Geometric chained inequalities for higher-dimensional systems

Marek Żukowski and Arijit Dutta
Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland
(Received 20 May 2014; published 11 July 2014)

For systems of an arbitrary dimension, a theory of geometric chained Bell inequalities is presented. The approach is based on chained inequalities derived by Pykacz and Santos. For maximally entangled states, the inequalities lead to a complete $0 = 1$ contradiction with quantum predictions. Local realism suggests that the probability for the two observers to have identical results is 1 (that is, a perfect correlation is predicted), whereas quantum formalism gives an opposite prediction: the local results always differ. This is so for any dimension. We also show that with the inequalities, one can have a version of Bell's theorem which involves only correlations arbitrarily close to perfect ones.

DOI: 10.1103/PhysRevA.90.012106 PACS number(s): 03.65.Ud, 03.67.−a

I. INTRODUCTION

Bell theorem states that there exists *no* local hidden-variable model of quantum theory. The hidden variables are hypothetical additional parameters which are beyond quantum formalism. They might be interpreted as (local) causes of events or, more narrowly, as hidden proper states of the systems. If one additionally assumes that the probabilities of measurement results depending on such local causes have a Kolmogorovian nature (and that there exists a procedure of random choice of local measurement settings, which is *independent* of anything else in the experiment), then one can derive Bell inequalities of some form. As individual detection events have, in quantum theory, an inherently spontaneous nature, such inequalities can be violated by quantum predictions. Also, quantum states do not describe the system but rather our knowledge about their preparation (and subsequent evolution), thus they have entirely different properties than the hypothetical (local) hidden proper states. There is no reason for quantum probabilities to satisfy inequalities based on the hypothesis of existence of the latter ones. If one insists on their existence, or existence of causes, one must abandon either locality or the independence of settings assumption (often provocatively put as *free will*).

Since the pioneering work of Bell, many new derivations of Bell inequalities have appeared in the literature (see, e.g., the latest review [1]) and many experimental tests have been performed, especially in the optical domain [2]. Very early, the so-called chained inequalities were found (see Ref. [3]; however, a more detailed analysis of their statistical behavior was first introduced in Ref. [4]). A different approach to obtain a similar kind of chained inequality was shown in Ref. [5]; later, the results were used to construct logical Bell inequalities for qubits [6]. The first ones were for dichotomic local outputs only, but, later on, generalizations followed [7,8], including one in the guise of a ladder Hardy-type argument [9]. The procedure of chaining rests on a derivation of an initial inequality, and then, by upper bounding some of the terms in this inequality by an inequality of a similar kind involving different settings, one can produce a new one. This iteration can be continued arbitrarily long.

All of this resembles the geometric triangle inequality for distances, which leads to a quadrangle one and, by iteration, to a polygon inequality of as many points as one wishes. In the works of Santos [10] and Pykacz [11], one can find a derivation of chained Bell inequalities based on geometrical concepts related to Kolmogorovian probabilities. The aim of our work is to extend their results to multidimensional systems and to show the full power of the geometric approach.

The derivations shown below are for systems of arbitrary dimensions (for different inequalities of this kind, see [7]), and there seems to be no obstacle to the generalization of the results to an infinite dimension. However, such cases will be studied elsewhere. As a bright squeezed vacuum resembles, in many respects, the Einstein-Podolski-Rosen (EPR) state, such states would probably violate generalizations of the inequalities to infinitely dimensional systems. As a matter of fact, a squeezed vacuum can be shown to violate a chained inequality of a different kind [12].

Chained inequalities are most interesting if we take into account correlations close to perfect ones. In this context, one can find a specific application of chained inequalities related to the problems of interpreting Franson-type [13] two-particle interferometry as a Bell experiment; see [14]. Here, we also shall concentrate on properties of quantum predictions for our chained inequalities, for predictions which are close to perfect correlations. For a very high number of chained settings, we approach a kind of Greenberger-Horne-Zeilinger (GHZ)-type contradiction, like in the case of [5,7]. We also show that one can use such inequalities to give a rigorous formulation of the heuristic approach to Bell's theorem given in [15]. One can have a Bell theorem involving only correlations that are infinitesimally close to a single perfect one.

II. DERIVATIONS

Within Kolmogorov theory of probability, one can introduce, for a pair of probabilistic events, a notion resembling distance (which can be called probabilistic separation). Let A and B be two events. Then, their separation $S(A,B)$ is defined as [10]

$$S(A,B) = P(A) + P(B) - 2P(A,B), \tag{1}$$

where $P(A,\, B)$ is the joint probability of the occurrence of both A and B. Obviously, $S(A,B) = S(B,A)$ and $S(A,B) \geqslant 0$. Most importantly, $S(A,B)$ satisfies a triangle inequality,

$$S(A,C) \leqslant S(A,B) + S(B,C). \tag{2}$$

This can be derived by using the definition given in (1). The triangle inequality reduces to

$$P(A,B) + P(B,C) \leqslant P(B) + P(A,C). \tag{3}$$

This relation can be easily proved with Venn diagrams or other methods. Note that if we have the triangle inequality, we can build a quadrangle and higher ones. It is important to note that the inequality in (3) cannot be used in quantum mechanics if one is interested in events related to the measurement of incompatible observables. Even for two separated observers, if, for example, one assumes that A is an event associated with observable $\hat{A}$ for Alice, say, getting the eigenvalue a', and C is an event related to obtain measurement result c' of a different observable $\hat{C}$ also by Alice, while B stands for getting b' when Bob measures $\hat{B}$, then we face the problem that $P(A,C)$ is associated with two noncommensurable observables of Alice, and has no quantum mechanical value. Nevertheless, there is no problem with using the inequality in the context of (stochastic) local hidden-variable theories, as in such a case complementarity does not apply.

Nevertheless, a quadrangle inequality, which is naturally implied by the triangle one, does not face this problem. We can denote Alice's events associated with her choice of settings of the local measuring apparatus by A_i and Bob's events by B_j, where $i,j \in 0,1$. We get

$$S(A_0,B_1) \leqslant S(A_0,B_0) + S(A_1,B_0) + S(A_1,B_1). \tag{4}$$

This is just the good-old Clauser-Horne (CH) inequality [16]:

$$\begin{aligned} &P(A_0,B_0) + P(A_1,B_0) + P(A_1,B_1) \\ &\quad - P(A_1) - P(B_0) - P(A_0,B_1) \leqslant 0. \end{aligned} \tag{5}$$

As it is violated by quantum predictions, we see that the notion of Kolmogorovian probability does not apply to quantum observations (Bohr's complementarity at work).

Note that we can generalize the above separation inequality (4). Let us consider n different experiments on each side. Let us give even indices i to Alice's measurement events at specific local settings, A_i, so that $i = 2k$; while for Bob's events B_j, we shall use odd indices, $j = 2k+1$. The following implication of the triangle inequality holds:

$$\begin{aligned} S(A_0,B_{2n-1}) &\leqslant S(A_0,B_1) + S(A_2,B_1) \\ &\quad + S(A_2,B_3) + \cdots + S(A_{2n-2},B_{2n-1}) \\ &= \sum_{|i-j|=1} S(A_i,B_j). \end{aligned} \tag{6}$$

This inequality also can be easily written in terms of probabilities.

However, we would try to derive from the above inequality a distancelike inequality for probability distributions of multivalued variables (assuming that the set of eigenvalues for observables of Alice and Bob is the same one; this can always be done, as eigenvalues related to clicks at specific detectors are a question of convention). Denote by $S(A^x,B^x)$ the Kolmogorovian separation of the following events: Alice, while measuring an observable $\hat{A}$, gets an eigenvalue a_x, and Bob, measuring an observable $\hat{B}$, gets an eigenvalue b_x. To make further notation easier, we assume the following convention for our eigenvalue assignment: $a_x = b_x$ for all $x = 1,\ldots,d$. With all that, one can write

$$\begin{aligned} S(A^x,B^x) &= P(A^x) - P(A^x,B^x) + P(B^x) - P(A^x,B^x) \\ &= P(A^x,\tilde{B}^x) + P(\tilde{A}^x,B^x), \end{aligned} \tag{7}$$

where $\tilde{B}^x$ denotes the event of B^x not occurring, and similarly $\tilde{A}^x$. Now, summing this over all possible d outcomes, one gets

$$\sum_{x=1}^{d} [P(A^x,\tilde{B}^x) + P(\tilde{A}^x,B^x)] = 2P(A \neq B), \tag{8}$$

where $P(A \neq B)$ denotes the probability that if Alice measures $\hat{A}$ while Bob $\hat{B}$, they get *different* results. Obviously,

$$P(A \neq B) = \frac{1}{2}\sum_x S(A^x,B^x). \tag{9}$$

By summing up inequalities (6) for all pairs A_i^x and B_j^x, over $x = 1,\ldots,d$, we see that $P(A_i \neq B_j)$ satisfy a polygon inequality of the following form:

$$\sum_{|i-j|=1} P(A_i \neq B_j) \geqslant P(A_0 \neq B_{2n-1}), \tag{10}$$

with $i = 0,2,\ldots,2n-2$ and $j = 1,3,\ldots,2n-1$. An inequality effectively equivalent to the one above was derived using a different method by Colbeck and Renner for dichotomic variables; see, e.g., [17].

A. Violation

Let us apply this inequality to entangled qudits. We shall first show this for a pair of qutrits, and further on present a calculation for an arbitrary dimension. The fact that two maximally entangled two-qubit states violate chained inequalities, which in the $d = 2$ case are equivalent to the ones presented, is well known.

Chained inequalities work well for measurements which are close to perfect correlations. Therefore, let us first assume that the state that Alice and Bob share is maximally entangled,

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{11}$$

and the observable that Alice measures, $\hat{A}_0$, has eigenstates $|0\rangle,|1\rangle$, and $|2\rangle$. However, Bob's observable $\hat{B}_1$ is *slightly* detuned, with eigenstates $|0'\rangle,|1'\rangle$, and $|2'\rangle$. They satisfy

$$\hat{U}_B(\theta)|j\rangle = |j'\rangle, \tag{12}$$

and one has a formal equivalence,

$$\hat{B}_1 = \hat{U}_B(\theta)\hat{B}_0[\hat{U}_B(\theta)]^{-1}, \tag{13}$$

where in this formula only $\hat{B}_0$ stands for an operator for Bob's subsystem which has, in his computational basis, the same representation as $\hat{A}_0$ of Alice. Thus, in such a case,

$$P(i,j) = |{}_A\langle i|_B\langle j'|\psi\rangle_{AB}|^2, \tag{14}$$

which is equal to

$$P(i,j|A_0,B_1) = \left|{}_A\langle i|_B\langle j|\hat{U}_B^{-1}(\theta)|\psi\rangle_{AB}\right|^2. \tag{15}$$

This, in turn, because of the specific properties of the maximally entangled state (11), can be put as

$$P(i,j|A_0,B_1) = |{}_A\langle i|_B\langle j|\hat{U}_A^{*}(\theta)|\psi\rangle_{AB}|^2, \tag{16}$$

where $\hat{U}_A(\theta)$ stands for an operator for Alice's subsystem which has, in her computational basis, the same representation as $\hat{U}_B(\theta)$ of Bob. This is because

$$\sum_{j=1}^{d} |j\rangle_A \hat{U}_B^{-1}(\theta)|j\rangle_B = \sum_{i=1}^{d} \hat{U}_A^{*}(\theta)|i\rangle_A|i\rangle_B. \tag{17}$$

Of course, this a general relation holding for any unitary transformation. Note, however, that for unitary transformations which are real (orthogonal), one has

$$\sum_{j=1}^{d} |j\rangle_A \hat{U}_B^{T}(\theta)|j\rangle_B = \sum_{i=1}^{d} \hat{U}_A(\theta)|i\rangle_A|i\rangle_B. \tag{18}$$

From now on, because of this property, we shall use orthogonal $\hat{U}_B(\theta)$.

The next pair of measurements can be $\hat{A}_2$ and $\hat{B}_1$ with the eigenstates of $\hat{A}_2$ given by $|i''\rangle = \hat{U}_A(\theta)^2|i\rangle_A$. Thus,

$$P(i,j|A_2,B_1) = \left|{}_A\langle i|_B\langle j|\left[\hat{U}_A^T(\theta)\right]^2\hat{U}_B^T(\theta)|\psi\rangle_{AB}\right|^2. \tag{19}$$

One has

$$\hat{U}_A^T(\theta)^2\hat{U}_B^T(\theta)|\psi\rangle_{AB} = \hat{U}_A^T(\theta)|\psi\rangle_{AB}, \tag{20}$$

and

$$\begin{aligned} P(i,j|A_2,B_1) &= \left|{}_A\langle i|_B\langle j|\hat{U}_A^T(\theta)|\psi\rangle_{AB}\right|^2 \\ &= |{}_A\langle i|_B\langle j|\hat{U}_B(\theta)|\psi\rangle_{AB}|^2. \end{aligned} \tag{21}$$

Let us now introduce specific transformation for the case of two qutrits:

$$\hat{U}(\theta) = \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}, \tag{22}$$

where $x = \frac{1}{3}(1 + 2\cos\theta)$, $y = \frac{1}{3}(1 - \cos\theta - \sqrt{3}\sin\theta)$, $z = \frac{1}{3}(1 - \cos\theta + \sqrt{3}\sin\theta)$, and we assume that $\theta = \frac{2\pi}{3(2n-1)}$.

An important property of $\hat{U}(\theta)$ is that $[\hat{U}(\theta)]^k = \hat{U}(k\theta)$. This is because $\hat{U}(\theta)$ is an orthogonal matrix, representing a rotation with respect to the axis given by vector (1,1,1) by an angle θ. Obviously, in such a case, $\hat{U}(\theta)\hat{U}(\theta') = \hat{U}(\theta + \theta')$. Therefore,

$$\hat{U}_B^{2n-1}(\theta) = \hat{U}_B\left(\frac{2\pi}{3}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{23}$$

It transforms the state from $|0\rangle$ to $|2\rangle$, $|1\rangle$ to $|0\rangle$, and $|2\rangle$ to $|1\rangle$.

Now define

$$\begin{aligned} \hat{A}_k &= \hat{U}_A(\theta)^k\hat{A}_0\hat{U}_A(\theta)^{-(k)}, \\ \hat{B}_k &= \hat{U}_B(\theta)^{2n-1}\hat{A}_0\hat{U}_B(\theta)^{-(k)}. \end{aligned} \tag{24}$$

The measurement of $\hat{A}_{2n-2} \otimes \hat{I}$ and $\hat{I} \otimes \hat{B}_{2n-3}$ results in the following probabilities:

$$P(i,j|A_{2n-2},B_{2n-3}) = \left|{}_A\langle i|_B\langle j|\hat{U}_A^T(\theta)|\psi\rangle_{AB}\right|^2, \tag{25}$$

whereas

$$P(i,j|A_{2n-2},B_{2n-1}) = \left|{}_A\langle i|_B\langle j|\hat{U}_B^T(\theta)|\psi\rangle_{AB}\right|^2. \tag{26}$$

Note that since in both cases

$$P(i \neq j) = 1 - \frac{1}{3}\sum_i |U(\theta)_{ii}|^2,$$

where $U(\theta)_{ij}$ stand for matrix elements of $\hat{U}(\theta)_{A/B}$, one has, for both formulas,

$$\begin{aligned} P(i \neq j) &= 1 - \frac{1}{9}(1 + 2\cos\theta)^2 \\ &= \frac{1}{9}\left(8\sin^2\frac{\theta}{2} + 4\sin^2\theta\right). \end{aligned} \tag{27}$$

Concerning the last pair of observables, $\hat{A}_0$ and $\hat{B}_{2n-1}$, from the above discussion, it is obvious that

$$\hat{B}_{2n-1} = [\hat{U}_B(\theta)]^{2n-1}\hat{B}_0[\hat{U}_B(\theta)]^{-(2n-1)}. \tag{28}$$

The idea is to obtain perfect correlations for the pair of observables $\hat{A}_0, \hat{B}_{2n-1}$ which are completely opposite to the ones for $\hat{A}_0$ and $\hat{B}_0$. This is the reason why the total angle of "rotation" on Bob's subsystem for the last measurement, $\hat{B}_{2n-1}$, must be $\frac{2\pi}{3}$. This leads to the optimal value of θ given by $\theta = \frac{2\pi}{3(2n-1)}$. The probabilities read

$$P(i,j|A_0,B_{2n-1}) = \left|{}_A\langle i|_B\langle j|\hat{U}_B^T\left(\frac{2\pi}{3}\right)|\psi\rangle_{AB}\right|^2. \tag{29}$$

However,

$$\hat{U}_B^T\left(\frac{2\pi}{3}\right)|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|02\rangle + |10\rangle + |21\rangle). \tag{30}$$

Therefore,

$$P(i \neq j|A_0,B_{2n-1}) = 1. \tag{31}$$

Thus, summing over all probabilities on the left-hand side of the chained inequality (10) and comparing them with the supposedly lower value of the right-hand side, which is by (31) equal to one, we get

$$\frac{N-1}{9}\left(8\sin^2\frac{\theta}{2} + 4\sin^2\theta\right) \geqslant 1, \tag{32}$$

where N is equal to $2n$. This inequality cannot hold already for $N = 2$, and for all higher values of it. Moreover, the left-hand side tends to zero when N goes to infinity. This is because the rule $\frac{\sin x}{x} \to 1$, for $x \to 0$, can be applied in both terms. With $N \to \infty$, one has $0 \geqslant 1$. In this limit, if local realism holds, the right-hand side of the inequality (10) as it approaches zero implies that for measurements of $\hat{A}_0$ and $\hat{B}_{2n-1}$, one should expect a perfect correlation, that is, $P(i = j) = 1$. However, quantum mechanics predicts a perfect correlation satisfying $P(i = j + 1) = 1$ (modulo 3). We have a kind of GHZ contradiction in the limit of infinitely many infinitely close settings.

III. ARBITRARY DIMENSIONS

Let us now extend the above results to an arbitrary dimension d. The case of $d = 2$ is well known, but it can be recovered from what we put here for $d = 4$, which we discuss first.

A. Four-dimensional systems

One can get similar results as for $d = 3$ with the use of the following simple unitary (orthogonal) matrix:

$$\hat{U}(\theta_1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \tag{33}$$

where $\theta_1 = \frac{\pi}{2(2n-1)}$. After $(2n-1)$ iterations, this gives

$$\hat{U}(\theta_1)^{2n-1} = \hat{U}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{34}$$

The operation fully permutes initial computational basis states (although this is not a cyclic permutation). We apply these unitary transformations to four-dimensional observables on Alice's and Bob's sides; the formal relations of consecutive measurements are the same as in the case of $d = 3$. However, now the probabilities entering into each term on the left-hand side of (10) are, because of the form of the unitary operation (33), equal to $\sin^2\theta_1$. Because of the permutation given by (34), the right-hand side of (10) is always 1. So, in the end, the quantum mechanical values of (10) are given by

$$(N-1)\sin^2\left[\frac{\pi}{2(N-1)}\right] \geqslant 1, \tag{35}$$

where N is equal to $2n$. Again, this is a contradiction, and with large N, it approaches a $0 \geqslant 1$ one, which can be given a GHZ interpretation.

B. Higher dimensions

Dimension $d = 5$ holds the key to all higher ones. The appropriate unitary matrix can be put as

$$\hat{U}(\theta_1,\theta_2) = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & x & y & z \\ 0 & 0 & z & x & y \\ 0 & 0 & y & z & x \end{pmatrix}, \tag{36}$$

where $a = \cos\theta_1$, $b = \sin\theta_1$, $x = \frac{1}{3}(1 + 2\cos\theta_2)$, $y = \frac{1}{3}(1 - \cos\theta_2 - \sqrt{3}\sin\theta_2)$, and $z = \frac{1}{3}(1 - \cos\theta_2 + \sqrt{3}\sin\theta_2)$. This matrix follows all of the properties mentioned previously in the case of qutrit rotation, $d = 3$, and the qubit case, $d = 2$. Namely, if we put the angles θ_1, θ_2 as $\frac{\pi}{2(2n-1)}$ and $\frac{2\pi}{3(2n-1)}$, respectively, we have

$$\hat{U}(\theta_1,\theta_2)^{2n-1} = \hat{U}\left(\frac{\pi}{2},\frac{2\pi}{3}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{37}$$

This matrix permutes initial computational basis states for $d = 5$. Each of the probabilities on the left-hand side of inequality (10) is equal to $1 - \frac{1}{5}[2\cos^2\theta_1 + \frac{3}{9}(1 + 2\cos\theta_2)^2]$.

After adding up all of the functions in (10), we get

$$(N-1) - \frac{N-1}{5}\left(2\cos^2\left[\frac{\pi}{2(N-1)}\right] + \frac{3}{9}\left\{1 + 2\cos\left[\frac{2\pi}{3(N-1)}\right]\right\}^2\right) \geqslant 1, \tag{38}$$

where again N is equal to $2n$. This again leads to a contradiction, which with $N \to \infty$ can be put as $0 \geqslant 1$.

We can generalize this approach to an arbitrary dimension d. We can always express any number d, which is greater than one, in terms of 2 and 3, i.e., one can always write

$$d = m2 + s3, \tag{39}$$

where $s = \frac{d-2m}{3}$, and m and s must be positive integers. In such a case, we can apply in a generalization of (36) a qubitlike transformation to m pairs of dimensions, and a qutritlike one to s triples of dimensions. Of course, for odd-dimensional systems, the easiest choice is to put m in such a way that $s = 1$, whereas for even dimensions, one simply has m qubitlike transformations. The basic unitary operation that we need can be constructed like (36), but now with m 2×2 blocks of qubitlike form (defined by a and b), and the last 3×3 block just as in (36). Obviously, $2n - 1$ applications of such a matrix lead to a complete permutation of basis states. Under such operations, the chained inequality (10) leads to

$$(N-1) - \frac{N-1}{d}\left(2m\cos^2\left[\frac{\pi}{2(N-1)}\right] + \frac{d-2m}{9}\left\{1 + 2\cos\left[\frac{2\pi}{3(N-1)}\right]^2\right\}\right) \geqslant 1. \tag{40}$$

In (40), as N tends to infinity, the left-hand side of the inequality goes to zero, although the right-hand side is always 1. So, distancelike inequalities for any local realistic description are violated by quantum mechanics. As before, any local realistic prediction based on the left-hand side implies a perfect correlation of a completely different kind, $P(i = j|\hat{A}_0,\hat{B}_{2n-1}) = 1$, than quantum prediction for the right-hand-side measurements—also a perfect correlation but with $P(i = j|\hat{A}_0,\hat{B}_{2n-1}) = 0$.

IV. GENERALIZATION

These results can be further amplified. We can abandon the constraint to our walk on the polygon, which is, in the case of inequality (10), from A_0 to B_1, next from B_1 to A_2, and so on until we reach the next-to-last step from A_{2n-2} to B_{2n-1} (all this is a "longer way" than directly from A_0 to B_{2n-1}). We can add one more step from B_{2n-1} to A_{2n}, and compare this with the separation of the first and the last event that is A_0 and A_{2n}. In this way, we get an inequality which holds for local hidden variables in the form of

$$\sum_{|i-j|=1}^{i=2n,j=2n-1} P(A_i \neq B_j) \geqslant P(A_0 \neq A_{2n}), \tag{41}$$

with $i = 0,2,\ldots,2n$ and $j = 1,3,\ldots,2n-1$. At first glance, this inequality seems as useless in quantum mechanics as the triangle one. However, if A_0 and A_{2n} are compatible, that is, they commute, then it can be compared with quantum predictions. The idea, therefore, is to use transformations U_1 which after $2n$ applications, that is, for U_1^{2n}, give a permutation of the original basis [this would mean for the ones introduced earlier, putting θ_1, θ_2 as $\frac{\pi}{2(2n)}$ and $\frac{2\pi}{3(2n)}$]. One can repeat all reasonings given earlier to get a $0=1$ contradiction for $P(A_0 \neq A_{2n})$. This directly implies an absolute contradiction in the local hidden-variable prediction, as for any theory one must definitely have $P(A_0 \neq A_{2n}) = 1$ since the difference between the two observables is just a permutation of the results (eigenvalues). Compare [17], where such a contradiction is explicitly shown for only $d = 2$.

However, we can start all that with an arbitrary $\hat{A}_0'$, redefine the computational basis such that it is now built out of eigenstates of $\hat{A}_0'$, and find a "conjugate" $\hat{B}_0'$, such that its eigenstates enter the Schmidt decomposition of the maximally entangled state involving eigenstates of $\hat{A}_0'$ (recall that a maximally entangled state has infinitely many equivalent Schmidt decompositions). With this, we can repeat all of the reasonings given above. This leads us to an absolute internal contradiction for a hidden-variable description of any observable.

As a matter of fact, one can derive a kind of Zeno paradox for any local hidden-variable description of observables describing a maximally entangled state. With the construction like above, even if $\hat{A}_0$ and $\hat{A}_{2n}$ are incompatible (in quantum theory), a local hidden-variable theory must give a definite prediction for $P(A_0 \neq A_{2n})$. If these are two *different* observables, one must have $P(A_0 \neq A_{2n}) > 0$ because $P(A_0 = B_{2n}) < 1$ and $P(A_{2n} = B_{2n}) = 1$. However, for a reasoning like above, in the limit of infinitesimally slow changes of the observables into the consecutive ones, in the chained inequality, the left-hand side always tends to zero, implying $P(A_0 \neq A_{2n}) = 0$. That is, up to sets of (probability) measure zero, one has identical local hidden-variable models of the two observables. We have no change if we move by infinitesimally small steps, even if they accumulate to a finite one. Thus, reasoning involving perfect correlations leads to absolutely absurd contradictions for local hidden-variable models.

V. CONTRADICTION INVOLVING NEIGHBORHOOD OF ONE PERFECT CORRELATION

Let us consider a maximally entangled state for a pair of qudits,

$$|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|kk\rangle. \tag{42}$$

The measurements that we shall consider will have two traits. First of all, they will be very close to one giving perfect correlations, and the unitary transformations leading us to other measurement settings would be constrained to just the first two basis vectors of each of the systems. Thus, in this sector, we have basically SU(2) transformations. The probabilities $P(\lambda_{A_i} \neq \lambda_{B_j})$ can, in such circumstances, be put as

$$P(\lambda_{A_i} \neq \lambda_{B_j}) = \frac{2}{d}P_q(\lambda_{A_i'} \neq \lambda_{B_j'}), \tag{43}$$

where the primed observables are effective qubit observables describing the effects of the measurements, and the probabilities P_q are the ones for a two-qubit system which effectively describes the sector within which our constrained transformations work.

In further considerations, we shall drop the subscript q and the primes. Thus, our calculations will be presented as if we were considering a two-qubit system in the ϕ^+ Bell state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{k=0}^{1}|kk\rangle. \tag{44}$$

We shall use the spin-1/2 approach to qubits, with local measurements described by the Pauli operators $\vec{a}\cdot\vec{\sigma}_1$ and $\vec{b}\cdot\vec{\sigma}_2$, where $\vec{a}$ and $\vec{b}$ are the local Bloch vectors defining the measurement direction. In such a case, the quantum predictions for the measurement results of Alice, $\lambda_A = \pm 1$, and Bob, $\lambda_B = \pm 1$, are given by

$$P(\lambda_A,\lambda_B) = \tfrac{1}{4}(1 + \lambda_A\lambda_B\vec{a}\cdot\hat{T}\vec{b}). \tag{45}$$

In the case of the ϕ^+ state, the correlation tensor on the z-x plane is written as

$$\hat{T} = \vec{z}\otimes\vec{z} + \vec{x}\otimes\vec{x} - \vec{y}\otimes\vec{y}. \tag{46}$$

If we use Bloch vectors defining the local settings, $\vec{a}$ and $\vec{b}$, which are constrained to the z-x plane, then only the first two terms matter.

A chained inequality for a pair of qubits with $2n$ settings reads

$$\sum_{|i-j|=1} P(\lambda_{A_i} \neq \lambda_{B_j}) \geqslant P\big(\lambda_{A_0} \neq \lambda_{B_{2n-1}}\big), \tag{47}$$

where $\lambda_{A_i},\lambda_{B_j}$ are the measurement outcomes on Alice's and Bob's sides, respectively, while measuring observables A_i and B_j.

For a pair of observables with dichotomic outcomes, one has

$$P\big(\lambda_{A_i} \neq \lambda_{B_j}\big) = \tfrac{1}{2}(1 - \vec{a}\cdot\hat{T}\vec{b}), \tag{48}$$

where $\vec{a},\vec{b}$ are measurement directions for Alice and Bob, respectively. Using the above, (47) can be written in the following form:

$$\begin{aligned}&\hat{T}\cdot[(\vec{a}_0+\vec{a}_2)\otimes\vec{b}_1 + (\vec{a}_4+\vec{a}_2)\\ &\quad\otimes\vec{b}_3\cdots + (\vec{a}_{2n-2}-\vec{a}_0)\otimes\vec{b}_{2n-1}] \leqslant (2n-2).\end{aligned} \tag{49}$$

Assume the following settings:

$$\begin{aligned}\vec{a}_{2i-2} &= \vec{z}\cos\left[\frac{\pi(2i-2)}{\gamma(2n)}\right] + \vec{x}\sin\left[\frac{\pi(2i-2)}{\gamma(2n)}\right],\\ \vec{b}_{2i-1} &= \vec{z}\cos\left[\frac{\pi(2i-1)}{\gamma(2n)}\right] + \vec{x}\sin\left[\frac{\pi(2i-1)}{\gamma(2n)}\right].\end{aligned} \tag{50}$$

Now, each pair of consecutive direction vectors is separated by the same angular separation, $(\frac{\pi}{\gamma}\frac{1}{2n})$. So, they follow the relation

$$\vec{a}_{2k} + \vec{a}_{2k+2} = 2\vec{b}_{2k+1}\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right), \tag{51}$$

where $k \in \{0,1,\dots,(n-2)\}$. Next we insert (51) in (47) to obtain a compact form,

$$2\hat{T}\cdot\left[\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right)\sum_{i=1}^{i=n-1}\vec{b}_{2i-1}\otimes\vec{b}_{2i-1}\right] + \hat{T}\cdot(\vec{a_{2n-2}}-\vec{a}_0)\otimes\vec{b}_{2n-1}\leqslant 2n-2. \quad (52)$$

Now, using (46), the left-hand side of (52) is reduced to

$$2\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right)\left\{\sum_{i=1}^{i=n-1}\cos^2\left[\frac{\pi(2i-1)}{\gamma(2n)}\right] + \sum_{i=1}^{i=n-1}\sin^2\left[\frac{\pi(2i-1)}{\gamma(2n)}\right]\right\} + \hat{T}\cdot(\vec{a}_{2n-2}-\vec{a}_0)\otimes\vec{b}_{2n-1} \leqslant 2n-2, \quad (53)$$

and this, of course, reduces to

$$2(n-1)\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right) + \hat{T}\cdot(\vec{a}_{2n-2}-\vec{a}_0)\otimes\vec{b}_{2n-1}\leqslant 2n-2. \quad (54)$$

Our next task is to estimate the value of $\hat{T}\cdot(\vec{a}_{2n-2}-\vec{a}_0)\otimes\vec{b}_{2n-1}$. According to (50),

$$\vec{a}_0 = \vec{z},$$
$$\vec{a}_{2n-2} = \vec{z}\cos\left[\frac{\pi(2n-2)}{\gamma(2n)}\right] + \vec{x}\sin\left[\frac{\pi(2n-2)}{\gamma(2n)}\right],$$
$$\vec{b}_{2n-1} = \vec{z}\cos\left[\frac{\pi(2n-1)}{\gamma(2n)}\right] + \vec{x}\sin\left[\frac{\pi(2n-1)}{\gamma(2n)}\right]. \quad (55)$$

Thus,

$$\hat{T}\cdot(\vec{a}_{2n-2}-\vec{a}_0)\otimes\vec{b}_{2n-1} = \left\{\cos\left[\frac{\pi(2n-2)}{\gamma(2n)}\right]-1\right\}\cos\left[\frac{\pi(2n-1)}{\gamma(2n)}\right] + \sin\left[\frac{\pi(2n-2)}{\gamma(2n)}\right]\sin\left[\frac{\pi(2n-1)}{\gamma(2n)}\right] = \cos\left[\frac{\pi}{\gamma(2n)}\right] - \cos\left[\frac{\pi(2n-1)}{\gamma(2n)}\right]. \quad (56)$$

After adding up all of the terms in (52), we get

$$2(n-1)\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right) + \cos\left[\frac{\pi}{\gamma(2n)}\right] - \cos\left[\frac{\pi(2n-1)}{\gamma(2n)}\right] \leqslant 2n-2, \quad (57)$$

or

$$2(n-1)\left[\cos\left(\frac{\pi}{\gamma}\frac{1}{2n}\right)-1\right] + \cos\left[\frac{\pi}{\gamma(2n)}\right] - \cos\left[\frac{\pi(2n-1)}{\gamma(2n)}\right]\leqslant 0. \quad (58)$$

But, when n tends to infinity for any fixed finite γ, the left-hand side of (57) is sooner or later greater than zero. Hence, the inequality in (47) is violated without going through the entire Bloch sphere. For very large γ, the derivation involves basically only perfect correlations. Thus, we have a kind of an approximate GHZ contradiction for maximally entangled two-system states—as it is based on correlations which can be infinitesimally close to a (single) perfect one. However, it is not "all or nothing." Nevertheless, the interesting aspect is that it is based on correlation in the "epsilonic" neighborhood of the one single perfect one. This might be seen as a rigorous version of the heuristic argument given by Ballentine (Ref. [15], p. 587), and, as a bonus, one working for a system of arbitrary dimensions.

VI. CONCLUSIONS

The main results of our work can be summarized as follows. The Pykacz-Santos chained inequalities can be generalized to situations in which we have entangled systems of arbitrary dimension. This, in turn, in the limit of infinitely equally spaced settings, leads to a no-go theorem for a local realistic description involving perfect correlations only. Another result is that one can also have conclusions of a similar kind involving only correlations infinitesimally close to just one in which we have a perfect correlation.

As the discussed inequalities are valid for any dimension, the results can also be applied to the case of the dimension of the systems approaching infinity. In a forthcoming work, we shall analyze the so-called bright squeezed vacuum (BSV) with the methods presented here, seeking drastic consequences for the hypothesis of local realism. Note that the BSV is a (physical) approximation of the original (unphysical) EPR state.

ACKNOWLEDGMENTS

M.Z. is supported by the BRISQ2 EU project, while A.D. is supported by an MPD Project of FNP.

[1] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[2] J.-W. Pan, Z.-B. Chen, C.-Y. Lu, H. Weinfurter, A. Zeilinger, and M. Żukowski, Rev. Mod. Phys. **84**, 777 (2012).
[3] P. A. Pearle, Phys. Rev. D **2**, 1418 (1970).
[4] S. L. Braunstein and C. M. Caves, Ann. Phys. (NY) **202**, 22 (1990).
[5] L. Hardy, Phys. Lett. A **161**, 21 (1991).
[6] S. Abramsky and L. Hardy, Phys. Rev. A **85**, 062114 (2012).
[7] J. Barrett, A. Kent, and S. Pironio, Phys. Rev. Lett. **97**, 170409 (2006).
[8] P. Kurzynski and D. Kaszlikowski, Phys. Rev. A **89**, 012103 (2014).
[9] A. Cabello, Phys. Rev. A **58**, 1687 (1998).
[10] E. Santos, Phys. Lett. A **115**, 363 (1986).
[11] J. Pykacz and E. Santos, J. Math. Phys. **32**, 1287 (1991).

[12] K. Rosolek, M. Stobinska, M. Wiesniak, and M. Zukowski (unpublished).

[13] J. D. Franson, Phys. Rev. Lett. **62**, 2205 (1989).

[14] S. Aerts, P. Kwiat, J.-A. Larsson, and M. Zukowski, Phys. Rev. Lett. **83**, 2872 (1999).

[15] L. E. Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, Singapore, 1998).

[16] J. Clauser and M. Horne, Phys. Rev. D **10**, 526 (1974).

[17] R. Colbeck and R. Renner, Nat. Commun. **2**, 411 (2011); Phys. Rev. Lett. **101**, 050403 (2008).