# Wigner time delay for tunneling ionization via the electron propagator

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Recent attoclock experiments using the attosecond angular streaking technique enabled the measurement of the tunneling time delay during laser-induced strong-field ionization. One of the theoretical models for the tunneling time delay is the Wigner time delay, which is the asymptotic time difference between the quasiclassical and the Wigner trajectories. The latter is derived from the derivative of the phase of the electron steady-state wave function with respect to energy. Here, we present an alternative method for the calculation of the Wigner trajectory by using the fixed-energy propagator. The developed formalism is applied to the nonrelativistic regime as well as to the relativistic regime of the tunnel-ionization process from a zero-range potential. Finally, it is shown that the Wigner time delay is measurable in the near-threshold-tunneling regime within the current state of the momentum spectroscopy via detecting the induced electron momentum shift in a mixture of two gas species.

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## I. INTRODUCTION

The tunneling of an electron through a potential barrier is a typical quantum-mechanical phenomenon. It has been the focus of both theoretical and experimental attention since the formulation of quantum mechanics [1]. In particular the issue of whether the motion of a particle under a barrier is instantaneous or not is a long-standing and controversial problem in physics since MacColl's first attempt to consider it in 1932 [2] (for a review, see [3–6]). Recent interest in this problem has been renewed due to a unique opportunity offered by attosecond angular streaking techniques for measuring the tunneling time delay during laser-induced tunnel ionization [7–11]. This novel experimental technique offers the measurement of tunneling time with unprecedented resolution of tens of attoseconds [12].

For the generic problem of the tunneling time delay, due to the lack of a well-defined time operator in quantum mechanics, different definitions have been proposed, and the discussion of their relevance still continues [13–29]. These definitions can be grouped in two main categories: the operational approach, where various time operators are introduced and/or used [15,18,19,26,27,29], and the functional approach, where the tunneling time delay can be inferred from the information carried by the wave function [13,14,16,20–23,28]. In the former approach, generally, a quantum clock is attached to the particle, and the delay is inferred with some suitable measurement operators [17–19]. For instance, the precessing spin in a magnetic field can serve as a clock when a magnetic field is applied within the barrier region [17]. A well-known example for the latter approach is the Wigner time delay concept, where the phase shift of the wave function due to tunneling is linked to the tunneling time delay [13,14,16]. This concept was first introduced in the context of quantum scattering using the group velocity of the electron wave packet or, in physical terms, following the peak of the wave-packet motion. The close relationship of the Wigner time delay to the concept of the dwell time of a particle in a given spatial region has been proven [16]; that is, the Wigner time delay is the difference between the interacting and the free dwell time [23]. Generally, the results obtained for the tunneling time delay depend significantly on the definition used and reflect different physical aspects of the process, which could lead to controversies in physical interpretations [21,22,28] and even to questioning causality [30].

Rather than discussing the mentioned controversies, in this paper we propose a theoretical method for the calculation of the Wigner time delay and discuss its correspondence to the measurement of the attoclock experiment [10]. In the attoclock experiment the photoelectron momentum distribution is registered in the laser field of elliptical polarization (close to circular). Taking into account that there is a direct mapping of the time moment of the electron appearance in the continuum to the photoelectron emission angle at the detector, from the attoclock measurement one can deduce the time delay between the peak of the laser field (when the formation of the tunneling wave packet should be maximal) and the peak of the electron wave packet appearance in the continuum [31]. As the attoclock measurement follows the peak of the photoelectron distribution, it is quite natural to describe the mentioned time delay by the concept of the Wigner time delay, which is based on the time difference between the quasiclassical and the Wigner trajectories at a remote distance. A mathematical definition of the Wigner trajectory can be given via the derivative of the phase of the steady-state wave function of the tunneling particle with respect to energy.

Recently, an intuitive picture of the relativistic regime of the laser-induced tunnel ionization was developed in Refs. [32,33]. It was shown that the tunneling picture applies also in the relativistic regime with a modification that the energy levels become position dependent. This modification accounts for the electron kinetic-energy change during the tunneling connected with the electron motion along the laser field due to the effect of the magnetic-field-induced Lorentz force. Furthermore, as a relativistic feature of tunnel ionization, it was shown that the spin asymmetry in the tunnel ionization from the ground state is negligibly small [34].

The Wigner time delay for the tunnel-ionization process was investigated in Ref. [33], where the Wigner trajectory is calculated by explicitly extracting the phase of the continuum

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wave function, which is a rather cumbersome procedure, especially in the relativistic regime. In this paper, we present an alternative method for the calculation of the Wigner trajectory in terms of the phase of the fixed energy propagator. This method provides an easier way to calculate the Wigner time delay and the Wigner trajectory in particular cases. We apply the developed formalism in the nonrelativistic regime of tunnel ionization from a zero-range atomic potential under the effect of a constant and uniform electric field and in the relativistic regime under the effect of a constant and uniform crossed field. The latter field configuration corresponds to the relativistic quasistatic regime of strong-field ionization when the Keldysh parameter $\gamma = \omega\sqrt{2I_p}/E_0$ is small ($\gamma \ll 1$) [35], where $I_p$ is the atomic ionization potential and $\omega$ and $E_0$ are the laser frequency and the field amplitude, respectively.

In this paper we also discuss the observability issue of the Wigner delay time. A method for the detection of the Wigner delay time is proposed by employing the electron momentum shift induced by the time delay in a mixture of two gas species.

Atomic units (a.u.) and the metric convention $g = (+,-,-,-)$ are used throughout the paper.

## II. THE WIGNER TIME DELAY FOR TUNNEL IONIZATION

For the generic problem of tunneling time delay for tunnel ionization, we will apply the Wigner time delay [13,14,16], which is defined as follows:

$$\tau^w = \lim_{x\to\infty}[t^w(x) - t^c(x)], \tag{1}$$

where $t^w(x)$ and $t^c(x)$ stand for the Wigner trajectory and the quasiclassical trajectory, respectively. The quasiclassical trajectory $t^c(x)$ is first instantaneous under the barrier, and emerges with a zero initial momentum, and then obeys the Newton's law outside the tunneling barrier. The Wigner trajectory defined in Refs. [13,14,16] can be generalized to a tunnel-ionization problem as

$$t^w(x) = \left.\frac{\partial\Phi(x,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0}, \tag{2}$$

where $\Phi(x,\varepsilon)$ is the phase of the steady-state wave function of the tunneled electron $\psi(x)$, with $\varepsilon_0 = -I_p$ for the nonrelativistic regime of tunnel ionization, whereas $\varepsilon_0 = c^2 - I_p$ for the relativistic regime, with the ionization energy of the ground state of the hydrogenlike ion $I_p = c^2 - \sqrt{c^4 - c^2\kappa^2}$, where $\kappa = Z$ is the atomic momentum with the charge $Z$ of a hydrogenlike ion and $c$ is the speed of light.

Since the typical tunnel-ionization time is much shorter than the laser period, which is implied by the Keldysh condition, $\gamma \ll 1$, the quasistatic approximation can be applied by approximating the full space-time-dependent laser as a constant and uniform crossed field during the ionization process. Therefore, the ionization problem is reduced to a time-independent one, and the wave function is given by the solution of the time-independent Schrödinger equation,

$$H\psi(x) = \varepsilon\psi(x), \tag{3}$$

with $H$ being the Hamiltonian and $\varepsilon$ being the energy eigenvalue.

The tunnel-ionization process has a distinguished feature that may lead to an important simplification in the original definition of the Wigner trajectory equation (2). Since some portion of the bound state is already under the barrier even before the tunneling starts, the quasiclassical trajectory and the Wigner trajectory have to coincide initially at the matching point of the bound state and the continuum state $x_m$, which can be identified with the following initial condition:

$$t^w(x_m) = \left.\frac{\partial\Phi(x_m,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0} = 0. \tag{4}$$

Furthermore, the exact wave function of the tunnel-ionization process can be written as

$$\psi(x \geqslant x_m) = A(x)\exp[i\Phi(x)] = T\psi_c(x), \tag{5}$$

where $A(x)$ is the amplitude of the exact wave function, the transmission coefficient $T = |T|\exp[i\Phi_T]$, and the continuum wave function $\psi_c(x) = A_c(x)\exp[i\Phi_c(x)]$. The continuity of the wave function implies that

$$\Phi(x_m) = \Phi_c(x_m) + \Phi_T. \tag{6}$$

As a consequence, the original definition of Eq. (2) can be written in terms of the phase of the continuum wave function as

$$t^w(x) = \left.\frac{\partial\Phi_c(x,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0} - \left.\frac{\partial\Phi_c(x_m,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0}, \tag{7}$$

where we have used the initial condition of Eq. (4). Since the expression of Eq. (7) is written solely in terms of the phase of the continuum wave function, the exact Wigner trajectory for tunnel ionization can be calculated using the continuum wave function, rather than the exact wave function.

The investigation of the Wigner time delay in Ref. [33] is based on Eq. (7). Although expression (7) provides a great simplification in comparison to the original definition (2), it is still a rather cumbersome procedure to extract the phase of the continuum wave function, especially in the relativistic regime. Therefore, we will present an alternative method for the calculation of the Wigner trajectory in terms of the phase of the fixed-energy propagator, as we discuss in the next section.

## III. THE PHASE OF THE FIXED-ENERGY PROPAGATOR

Our approach for the Wigner trajectory follows from the relation between the space-time propagator

$$K(x,x';t.t') = \langle x|U(t,t')|x'\rangle \tag{8}$$

and the fixed-energy propagator (the retarded Green's function of the time-independent Schrödinger equation)

$$G(x,x';\varepsilon) = \langle x|\frac{1}{\varepsilon - H + i\epsilon_F}|x'\rangle, \tag{9}$$

with the time evolution operator $U(t,t')$ and the standard Feynman $i\epsilon$ prescription, $\epsilon_F > 0$ [36]. First, the space-time propagator can be written as

$$K(x,x';t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\varepsilon\, \exp(-i\varepsilon t)G(x,x';\varepsilon), \tag{10}$$

where we set the initial time to zero, $t' = 0$. Then, the fixed-energy propagator is split into its phase $\phi$ and its amplitude $a$

such that the space-time propagator reads

$$K(x,x';t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\varepsilon\, a(x,x',\varepsilon)\exp[-i\varepsilon t + i\phi(x,x',\varepsilon)]\,. \tag{11}$$

The propagator (11) connects the space points $x$ and $x'$ in a time interval $t$, and its value will be maximal when the phase of the fixed-energy propagator $\phi(x,x',\varepsilon)$ fulfills the stationary phase condition

$$t - \frac{\partial\phi(x,x',\varepsilon)}{\partial\varepsilon} = 0\,. \tag{12}$$

Then a trajectory which fulfills condition (12) with a certain energy of the incoming wave $\varepsilon_0$ maximizes the propagator. In other words, the trajectory given by the relation

$$t^w(x,x') = \left.\frac{\partial\phi(x,x',\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0} \tag{13}$$

traces the maximum of the propagator. In fact, the trajectory (13) is nothing but the definition of the so-called Wigner trajectory previously given in Eq. (2). In contrast to the original definition (2), definition (13) has an additional argument $x'$, which indicates the initial propagation point. Therefore, since the trajectory starts at the matching point $x_m$, the Wigner trajectory for a tunnel-ionization process may be written as

$$t^w(x) = \left.\frac{\partial\phi(x,x_m,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0}, \tag{14}$$

with the initial condition $t^w(x_m) = 0$. Here, it should be underlined that the phase for the tunnel-ionization process is the phase of the exact propagator for the binding potential combined with the electromagnetic field, which is, in general, difficult to derive. However, since we are interested in the propagation from the matching point $x_m$ to infinity, we need solely the propagator for the electromagnetic field. Then, with the same analogy of the derivation of Eq. (7), the Wigner trajectory in terms of the phase of the continuum propagator $\phi_c(x,x',\varepsilon)$ can be written as

$$t^w(x) = \left.\frac{\partial\phi_c(x,x_m,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0} - \left.\frac{\partial\phi_c(x_m,x_m,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0}, \tag{15}$$

so that the Wigner trajectory can be calculated by employing the propagator in the electromagnetic field only.

Before going further, we want to comment on the existence of the fixed-energy propagator for tunnel ionization. In the quasistatic limit $\gamma \ll 1$, there exists a gauge-invariant energy operator for time-independent fields, which allows us to identify the tunneling barrier and hence the classically forbidden and allowed regions without any ambiguity [33]. Furthermore there exists a certain gauge where the corresponding Hamiltonian coincides with the energy operator. This gauge is the length gauge (Göppert-Mayer gauge) for the nonrelativistic (relativistic) regime of tunnel ionization [33]. As a consequence, it is always possible to define a fixed-energy propagator for a tunnel-ionization process, and hence, we will calculate the fixed-energy propagator within the aforementioned gauge.

The phase of the fixed-energy propagator can be inferred by the inverse Fourier transform of the space-time propagator as

$$G(x,x';\varepsilon) = -i\int_0^{\infty} dt\, \exp(i\varepsilon t - \epsilon_F t)K(x,x';t)\,. \tag{16}$$

Thus, the alternative method for the calculation of the Wigner trajectory via Eq. (14) or Eq. (15) employs the phase of the fixed-energy propagator, rather than the phase of the wave function as in the original Wigner method [see Eq. (2) or Eq. (7)]. The advantage of our method is that the calculation and investigation of the phase of the fixed-energy propagator is relatively easier in certain cases. Moreover, if the fixed-energy propagator (16) is calculated by the saddle-point approximation, then the Wigner trajectory (15) coincides with the quasiclassical trajectory $t^c(x)$, which indicates that the Wigner trajectory, well defined in terms of the phase of the fixed-energy propagator, goes beyond the quasiclassical description. This further allows us to calculate an analytical formula for the Wigner time delay, which is done in Sec. IV.

In the following sections, we apply the developed formalism in the nonrelativistic regime as well as in the relativistic regime of tunnel ionization from a zero-range potential. Moreover, the Wigner time delay is investigated in two distinct regimes: the deep-tunneling and the near-threshold-tunneling regimes. The former regime corresponds to a tunnel-ionization process with a small tunneling probability and mathematically is identified by the condition $(E_0/E_a)^{2/3} \ll 1$ in the case of a zero-range atomic potential with the atomic field $E_a = (2I_p)^{2/3}$ (see the paragraph after Eq. (127) in Ref. [33]). In the opposite case, when the parameter $(E_0/E_a)^{2/3}$ is not much less than or is comparable to 1, the ionization is in the near-threshold-tunneling regime. However, to avoid over-the-barrier ionization and to secure the regime as tunneling, it is required that the energy bandwidth of the tunneling state $\Delta\varepsilon \sim 1/\tau_K$, with the Keldysh time $\tau_K = \gamma/\omega$, is less than the ionization energy $\Delta\varepsilon < I_p$. This condition imposes an additional restriction on the laser field, which for a zero-range potential reads

$$\frac{E_0}{E_a} < 1\,. \tag{17}$$

In the following, we use $E_0/E_a = 1/7$ for the deep-tunneling regime and $E_0/E_a = 1/2$ for the near-threshold-tunneling regime, which fulfills the above-mentioned conditions.

### A. Nonrelativistic case: In a constant and uniform electric field

Let us apply the developed formalism first in the nonrelativistic regime of tunnel ionization from a zero-range potential by neglecting the magnetic field of the laser and considering solely the electric-field component

$$\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}\,. \tag{18}$$

As we argued in the previous section, the Wigner trajectory can be calculated using the propagator solely for a electric field. Since in the quasistatic limit the Lagrangian is a quadratic function of the coordinate and velocity, the classical path dominates the Feynman path integral [37–39], and the exact

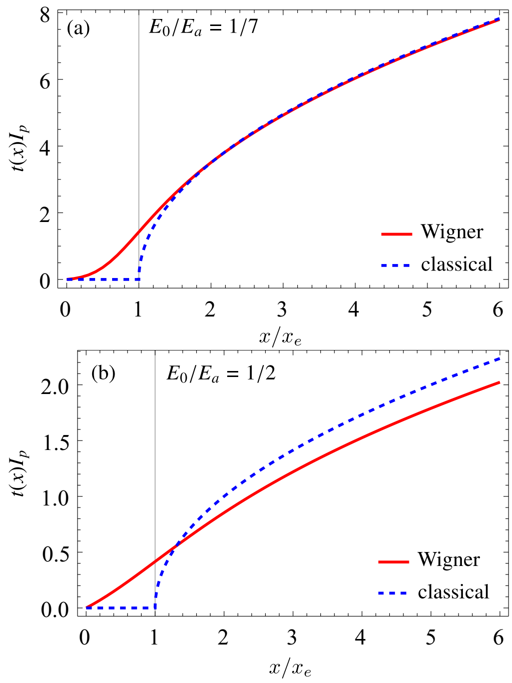


FIG. 1. (Color online) Comparison of the Wigner (red solid line) and the classical trajectories (blue dashed line) for nonrelativistic tunnel ionization from a zero-range potential under the effect of a constant and uniform electric field; (a) the deep-tunneling regime with $E_0/E_a = 1/7$ and (b) the near-threshold-tunneling regime with $E_0/E_a = 1/2$. The vertical black line indicates the exit coordinate, and the applied parameter is $\kappa = 1$.

space-time propagator can be given by

$$K(x,x';t) = \sqrt{\frac{-\partial^2 S_c}{(2\pi i)\,\partial x \partial x'}}\exp[i\,S_c(x,x',t)]\,, \tag{19}$$

with $S_c$ being the classical action, which is the action evaluated along the classical trajectory. Then, the fixed-energy propagator solely for the electric field in the length gauge $A^\mu = (-xE_0,0)$ can be written as

$$G(x,0;\varepsilon) = \frac{-i}{\sqrt{2\pi i}}\int_0^\infty dt\,\frac{\exp\left(\frac{ix^2}{2t} + \frac{iE_0tx}{2} - \frac{iE_0^2t^3}{24} + i\varepsilon t - \epsilon_F t\right)}{\sqrt{t}}\,. \tag{20}$$

Here, the matching point for the tunnel ionization from a zero-range potential can be set equal to zero, i.e., $x_m = 0$ [40]. Then using definition (15), we obtain the Wigner trajectory (see Fig. 1). The classical trajectory, on the other hand, is

$$x(t) = x_e + \tfrac{1}{2}E_0t^2\,, \tag{21}$$

where the initial velocity at the tunnel exit is zero and the tunnel exit point is given by $x_e = I_p/E_0$.

Now, we can compare the classical trajectory (21) with the Wigner trajectory (15) for nonrelativistic tunnel ionization from a zero-range potential. We did a comparison for the deep-tunneling regime and the near-threshold-tunneling regime with the applied parameter $\kappa = 1$. Figure 1 demonstrates that for the deep-tunneling regime the Wigner time delay vanishes, while for the near-threshold-tunneling regime it persists and is detectable at remote distances. The results are consistent with Ref. [33].

### B. Relativistic case: In a constant and uniform crossed field

In this section, we consider the relativistic tunnel ionization from a zero-range potential. As shown in Ref. [34], the spin asymmetry is negligibly small in the tunnel-ionization case. Therefore, we neglect the spin interaction when investigating the Wigner time delay in the relativistic regime.

In the quasistatic limit of the relativistic tunnel ionization, the laser field can be approximated by a constant and uniform crossed field:

$$\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}\,,\quad \boldsymbol{B} = E_0\,\hat{\boldsymbol{y}}\,,\quad \hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}\,, \tag{22}$$

with $\hat{\boldsymbol{k}}$ being the propagation direction. The $(3+1)$-dimensional relativistic space-time propagator can be calculated within the proper time method [41–44]. As in the case of the nonrelativistic regime we calculate the corresponding space-time propagator solely for the crossed field (22). Then the propagator in the Göppert-Mayer gauge $A^\mu = -xE_0k^\mu$ can be written as follows [45]:

$$\begin{aligned} K(x^\alpha,0) = &-\frac{i}{2}\exp\left(\frac{iE_0}{2c}\epsilon_\mu x^\mu k_\nu x^\nu\right)\int_0^\infty d\tau\,\frac{1}{(2\pi\tau)^2} \\ &\times\exp\left(-i\frac{x_\mu x^\mu}{2\tau} - \frac{i\tau}{24c^2}x_\mu F^\mu_\nu F^\nu_\rho x^\rho \right. \\ &\left. -\frac{ic^2\tau}{2} - \epsilon_F\tau\right), \end{aligned} \tag{23}$$

with the wave vector $k^\mu = (1,0,0,1)$, the polarization vector $\epsilon^\mu = (0,1,0,0)$, and $F_{\mu\nu}$ being the field strength tensor, where the summation convention was used, and further, we set the initial space-time point $x'^\mu = 0$, similar to the nonrelativistic case.

The propagator with a fixed energy and a fixed transversal momentum along the laser polarization direction can be calculated via the Fourier transform

$$\begin{aligned} &G(x,0;\varepsilon,p_z,p_y) \\ &\quad = \int dt\,dz\,dy\,K(x^\alpha,0)\exp(i\varepsilon t - ip_z z - p_y y)\,, \end{aligned} \tag{24}$$

which can be given by

$$\begin{aligned} G(x;\varepsilon,p_z,p_y) = &-\frac{(-1)^{3/4}}{2c}\int_0^\infty d\tau\,\frac{1}{\sqrt{2\pi\,\tau}}\exp\left\{\frac{ix^2}{2\tau}\right. \\ &-\frac{i\tau\left(c^2+p_y^2+p_z^2\right)}{2} \\ &+\frac{i\tau[cE_0p_zx + \varepsilon(\varepsilon - E_0x)]}{2c^2} \\ &\left.-\frac{i\tau^3E_0^2(\varepsilon - cp_z)^2}{24c^4} - \epsilon_F\tau\right\}. \end{aligned} \tag{25}$$

In order to be able to plot the Wigner trajectory, we have to define $p_y$ as well as $p_z$ in Eq. (25). First, one can set $p_y = 0$ without loss of generality for the maximal tunneling

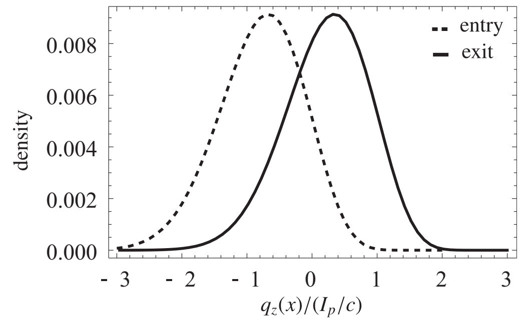


FIG. 2. Tunneling probability vs the kinetic momentum along the propagation direction of the crossed field at the tunnel entry (dashed line) and at the tunnel exit (solid line). In the case of tunnel ionization from a zero-range potential, the values are independent from the barrier suppression parameter $E_0/E_a$, and the transversal momentum transfer is $I_p/c$. The applied parameters are $E_0/E_a = 1/7$ and $\kappa = 90$.

probability. However, in contrast to the nonrelativistic tunnel-ionization process, the existence of a magnetic field leads to a nonzero value for $p_z$ for the maximal tunneling probability, which is a consequence of the fact that there is a momentum transfer along the propagation direction of the laser in the relativistic regime [32,33]. To determine the value of $p_z$ for the maximal tunneling probability, we calculate the tunneling probability for a given energy $\varepsilon_0 = c^2 - I_p$ as

$$|T|^2 = \frac{|G(x_e;\varepsilon_0,p_z)|^2}{|G(0;\varepsilon_0,p_z)|^2}, \tag{26}$$

where the quasiclassical tunnel exit point $x_e$ can be calculated via the condition

$$(c^2 - I_p - x_e E_0)^2 = c^2\left(p_z - x_e\frac{E_0}{c}\right)^2 + c^4, \tag{27}$$

which yields

$$x_e = \frac{I_p^2 - c^2\left(2I_p + p_z^2\right)}{2E_0(c^2 - I_p - cp_z)}. \tag{28}$$

The transition probability (26) is maximal at a transversal momentum $p_z = -2I_p/(3c)$. This indicates that during the tunneling there is a momentum transfer along the propagation direction of the crossed field (see Fig. 2). The kinetic momentum $q_z(x) = p_z - xE_0/c$ with the maximal tunneling probability at the tunneling entry is $q_z(0) = p_z \sim -2I_p/(3c)$, whereas at the exit it is $q_z(x_e) \sim I_p/(3c)$. As a consequence, the momentum transfer along the laser propagation direction is $I_p/c$. Furthermore, in contrast to tunnel ionization from a Coulomb potential case, the momenta at the entry and the exit, and hence the momentum transfer, are independent from the barrier suppression parameter $E_0/E_a$ for a zero-range potential [32,33].

For the comparison of the Wigner trajectory with the classical one, we need to evaluate the classical equations of motion. Let us calculate the classical trajectory for a relativistic particle via the proper time parametrization $\tau$. The classical equations of motion are governed by

$$\ddot{x}(\tau)^\mu = -\frac{1}{c}F^\mu_\nu(x)\dot{x}(\tau)^\nu. \tag{29}$$

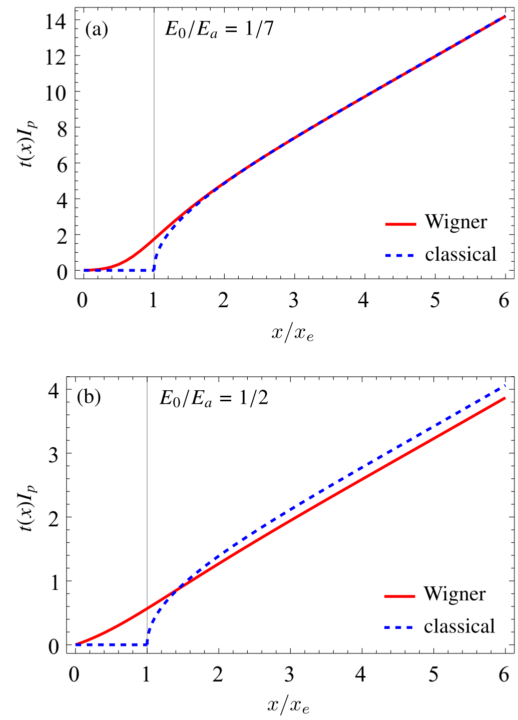


FIG. 3. (Color online) Comparison of the Wigner (red solid line) and the classical trajectories (blue dashed line) for relativistic tunnel ionization from a zero-range potential under the effect of a constant and uniform crossed field; (a) the deep-tunneling regime with $E_0/E_a = 1/7$ and (b) the near-threshold-tunneling regime with $E_0/E_a = 1/2$. The vertical black line indicates the exit coordinate, and the applied parameter is $\kappa = 90$.

For a constant and uniform crossed field (22), the solutions are given by

$$\frac{1}{c}x^0(\tau) = \frac{\tau}{6c^3}\sqrt{c^2 + v_{z0}^2}\left(6c^2 + E_0^2\tau^2\right) - \frac{E_0^2\tau^3 v_{z0}}{6c^3}, \tag{30}$$

$$x^1(\tau) = \frac{E_0\tau^2}{2c}\left(v_{z0} - \sqrt{c^2 + v_{z0}^2}\right) + x_e, \tag{31}$$

$$x^3(\tau) = \frac{E_0^2\tau^3}{6c^2}\left(\sqrt{c^2 + v_{z0}^2} - v_{z0}\right) + \tau v_{z0}, \tag{32}$$

with the initial conditions $x^\mu(0) = (0,x_e,0)$ and $\dot{x}^\mu(0) = (\sqrt{c^2 + v_{z0}^2},0,v_{z0})$, where we used the Lorentz invariant relation for the four-velocity $\dot{x}^\mu\dot{x}_\mu = c^2$. Moreover, the initial velocity along the propagation direction of the crossed field can be written as $v_{z0} = c\,q_z(x_e)/\sqrt{c^2 + q_z(x_e)^2}$, with $q_z(x_e) = p_z - x_e E_0/c$, and the tunnel exit is given by

$$x_e = -\frac{I_p}{E_0}\left(\frac{18c^2 - 5I_p}{18c^2 - 6I_p}\right), \tag{33}$$

where we have used $p_z = -2I_p/(3c)$ in Eq. (28).

Similar to the previous case, we compared the Wigner trajectory with the classical trajectory in the two distinct regimes with the applied parameter $\kappa = 90$ (see Fig. 3). While in the deep-tunneling regime the Wigner time delay vanishes, for the near-threshold-tunneling regime the delay is detectable.

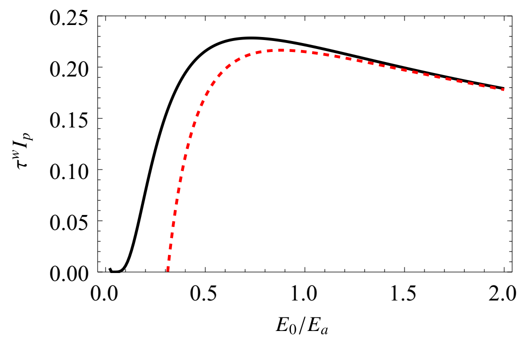


FIG. 4. (Color online) The scaled Wigner time delay $\tau^w I_p$ vs the barrier suppression parameter $E_0/E_a$ for a certain ionization energy $I_p$. The black solid curve indicates the numerical calculation for Eq. (37), whereas the red dashed curve is for the asymptotic simple expression (38).

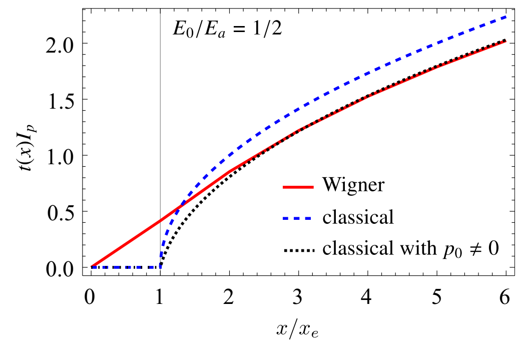


FIG. 5. (Color online) Comparison of the Wigner (red solid line), the classical (blue dashed line), and the classical with a nonzero initial momentum $p_0$ (black dotted line) trajectories for nonrelativistic tunnel ionization from a zero-range potential in the near-threshold-tunneling regime. It is shown that the Wigner trajectory coincides with the classical trajectory with a nonzero initial momentum along the tunneling direction. The vertical black line indicates the exit coordinate, and the applied parameters are $E_0/E_a = 1/2$ and $\kappa = 1$.

## IV. HOW TO MEASURE THE WIGNER TIME DELAY

In this section, we discuss a possible experimental setup for the measurement of the Wigner time delay. Figures 1(b) and 3(b) indicate that the Wigner time delay is inversely proportional to ionization energy $I_p$. Consequently, an optimal experimentally feasible delay can be obtained within the nonrelativistic configuration.

The Wigner time delay $\tau^w$ previously defined in Eq. (1) can be simplified analytically in the following way. First, it can be written in terms of the phase of the fixed-energy propagator as

$$\tau^w = \lim_{x\to\infty}\left(\frac{\partial\phi_c(x,x_m,\varepsilon_0)}{\partial\varepsilon} - \frac{\partial\phi_c(x_m,x_m,\varepsilon_0)}{\partial\varepsilon} - \frac{\partial\phi_c^{SP}(x,x_m,\varepsilon_0)}{\partial\varepsilon} + \frac{\partial\phi_c^{SP}(x_m,x_m,\varepsilon_0)}{\partial\varepsilon}\right). \quad (34)$$

Here, we have used the definition of the Wigner trajectory (15) and identify the quasiclassical trajectory as the saddle-point-approximated Wigner trajectory $t^c(x) = t^w_{SP}(x)$, where the symbol $SP$ indicates that the fixed-energy propagator (16) is calculated with the saddle-point approximation.

Then, using the expressions

$$\lim_{x\to\infty}\frac{\partial\phi_c(x,x_m,\varepsilon_0)}{\partial\varepsilon} = \lim_{x\to\infty}\frac{\partial\phi_c^{SP}(x,x_m,\varepsilon_0)}{\partial\varepsilon}, \quad (35)$$

$$\frac{\partial\phi_c^{SP}(x_m,x_m,\varepsilon_0)}{\partial\varepsilon} = 0, \quad (36)$$

the Wigner time delay reads

$$\tau^w = -\frac{\partial\phi_c(x_m,x_m,\varepsilon_0)}{\partial\varepsilon}. \quad (37)$$

Now if we plot the Wigner time delay (37) in terms of different barrier suppression parameters $E_0/E_a$ for a certain ionization energy $I_p$ for a zero-range potential, we observe the following. First, the delay is negligible for very small $E_0/E_a$ values, which corresponds to the deep-tunneling regime. Then, it increases as the $E_0/E_a$ value increases in a certain region which can be identified as the near-threshold-tunneling regime (see the black solid curve in Fig. 4). Finally, after a peak around a value $E_0/E_a < 1$, it gradually decreases, which can be interpreted as the contribution of the over-the-barrier ionization to the ionization process. This also agrees with condition (17).

Furthermore, since the Wigner time delay converges for large values of $E_0/E_a$ as shown in Fig. 4, we can do an asymptotic expansion for Eq. (37) around a large $E_0/E_a$ value, and hence, we obtain the following simple Wigner time-delay formula for a zero-range potential in the nonrelativistic regime:

$$\tau^w \approx -\frac{3^{5/6}\sqrt{\pi}}{E_0^{4/3}\Gamma[1/6]^2}\left(E_0^{2/3}\Gamma[1/6] - 2I_p 3^{1/3}\sqrt{\pi}\right), \quad (38)$$

where $\Gamma[x]$ is the gamma function. The validity of Eq. (38) in the near-threshold-tunneling regime can also be seen in the comparison in Fig. 4.

For the parameters $E_0/E_a = 1/2$ and $\kappa = 1$ the Wigner time delay amounts approximately to $\tau^w \approx 10$ as. Although it is very challenging for direct measurement, this delay is measurable in the photoelectron momentum spectrum in the following way. First, we note that, for a Wigner trajectory, there exist not only a time spent under the barrier but also a *nonzero* initial momentum at the tunnel exit along the tunneling direction. While the former retards the Wigner trajectory, the latter advances it. In the deep-tunneling regime their effects cancel each other, and we have the vanishing Wigner time delay. Nonetheless, in the near-threshold-tunneling regime the effect of the latter is bigger than the former, which leads to a negative Wigner time delay. We can further map this delay to an effective initial momentum for the quasiclassical trajectory so that it later coincides with Wigner trajectory (see the dotted black curve in Fig. 5). Moreover, the effective initial momentum can be estimated as

$$p_0 \approx -E_0\,\tau^w \approx \frac{3^{5/6}\sqrt{\pi}}{{E_0}^{1/3}\Gamma[1/6]^2}\left({E_0}^{2/3}\Gamma[1/6] - 2I_p 3^{1/3}\sqrt{\pi}\right), \quad (39)$$

which depends on $E_0$ as well as $I_p$. Then, the final momentum of the ionized electron from a zero-range potential at a detector can be given by

$$p_f(E_0,I_p) = p_c(E_0) + p_0(E_0,I_p)\,, \tag{40}$$

with $p_c(E_0)$ being the classical momentum, which the electron gains in the electric field so that it only depends on the external field $E_0$ for a zero-range potential.

In fact, inferring the time delay via the momentum spectrum of the ionized electron lies at the heart of attosecond angular streaking techniques [7–11], which map the experimentally measured momentum of the ionized electron to the phase of the electric field $\theta_i$. Then, the time delay is defined as $\theta_i/\omega$, with $\omega$ being the laser frequency. Here, in contrast to the attoclock technique, we propose to employ the $I_p$ dependency of the final momentum of the ionized electron. Namely, we consider a tunnel-ionization scenario for a gas mixture of two different species of ions with different ionization energies. Since the corresponding final momenta of the ionized electrons depend on the ionization energy $I_p$, there will be two different momentum peaks in the spectrum for the same field strength $E_0$, which is a signature of the Wigner time delay. Furthermore, the difference can be estimated as

$$\Delta p_f \approx \frac{0.73}{E_0^{1/3}}\,\Delta I_p\,. \tag{41}$$

For instance, when a mixture of helium and neon gases is used, $\Delta I_p \approx 3$ eV, and laser intensity is $10^{15}$ W/cm$^2$ ($E_0 \approx 0.2$), one has $\Delta p_f \approx 0.1$ a.u., which is in the measurable range of current momentum spectroscopy techniques [46]. This estimation is valid in the case of a zero-range potential. In the realistic Coulomb potential case a more careful investigation is inevitable because the classical momentum $p_c$ will also depend on the ionization energy due to the Coulomb focusing effect, which makes it harder to determine the momentum difference due to the time delay.

## V. CONCLUSION

In this paper, we present an alternative method for calculating the Wigner trajectory which employs the fixed-energy propagator. The developed method is applied to nonrelativistic as well as relativistic tunnel ionization from a zero-range potential. We compare the quasiclassical trajectory with the Wigner trajectory for the ionization processes in the deep-tunneling regime as well as in the near-threshold-tunneling regime and calculate the Wigner time delay. It is shown that the Wigner time delay is detectable in the latter case, and the results are in accordance with those of Ref. [33].

Furthermore, we mapped the Wigner time delay to the initial momentum along the tunneling direction. This implies that in the deep-tunneling regime the time spent under the barrier of the electron is compensated by its initial velocity at the tunnel exit, and hence we obtain a vanishing Wigner time delay. In the near-threshold-tunneling regime, however, this is not the case and the Wigner trajectory overtakes the classical one so that we observe a negative Wigner time delay.

Finally, we have proposed an experimental method for the detection of the Wigner time delay in a mixture of different atomic species. The method is based on the dependence of the Wigner time delay on the atomic ionization energy. In this scheme the difference of the Wigner time delay for the different atoms in a two-component gas mixture can be measured as the difference in momentum of two peaks which will appear in the photoelectron spectrum.

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