,t}|^2$

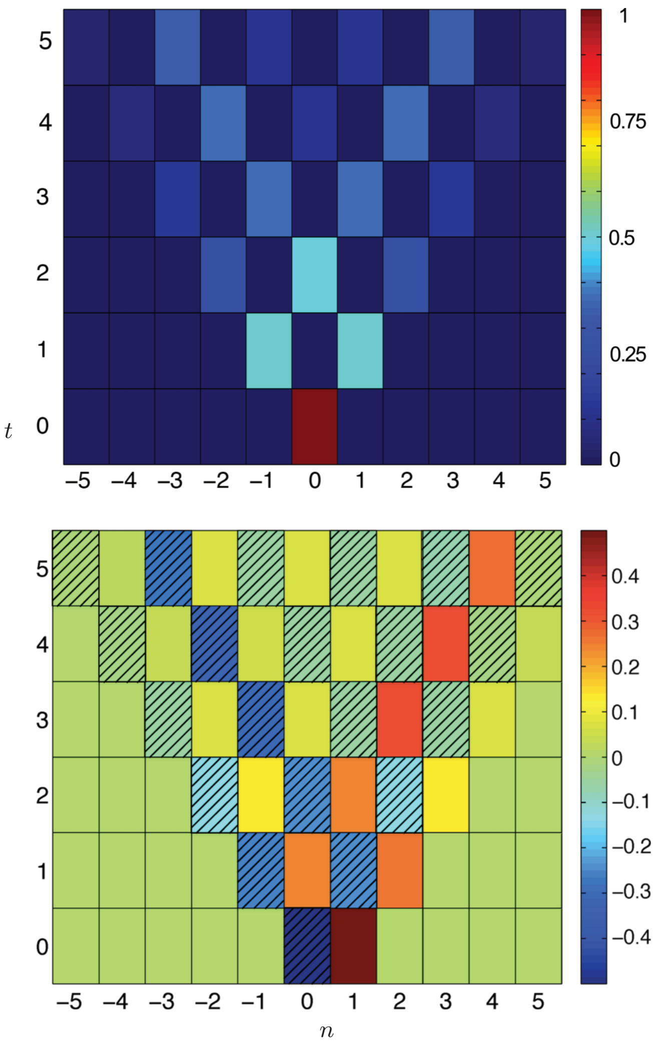

FIG. 2. (Color online) For a QW which is initialized to yield a spatially symmetric walk, where $(\alpha_{n,0},\beta_{n,0}) = \frac{1}{\sqrt{2}}(1,i)\delta_{n,0}$, i.e., where the walker is initially located at the origin, and using $\theta = \pi/4, \delta = 0$ (a Hadamard coin), we plot (top) the probability for the walker to be found at spatial location n at time t, i.e., $\rho(n,t)$, and (bottom) the probability current density $j(n,t)$. We see that although the probability evolves in a spatially symmetric fashion the forward-difference defined probability current density breaks this symmetry.

and then use the recurrence equations (6) to get $\rho(n,t+1)$:

$$\begin{aligned}
-\triangle_t\, \rho(n,t) &= |\alpha_{n,t}|^2 + |\beta_{n,t}|^2 - \cos^2\theta|\alpha_{n-1,t}|^2 \\
&\quad - \sin^2\theta|\beta_{n-1,t}|^2 - \sin^2\theta|\alpha_{n+1,t}|^2 - \cos^2\theta|\beta_{n+1,t}|^2 \\
&\quad + \sin 2\theta\, \mathrm{Re}\{e^{i\delta}(\alpha^*_{n+1,t}\beta_{n+1,t} - \alpha^*_{n-1,t}\beta_{n-1,t})\} \qquad (9)
\end{aligned}$$

This equation can be arranged in a more suggestive form as

$$\begin{aligned}
-\triangle_t\, \rho(n,t) &= (\cos^2\theta|\alpha_{n,t}|^2 - \sin^2\theta|\alpha_{n+1,t}|^2) \\
&\quad - (\cos^2\theta|\alpha_{n-1,t}|^2 - \sin^2\theta|\alpha_{n,t}|^2) \\
&\quad + (\sin^2\theta|\beta_{n,t}|^2 - \cos^2\theta|\beta_{n+1,t}|^2) \\
&\quad - (\sin^2\theta|\beta_{n-1,t}|^2 - \cos^2\theta|\beta_{n,t}|^2) \\
&\quad + \sin 2\theta\, \mathrm{Re}[e^{i\delta}(\alpha^*_{n+1,t}\beta_{n+1,t} + \alpha^*_{n,t}\beta_{n,t})] \\
&\quad - \sin 2\theta\, \mathrm{Re}[e^{i\delta}(\alpha^*_{n,t}\beta_{n,t} + \alpha^*_{n-1,t}\beta_{n-1,t})], \qquad (10)
\end{aligned}$$

and from this the following expression for the probability current density can be read:

$$\begin{aligned}
j(n,t) &= \cos^2\theta(|\alpha_{n-1,t}|^2 - |\beta_{n,t}|^2) \\
&\quad - \sin^2\theta(|\alpha_{n,t}|^2 - |\beta_{n-1,t}|^2) \\
&\quad + \sin 2\theta\, \mathrm{Re}[e^{i\delta}(\alpha^*_{n-1,t}\beta_{n-1,t} + \alpha^*_{n,t}\beta_{n,t})]. \qquad (11)
\end{aligned}$$

It is interesting to note that the current density $j(n,t)$ in Eq. (11) is more involved than had been initially supposed [Eq. (7)]. Interestingly we find a dependence on the interferences between $\alpha_{n,t}$ and $\beta_{n,t}$ but since the amplitudes $\alpha_{n,t}$ and $\beta_{n,t}$ oscillate forever, $j(n,t)$ does not achieve a steady state. In the next section we instead consider the total cumulative current over the entire 1D lattice and find that this indeed does achieve a steady state. As an illustration we show in Fig. 2 the probability current density for the Hadamard QW ($\delta = 0$) as a function of time for a localized symmetric initial coin state $(\alpha_{n,0},\beta_{n,0}) = \frac{1}{\sqrt{2}}(1,i)\delta_{n,0}$. We observe that the probability current density as defined using the forward differences has some peculiarities; i.e., although the probability distribution for the evolving QW is spatially symmetric about the origin, the introduction of the forward difference has introduced an apparent symmetry breaking into the associated probability current density. Despite this, the forward difference probability current density still satisfies the associated continuity equation (8). This apparent asymmetry is remedied later on when we describe how to use a central difference form of Eq. (8).

IV. STEADY-STATE CURRENT

We now examine the asymptotic behavior of the "total" current on the entire 1D lattice. Let the total cumulative current $J(t)$ be defined as

$$J(t) = \sum_{n=-\infty}^{\infty} j(n,t). \qquad (12)$$

By using the dynamical recurrence equations we can derive a very compact analytical expression for the steady-state value of this current, e.g., $J_\infty \equiv \lim_{t\to+\infty} J(t)$. We also investigate the steady-state value J_∞ by numerical simulation using Eq. (6) and find perfect agreement with our analytic formula. We now define some useful quantities, the global probability amplitudes and global interference terms, as $\rho_+(t) \equiv \sum_{n=-\infty}^{\infty}|\alpha_{n,t}|^2$, $\rho_-(t) \equiv \sum_{n=-\infty}^{\infty}|\beta_{n,t}|^2$, and $Q(t) \equiv \sum_{n=-\infty}^{\infty}\alpha^*_{n,t}\beta_{n,t}$. From Eq. (6), we can find

$$\begin{aligned}
|\alpha_{n,t+1}|^2 &= \cos^2\theta|\alpha_{n-1,t}|^2 + \sin^2\theta|\beta_{n-1,t}|^2 \\
&\quad + \sin 2\theta\, \mathrm{Re}(e^{i\delta}\alpha^*_{n-1,t}\beta_{n-1,t}), \\
|\beta_{n,t+1}|^2 &= \sin^2\theta|\alpha_{n+1,t}|^2 + \cos^2\theta|\beta_{n+1,t}|^2 \\
&\quad - \sin 2\theta\, \mathrm{Re}(e^{i\delta}\alpha^*_{n+1,t}\beta_{n+1,t}), \qquad (13)
\end{aligned}$$

and summing over space in Eqs. (13) gives

$$\begin{aligned}\rho_+(t+1) &= \cos^2\theta\rho_+(t) + \sin^2\theta\rho_-(t)\\ &\quad + \sin 2\theta\,\mathrm{Re}(e^{i\delta}Q(t)),\\ \rho_-(t+1) &= \sin^2\theta\rho_+(t) + \cos^2\theta\rho_-(t)\\ &\quad - \sin 2\theta\,\mathrm{Re}(e^{i\delta}Q(t)). \end{aligned} \tag{14}$$

In the long-time limit, we define the following: $\rho_+(t \to \infty) = \Omega_+$, $\rho_-(t \to \infty) = \Omega_-$, $Q(t \to \infty) = Q_0$, and $J(t \to \infty) = J_\infty$.

In this limit, Eqs. (14), in matrix form, become

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \Omega_+ \\ \Omega_- \end{pmatrix} = 2\cot\theta\,\mathrm{Re}(e^{i\delta}Q_0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{15}$$

which, with $\Omega_+ + \Omega_- = 1$, has the solution

$$\begin{pmatrix} \Omega_+ \\ \Omega_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + 2\cot\theta\,\mathrm{Re}(e^{i\delta}Q_0) \\ 1 - 2\cot\theta\,\mathrm{Re}(e^{i\delta}Q_0) \end{pmatrix}. \tag{16}$$

This remarkable result was previously derived by Romanelli [23]. The steady-state current J_∞ then easily proceeds from Eqs. (11), (12), and (16) as

$$J_\infty = 2\cot\theta\,\mathrm{Re}(e^{i\delta}Q_0). \tag{17}$$

The above equation clearly indicates that the total current in our quantum walk depends on a number of factors including (a) the interferences through Q_0 which ultimately depend on the initial state $|\psi(t=0)\rangle$ and (b) the bias parameter δ.

The global interference term Q_0 can be computed using a Fourier series method and this same method has been quite successful in analyzing the asymptotics of quantum walks [1,24,25]. Using this method and assuming a sharply localized initial state $|\psi(t=0)\rangle$ at $n=0$ as

$$|\psi(t=0)\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ e^{i\gamma}\sin\frac{\phi}{2} \end{pmatrix} \otimes |n=0\rangle\,, \tag{18}$$

where $\phi \in [0,\pi]$ and $\gamma \in [0,2\pi]$, with some effort (see the Appendix), one can derive Q_0 to be

$$Q_0 = \frac{(1-\sin\theta)e^{-i\delta}\tan\theta}{2}\left[\cos\phi + \sin\phi\left(e^{-i(\delta+\gamma)}\tan\theta + i\sin(\delta+\gamma)\frac{\cos\theta}{1-\sin\theta}\right)\right]. \tag{19}$$

Through numerical simulation one can observe the asymptotic approach of $Q(t)$ to Q_0 (see Fig. 3) in the long-time limit.

From this we can express the steady-state total current J_∞, for the particular case of the initial state (18), as

$$J_\infty = (1-\sin\theta)\,[\cos\phi + \sin\phi\cos(\delta+\gamma)\tan\theta]\,, \tag{20}$$

which depends not only on the initial state through ϕ,γ, but also on the nature of the dynamics through θ and δ as shown in Fig. 4. The term $\cos(\delta+\gamma)$ indicates that J_∞ can be controlled in an identical fashion either by the coin bias factor δ or by the phase of the initial state. One has the freedom to adjust J_∞ dynamically irrespective of the initial state.

Finally it is curious to note that the sinusoidal dependence of the current on the phase δ (or γ) is very reminiscent of the sinusoidal current-phase relationship (CPR) found in Josephson junctions [26]. The highly nonlinear dependence of the Josephson current on the phase difference across the junction has led to numerous quantum devices, the most prominent being the superconducting quantum interference device (SQUID). Indeed, based on the Josephson CPR the total current in a SQUID varies sinusoidally with the magnitude of the trapped flux.

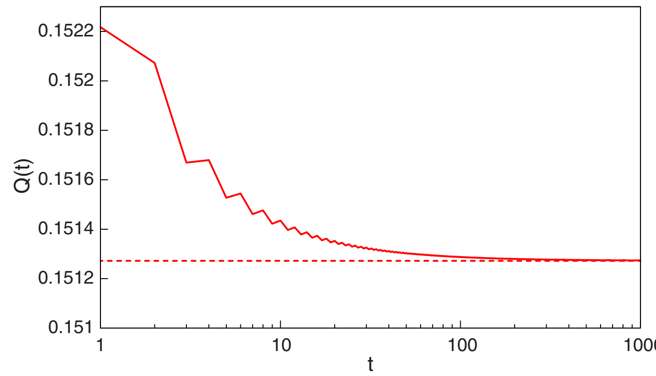

FIG. 3. (Color online) Global interference term $Q(t)$ for a symmetric localized initial coin state $(\alpha_{n,0},\beta_{n,0}) = \frac{1}{\sqrt{2}}(1,i)\delta_{n,0}$ and with coin parameters $\theta = \pi/4$, $\delta = 5\pi/12$, smoothed in time using a moving window average of width 10. The solid line is the numerical result while the dashed line is the asymptotic value Q_0.

V. SYMMETRIC PROBABILITY CURRENT DENSITY

Above we derived a probability current density according to a forward difference approximation to the continuity equation (8), and we noted that this lack of symmetry in this discretization led to peculiar asymmetric behaviors in the associated probability current density. We now show how this can be remedied by choosing a central difference version of the continuity equation, where we now choose

$$-\triangle_t^C\,\rho(n,t) = \triangle_n^C\,j^C(n,t), \tag{21}$$

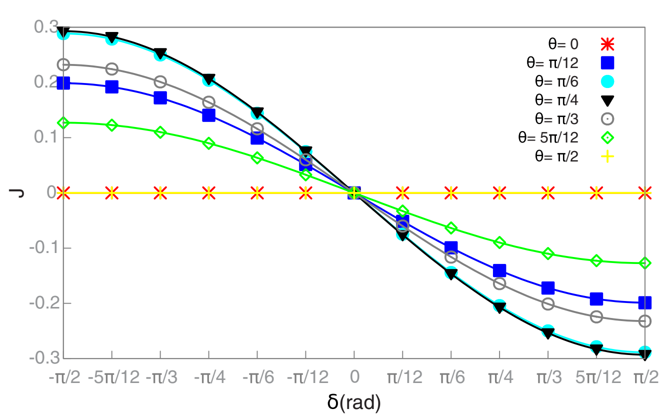

FIG. 4. (Color online) Total "steady" current J_∞ against δ for various coin values θ for a symmetric localized initial coin state $(\alpha_{n,0},\beta_{n,0}) = \frac{1}{\sqrt{2}}(1,i)\delta_{n,0}$. The points are the results of numerical simulation using Eq. (11) while the lines are the analytical result (20). We note that $J_\infty(-\delta) = -J_\infty(\delta)$; i.e., $J_\infty(\delta)$ is an odd function of δ.

where $\triangle^C_{n,t}$ are the central difference operators in space and time given as $\triangle^C_t\ \rho(n,t) \equiv [\rho(n,t+1) - \rho(n,t-1)]/2$, and $\triangle^C_n\ j(n,t) \equiv [j^C(n+1,t) - j^C(n-1,t)]/2$. We find that

$$j^C(n,t) = \cos^2\theta(|\alpha_{n,t}|^2 - |\beta_{n,t}|^2) + \sin 2\theta\ \mathrm{Re}[e^{i\delta}\alpha^*_{n,t}\ \beta_{n,t}]. \quad (22)$$

To show this we use the recurrence relations (6) to express $\rho(n,t+1)$ in terms of the amplitudes $\alpha_{k,t}$ and $\beta_{l,t}$, at time step t. We then note that these same recurrence relations can be recast in the matrix format

$$\begin{pmatrix}\alpha_{n+1,t}\\ \beta_{n-1,t}\end{pmatrix} = \begin{pmatrix}\cos\theta & e^{i\delta}\sin\theta\\ e^{-i\delta}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha_{n,t-1}\\ \beta_{n,t-1}\end{pmatrix}, \quad (23)$$

and as the transformation matrix in Eq. (23) is unitary we straightaway can reexpress $\rho(n,t-1)$ in terms of quantities at time t,

$$\rho(n,t-1) = |\alpha_{n,t-1}|^2 + |\beta_{n,t-1}|^2 = |\alpha_{n+1,t}|^2 + |\beta_{n-1,t}|^2. \quad (24)$$

With Eq. (24), one can write $\rho(n,t+1) - \rho(n,t-1)$, involving quantities expressed only at time step t, to be

$$\begin{aligned}
\triangle_t\ \rho &= \rho(n,t+1) - \rho(n,t-1)\\
\triangle_t \rho &= \cos^2\theta|\alpha_{n-1,t}|^2 + (\sin^2\theta - 1)|\beta_{n-1,t}|^2\\
&\quad + \sin 2\theta\ \mathrm{Re}[e^{i\delta}\alpha^*_{n-1,t}\beta_{n-1,t}]\\
&\quad + (\sin^2\theta - 1)|\alpha_{n+1,t}|^2 + \cos^2\theta|\beta_{n+1,t}|^2\\
&\quad - \sin 2\theta\ \mathrm{Re}[e^{i\delta}\alpha^*_{n+1,t}\ \beta_{n+1,t}]\\
&= \cos^2\theta(|\alpha_{n-1,t}|^2 - |\alpha_{n+1,t}|^2)\\
&\quad + \cos^2\theta(|\beta_{n+1,t}|^2 - |\beta_{n-1,t}|^2)\\
&\quad + \sin 2\theta\ \mathrm{Re}[e^{i\delta}(\alpha^*_{n-1,t}\beta_{n-1,t} - \alpha^*_{n+1,t}\beta_{n+1,t})]\\
&= \cos^2\theta|\alpha_{n-1,t}|^2 - \cos^2\theta|\beta_{n-1,t}|^2\\
&\quad + \sin 2\theta\ \mathrm{Re}[e^{i\delta}\alpha^*_{n-1,t}\ \beta_{n-1,t}]\\
&\quad -[\cos^2\theta|\alpha_{n+1,t}|^2 - \cos^2\theta|\beta_{n+1,t}|^2\\
&\quad + \sin 2\theta\ \mathrm{Re}[e^{i\delta}\alpha^*_{n+1,t}\ \beta_{n+1,t}]]. \quad (25)
\end{aligned}$$

Now using the central difference form for the continuity equation as $\rho(n,t+1) - \rho(n,t-1) = -(j^C(n+1,t) - j^C(n-1,t))$, by inspection we obtain the central difference probability current density (22).

In Fig. 5 we plot out the behavior of $j^C(n,t)$ for the same symmetric initial state as in Fig. 2, and now we observe that the spatial symmetry of the QW's evolution is maintained by j^C. We also can define the total symmetric probability current $J^C(t) \equiv \sum_{n=-\infty}^{\infty} j^C(n,t)$, and we find that $J^C(t)$ reaches a steady-state value which is identical to that found in the forward difference case.

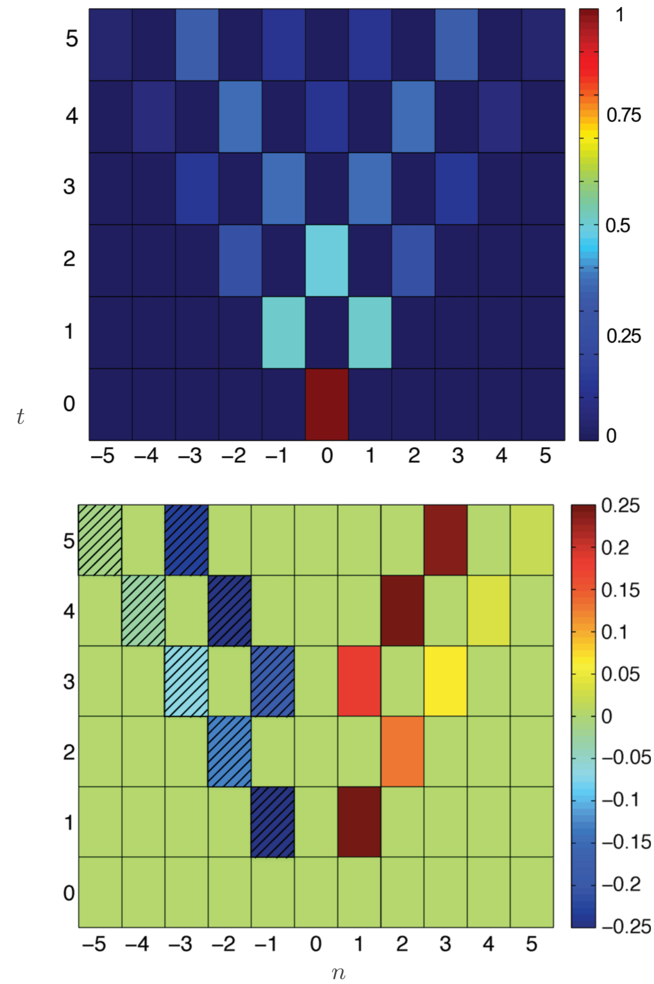

FIG. 5. (Color online) For a QW which is initialized in an identical fashion to Fig. 2, we plot (top) the probability density $\rho(n,t)$ and (bottom) the symmetric probability current density $j^C(n,t)$. We see that now the current density is symmetric in space.

VI. DO WE EXPECT THE TOTAL PROBABILITY CURRENT TO HAVE A STEADY STATE?

We have seen that for the coined QW the total current attains a steady state. We now ask the question whether this is typical or not? In classical mechanics, we associate a steady-state current or momentum with terminal velocity, i.e., acceleration in a dissipative medium. Our biased-coined QW is completely unitary and thus one may pose the question: in a purely Hamiltonian system (classical or quantum), what type of dynamics will result in steady-state momenta or probability current? We show that such conservative dynamics are possible and may yield a continuous space-time analog of our biased or directed quantum walk.

Looking now within classical mechanics, a steady-state current typically corresponds to a terminal velocity or momentum. Considering a massive particle moving in one dimension, and assuming that its momentum attains a terminal value of p_f as $t \to \infty$, and that we have no interest in the detailed dynamics before steady state, we can phenomenologically model the long-time dynamics as

$$\dot p = \lambda(p_f - p), \quad (26)$$

where p_f is the terminal momentum; i.e., at $t = \infty$, $p_\infty = p_f$ and $\lambda > 0$. Now taking the classical Hamiltonian to be

$$H = \frac{p^2}{2m} + \lambda W(x,p), \quad (27)$$

where $W(x,p)$ depends on *both* x and p, using Hamilton's equations of motion, $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$, we can construct a sample H as

$$H = \frac{p^2}{2m} - \lambda x(p_f - p), \tag{28}$$

from which the equations of motion are $\dot{x} = \frac{p}{m} + \lambda x$, $\dot{p} = \lambda(p_f - p)$. The solution to this dynamics is

$$p(t) = p_f(1 - e^{-\lambda t}), \quad x(t) = \frac{p_f e^{\lambda t}}{2m\lambda}(1 - e^{-\lambda t})^2, \tag{29}$$

with the initial conditions $p(0) = 0$, $x(0) = 0$. With this sample Hamiltonian the particle reaches a steady-state momentum p_f as it travels to $x \to \mathrm{sgn}(p_f) \times \infty$.

The quantization of this sample classical Hamiltonian turns out to be somewhat ambiguous. If we shift p by p_f, i.e., $p \to p_f + p$, the classical Hamiltonian becomes

$$H = \frac{(p + p_f)^2}{2m} + \lambda x p. \tag{30}$$

To quantize this equation, we can let $p \to \hat{p}$, and $\hat{x}\hat{p} \to \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ to have the Hamiltonian Hermitian. However, there are many ways of promoting the classical quantity xp to a Hermitian operator, and many issues relating to these difficulties have been addressed before in the literature under the so-called $H = xp$ model [27,28].

VII. CONCLUSION

Both the theory pertaining to, and the experimental implementation of, quantum walks continues to attract widespread interest. Despite this, little research has been done to understand their transport properties. In this work we derived expressions for the probability current density based on a discretization of the probability continuity equation. For the specific case of an initial localized state we were able to derive an analytical expression for the total spatial current and showed that it reached a steady state. Curiously this steady-state total current satisfied a type of sinusoidal current-phase relationship akin to the current behavior in Josephson junctions. With some effort we were able to derive an expression for the current density when we used central difference approximations and found that this symmetric probability current density behaved more intuitively. Finally we asked the question whether one can find conservative classical or quantum continuous systems whose dynamics leads to steady-state currents or momenta and found a wide class of such Hamiltonians.

APPENDIX

We show here the derivation of the global interference term Q_0 which we defined as $Q_0 = \lim_{t\to\infty} \sum_{n=-\infty}^{\infty} \alpha_{n,t}^* \beta_{n,t}$ in real space. We derive the expression for Q_0 directly in Fourier space as it is more convenient and has been reportedly successful in analyzing the properties of quantum walks.

Let the amplitudes $\alpha_{n,t}$ and $\beta_{n,t}$ be grouped together as $\psi_n(t)$ written as a column vector,

$$\psi_n(t) = \begin{pmatrix} \alpha_{n,t} \\ \beta_{n,t} \end{pmatrix} = \alpha_{n,t}\,|0\rangle + \beta_{n,t}\,|1\rangle\,, \tag{A1}$$

where vectors $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$ are introduced for notational convenience in the derivation of Q_0. Note that the vectors $|0\rangle$ and $|1\rangle$ are not necessarily related to the coin states $\{|\uparrow\rangle, |\downarrow\rangle\}$.

Let the Fourier transform of $\psi_n(t)$ be $\tilde{\psi}_k(t)$ given as

$$\tilde{\psi}_k(t) = \sum_{n=-\infty}^{\infty} \psi_n(t) e^{ikn} \tag{A2}$$

with $k \in [-\pi,\pi]$. The recurrence relation Eq. (6) in Fourier space becomes

$$\begin{pmatrix} \tilde{\alpha}_k(t) \\ \tilde{\beta}_k(t) \end{pmatrix} = \begin{pmatrix} e^{ik}\cos\theta & e^{i(k+\delta)}\sin\theta \\ e^{-i(k+\delta)}\sin\theta & -e^{-ik}\cos\theta \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_k(t-1) \\ \tilde{\beta}_k(t-1) \end{pmatrix}, \tag{A3}$$

i.e.,

$$\tilde{\psi}_k(t) = \hat{U}_k \tilde{\psi}_k(t-1), \tag{A4}$$

and iterating the "Markovian" equation recursively gives

$$\tilde{\psi}_k(t) = \hat{U}_k^t \tilde{\psi}_k(t=0), \tag{A5}$$

where $\tilde{\psi}_k(0)$ is the Fourier transform of the localized initial state $\psi_n(0)$. We assume here that our initial state $\psi_n(0)$ is localized at the origin of our lattice and is given as $\psi_n(0) = \binom{\cos\frac{\phi}{2}}{e^{i\gamma}\sin\frac{\phi}{2}}\delta_{n,0}$ as in Eq. (18), situated on a Bloch sphere, with $\phi \in [0,\pi]$ and $\gamma \in [0,2\pi]$. Its Fourier transform is $\tilde{\psi}_k(0) = \binom{\cos\frac{\phi}{2}}{e^{i\gamma}\sin\frac{\phi}{2}}$.

The Fourier transform of Q_0 becomes

$$Q_0 = \lim_{t\to\infty} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \tilde{\alpha}_k^*(t)\tilde{\beta}_k(t). \tag{A6}$$

As U is unitary, it can be diagonalized with complex eigenvalues and eigenvectors and written in *spectral representation* as

$$U_k = \left(\lambda_k^1\right) |\phi_1(k)\rangle\,\langle\phi_1(k)| + \left(\lambda_k^2\right) |\phi_2(k)\rangle\,\langle\phi_2(k)|, \tag{A7}$$

from which the expression for U_k^t follows as

$$U_k^t = \left(\lambda_k^1\right)^t |\phi_1(k)\rangle\,\langle\phi_1(k)| + \left(\lambda_k^2\right)^t |\phi_2(k)\rangle\,\langle\phi_2(k)|, \tag{A8}$$

where $\lambda_k^1 = e^{i\omega_k}$ and $\lambda_k^2 = -e^{-i\omega_k}$ are the eigenvalues with their respective eigenvectors $|\phi_1(k)\rangle$ and $|\phi_2(k)\rangle$ as

$$|\phi_1(k)\rangle = N(k) \begin{pmatrix} e^{i(k+\delta)}\sin\theta \\ e^{i\omega_k} - e^{-ik}\cos\theta \end{pmatrix},$$

$$|\phi_2(k)\rangle = N(\pi - k) \begin{pmatrix} e^{i(k+\delta)}\sin\theta \\ -e^{-i\omega_k} - e^{ik}\cos\theta \end{pmatrix}, \tag{A9}$$

where ω_k is determined from $\sin\omega_k = \cos\theta\sin k$. $N(k)$ given as

$$N(k) = \frac{1}{\sqrt{2 - 2\cos\theta\cos(\omega_k - k)}} \tag{A10}$$

is a normalization factor that ensures the orthonormality of the eigenvectors $|\phi_1(k)\rangle$ and $|\phi_2(k)\rangle$.

From Eq. (A5) we have

$$|\tilde{\psi}_k(t)\rangle = \tilde{\alpha}_k(t)\,|0\rangle + \tilde{\beta}_k(t)\,|1\rangle$$
$$= \cos\frac{\phi}{2}\hat{U}_k^t\,|0\rangle + e^{i\gamma}\sin\frac{\phi}{2}\hat{U}_k^t\,|1\rangle, \tag{A11}$$

from which we could easily identify the amplitudes $\tilde{\alpha}_k(t)$ and $\tilde{\beta}_k(t)$ to be given as

$$\tilde{\alpha}_k(t) = \cos\frac{\phi}{2}\langle 0|\, U_k^t\, |0\rangle + e^{i\gamma}\sin\frac{\phi}{2}\langle 0|\, \hat{U}_k^t\, |1\rangle,$$
$$\tilde{\beta}_k(t) = \cos\frac{\phi}{2}\langle 1|\, U_k^t\, |0\rangle + e^{i\gamma}\sin\frac{\phi}{2}\langle 1|\, \hat{U}_k^t\, |1\rangle, \quad \text{(A12)}$$

where $\tilde{\alpha}_k(t) = \langle 0\,|\tilde{\psi}_k(t)\rangle$ and $\tilde{\beta}_k(t) = \langle 1\,|\tilde{\psi}_k(t)\rangle$.

With the expression we have for U_k^t, we could compute the expressions for $\tilde{\alpha}_k(t)$ and $\tilde{\beta}_k(t)$ as

$$\tilde{\alpha}_k(t) = \cos\frac{\phi}{2}a_k(t) + e^{i\gamma}\sin\frac{\phi}{2}b_k(t),$$
$$\tilde{\beta}_k(t) = \cos\frac{\phi}{2}c_k(t) + e^{i\gamma}\sin\frac{\phi}{2}d_k(t), \quad \text{(A13)}$$

where $a_k(t)$, $b_k(t)$, $c_k(t)$, and $d_k(t)$ are certain oscillatory functions which we list below:

$$a_k(t) = e^{i\omega_k t}\sin^2\theta N^2(k) + (-1)^t e^{-i\omega_k t}\sin^2\theta N^2(\pi - k),$$
$$b_k(t) = e^{i\omega_k t}e^{i(k+d)}(e^{-i\omega_k} - e^{-ik}\cos\theta)\sin\theta N^2(k)$$
$$-(-1)^t e^{-i\omega_k t}e^{i(k+d)}(e^{i\omega_k}$$
$$+ e^{-ik}\cos\theta)\sin\theta N^2(\pi - k),$$
$$c_k(t) = e^{i\omega_k t}e^{-i(k+d)}(e^{i\omega_k} - e^{ik}\cos\theta)\sin\theta N^2(k)$$
$$-(-1)^t e^{-i\omega_k t}e^{-i(k+d)}(e^{-i\omega_k}$$
$$+ e^{ik}\cos\theta)\sin\theta N^2(\pi - k),$$
$$d_k(t) = e^{i\omega_k t}(1 - 2\cos\theta\cos(\omega_k - k) + \cos^2\theta)N^2(k)$$
$$+ (-1)^t e^{-i\omega_k t}(1 + 2\cos\theta\cos(\omega_k + k)$$
$$+ \cos^2\theta)N^2(\pi - k).$$

With all the above equations, we can evaluate Q_0 from Eq. (A6) as

$$Q_0 = \cos^2\left(\frac{\gamma}{2}\right)E_0 + \sin^2\left(\frac{\gamma}{2}\right)F_0 + \frac{e^{i\varphi}\sin\gamma}{2}G_0$$
$$+ \frac{e^{-i\varphi}\sin\gamma}{2}H_0, \quad \text{(A14)}$$

where

$$E_0 = \lim_{t\to\infty}\int_{-\pi}^{\pi}\frac{dk}{2\pi}a_k^*(t)c_k(t),$$
$$F_0 = \lim_{t\to\infty}\int_{-\pi}^{\pi}\frac{dk}{2\pi}b_k^*(t)d_k(t),$$
$$G_0 = \lim_{t\to\infty}\int_{-\pi}^{\pi}\frac{dk}{2\pi}a_k^*(t)d_k(t),$$
$$H_0 = \lim\int_{-\pi}^{\pi}\frac{dk}{2\pi}b_k^*(t)c_k(t).$$

The asymptotics of the integrals above are easy to calculate but lengthy. Using the stationary phase approximation (SPA), the integral of the form $\int f(k)e^{i\varphi(k)t}dk$ vanishes as $t^{-\frac{1}{2}}$ for a nonvanishing $\varphi''(k)$ as shown in [1] and hence every time-dependent part of the integrals E_0, F_0, G_0, and H_0 gives a negligible contribution and we can drop them in the the evaluation of the integrals to obtain

$$E_0 = \frac{e^{-i\delta}\tan\theta}{2}(1 - \sin\theta), \quad \text{(A15)}$$

$$F_0 = -\frac{e^{-i\delta}\tan\theta}{2}(1 - \sin\theta), \quad \text{(A16)}$$

$$G_0 = \frac{\sin\theta}{2}, \quad \text{(A17)}$$

$$H_0 = e^{-2i\delta}\left((1 - \sin\theta)\tan^2\theta - \frac{\sin\theta}{2}\right). \quad \text{(A18)}$$

Hence, Q_0 is given as

$$Q_0 = \frac{(1 - \sin\theta)e^{-i\delta}\tan\theta}{2}\left[\cos\phi + \sin\phi\left(e^{-i(\delta+\gamma)}\tan\theta\right.\right.$$
$$\left.\left. + i\sin(\delta + \gamma)\frac{\cos\theta}{1 - \sin\theta}\right)\right]. \quad \text{(A19)}$$

This result can also be checked if we compare it with numerical results of $Q(t)$ in the long-time limit as shown in Fig. 3.

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