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Response under arbitrary groups of point transformations of multipoint-correlation functions of local fluctuating quantities

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We derive an identity which determines the infinitesimal response of the cumulant average of the product of m arbitrary local fluctuating quantities (connected m-point functions) to changes in its arguments engendered by a member of an arbitrary continuous group of point transformations. We illustrate the use of the identity in a variety of cases, and in particular we show that there is an intimate connection between the covariance of the connected m-point functions under the special conformal group and under the group of dilations. By combining our identity with the assumptions of the operator algebra, we find that at the critical point the connected density m-point functions are covariant under both groups. We also give another derivation for an expression for the exponent $x=(1/2)(5-\eta)$, which was previously found by Green and Gunton.

I. INTRODUCTION

We have derived identities which determine the infinitesimal response of the cumulant average of the product of m arbitrary local fluctuating quantities (connected m-point functions) to changes in its arguments engendered by a member of an arbitrary continuous group of point transformations. It is the purpose of this paper to derive and exhibit this identity in its general form and to make certain applications, several of which belong to the theory of critical phenomena.

Our result may be thought of as a generalization of the relations among the molecular distribution functions, which arise from the equivalence of adding particles to those already contained in a fixed volume with decreasing the volume while keeping the number of particles it contains constant, as was found by Schofield.[1] Our result may also be seen as the analog for the statistical mechanics of fluids of the Ward-Takahashi identities of quantum field theory.[2] Like these identities, the left-hand side (lhs) of ours has an operator characteristic of an infinitesimal element of the group operating on an m-point correlation function and the right-hand side (rhs) has an $(m+1)$-point correlation function in which an additional local fluctuating quantity characteristic of both the group and the Hamiltonian appears.

In cases, like that of the translation or rotation group, in which the Hamiltonian may be invariant under the operations of the group the additional fluctuating quantity is zero and the identities express the covariance of the multipoint correlation function under the operations of the group. Among the cases in which the Hamiltonian is not invariant, the group of dilations and special conformal transformations is especially interesting because in both these cases the additional fluctuating quantity has a well-defined thermodynamic significance. It is the local virial or the instantaneous local pressure.[3] This fact alone emphasized that scale and conformal covariance are closely connected. As Schofield already noted, his identities are especially useful near the critical point.[1] With the aid of our identities and the concepts of the operator algebra[4] we are able to shed light on several interesting questions about critical phenomena. We are able to show that multipoint-density-correlation functions are covariant under both scale and special conformal transformations at the critical point. We are able to understand, without reference to the Migdal-Polyakov bootstrap, why scale and conformal covariance are concomitant at the critical point.[5] The numerical parameter which appears in the equations expressing scale and conformal covariance is nothing else than $x=\frac{1}{2}(5-\eta)$ which is the scaling exponent of the chemical potential.[6] The formal expression we obtain for this exponent is identical to that obtained by Green and Gunton by a somewhat different application of Schofield's relations coupled with the operator algebra. We are also able to give a tentative answer to the question, "Why scale covariance (or conformal covariance) at the critical point?" Since length scaling can be understood to be the source of thermodynamic scaling, this is a very important question in the theory of critical phenomena.[7] The answer seems to lie in the fact mentioned above that the additional fluctuating quantity for dilations and special conformal transformations is the pressure and that the isothermal density derivatives of pressure is zero at the critical point.

We have given only a few applications of our identities in the present paper. Since these identities are satisfied by arbitrary local fluctuating quantities, we believe that we have not exhausted their interesting applications to critical phenomena. In particular, we expect them to be of use in discussions of exponents other than x and concerning fluctuations other than the isothermal density fluctuations.

The plan of the paper is as follows. In Sec. II we recall some of our motivation and establish our plan for derivation of our identity, which is explicitly carried out in Sec. III. In Sec. IV we allow the reader to gain familiarity with the identity by applying it to a variety of two-point functions. In Sec. V we use it to show the intimate relation between the effects of the group of dilations and the special conformal group on the connected m-point functions. We then combine our identity with the operator algebra in order to see how covariance under both conformal transformations and dilations comes about at the critical point and in order to give an expression for η. Finally, we summarize our results.

II. MOTIVATION

Let us recall some of our motivation. At the critical point, the connected part of the two-point order-parameter correlation function is a homogeneous function of the distance separating the points when this distance is large. The infinitesimal form of this statement is

$$(\vec{x}_1\cdot\nabla_1+\vec{x}_2\cdot\nabla_2)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c=-(1+\eta)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c \tag{1}$$

for large $|\vec{x}_1-\vec{x}_2|$. While this is true at the critical point it is false away from it. Therefore, it is appropriate to seek the general form of the rhs of Eq. (1) in order to understand its simple form at the critical point and the forms that deviations from homogeneity may take near it. Now it is generally believed that at the critical point, not only does Eq. (1) hold but that all the connected m-point functions of local densities are homogeneous functions.[8] It has also been suggested that they are conformally covariant.[9]

In order to place these assertions in their most general context we consider the result of applying the operation $\sum_{i=1}^{m}(d/d\lambda)\vec{g}(\vec{x}_i,\lambda)|_{\lambda=0}\cdot\nabla_i$ to $\langle\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\rangle_c$, where $F_a(\vec{x})$ is a local fluctuating quantity whose character will be weakly circumscribed in Sec. III. $\vec{g}(\vec{x}_i,\lambda)$ is a member of an arbitrary Lie group of point transformations which are parametrized so that

$$\vec{g}(\vec{g}(\vec{x},\lambda_1),\lambda_2)=\vec{g}(\vec{x},\lambda_1+\lambda_2)$$

and (2)

$$\vec{g}(\vec{x},\lambda=0)=\vec{x}.$$

Henceforth we will denote $(d/d\lambda)\vec{g}(\vec{x},\lambda)|_{\lambda=0}$ by $\vec{g}'(\vec{x})$ and, more generally, throughout this paper a primed quantity will stand for the derivative with respect to λ of the corresponding unprimed quantity evaluated at $\lambda=0$.

Our approach to the evaluation of

$$\sum_{i=1}^{m}\vec{g}'(\vec{x}_i)\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}F_{a_j}(\vec{x}_j)F_{a_i}(\vec{x}_i)\prod_{j=i+1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c$$

is, in essence, simple. We construct a suitably transformed ensemble in which the cumulant averages of the transformed quantities equal $\langle\prod_{j=1}^{m}F_{a_j}(\vec{g}(\vec{x}_i,\lambda))\rangle_c$. We differentiate both expressions with respect to λ and then equate the results. However, since the value of m is immaterial we will execute this program by constructing a generating functional from which our identities can be obtained by functional differentiation. It is to this task that we now turn.

III. WARD IDENTITIES

Let us consider the functional $\mathfrak{z}(E,\lambda)$ and the definitions given below:

$$\mathfrak{z}(E,\lambda)\equiv\sum_{N=0}^{\infty}\int d\Gamma(N,W(q))\exp[-\beta H^N(q,p)+\alpha N+(E,F^N\cdot g_\lambda)], \tag{3a}$$

$$F_a^N(\vec{z})\equiv\sum_{k=1}^{N}\delta^3(\vec{z}-\vec{q}_k)f_{ak}^N(q,p), \tag{3b}$$

$$(F_a^N\cdot g_\lambda)(\vec{z})\equiv\sum_{k=1}^{N}\delta^3(\vec{g}(\vec{z},\lambda)-\vec{q}_k)f_{ak}^N(q,p)\equiv\mathfrak{F}_a^N(\vec{z}), \tag{3c}$$

$$(E,\mathfrak{F}^N)\equiv\sum_{a=1}^{A}\int d^3z\,E_a(\vec{z})\,\mathfrak{F}_a^N(\vec{z}), \tag{3d}$$

$$d\Gamma(N,W(\vec{q}))\equiv(h^{3N}N!)^{-1}\prod_{n=1}^{N}d^3p_n\,d^3q_n\,\Theta(-W(\vec{q}_n)). \tag{3e}$$

$F_a(\vec{z})$ is a fluctuating quantity which we require to be local in the sense that its value depends only on those particles which are close to it. More precisely, the functions $f_{ak}^N(\vec{q}_1,\vec{p}_1,\ldots,\vec{q}_N,\vec{p}_N)$ have the following property when any particle, say particle N, is far on a microscopic scale from particle k:

$$f_{ak}^N(\vec{q}_1,\vec{p}_1,\ldots,\vec{q}_N,\vec{p}_N)=f_{ak}^{N-1}(\vec{q}_1,\vec{p}_1,\ldots,\vec{q}_{N-1},\vec{p}_{N-1}).$$

$E_a(\vec{z})$ is the external field conjugate to the fluctuating quantity $F_a(\vec{z})$ and the surface $W(\vec{q})=0$ is the wall which confines our system. Clearly when $\lambda=0$ and E vanishes everywhere, $\mathfrak{z}$ is the grand partition function Ξ. We also see that

$$\frac{d}{d\lambda}\left[\prod_{j=1}^{m}\frac{\delta}{\delta E_{a_j}(\vec{x}_j)}\ln\left(\frac{\mathfrak{z}(E,\lambda)}{\Xi}\right)\right]_{\lambda=E=0} = \sum_{i=1}^{m}\vec{g}'(\vec{x}_i)\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}F_{a_j}(\vec{x}_j)F_{a_i}(\vec{x}_i)\prod_{j=i+1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c . \qquad (4)$$

We may now re-express $\mathfrak{z}(E,\lambda)$ by using the canonical transformation[10] induced by $\vec{g}(\vec{x},\lambda)$,

$$Q_a \equiv g_a(\vec{q},-\lambda), \qquad (5a)$$

$$P_a \equiv p_b\, g_{a,b}(\vec{g}(\vec{q},\lambda),-\lambda) \equiv G_a(\vec{q},\vec{p},-\lambda). \qquad (5b)$$

After this change of variables, Eq. (3a) becomes Eq. (6),

$$\mathfrak{z}(E,\lambda)=\sum_{N=0}^{\infty}\int d\Gamma(N,W(\vec{g}(\vec{Q},\lambda)))\times\exp[-\beta H^N(Q,P,\lambda)+\alpha N+(E,J^{-1}F_\lambda^N)], \qquad (6)$$

where

$$H^N(Q,P,\lambda)\equiv H^N(g(Q,\lambda),G(Q,P,\lambda)), \qquad (7a)$$

$$F_{a\lambda}^N(\vec{z})\equiv\sum_{k=1}^{N}\delta^3(\vec{z}-\vec{Q}_k)f_{ak}^N(g(Q,\lambda),G(Q,P,\lambda)), \qquad (7b)$$

and J is the Jacobian determinant of $\vec{g}(\vec{z},\lambda)$. It follows from Eq. (4) that $\mathfrak{z}'(E)/\mathfrak{z}(E)$ is the generating functional we seek. By inspecting Eq. (6) we see that $\mathfrak{z}'(E)$ is given below.

$$\mathfrak{z}'(E)=\sum_{N=1}^{\infty}\int d\Gamma(N,W(\vec{Q}))(E,F^{N\prime}-J'F^N)\exp[-\beta H^N+\alpha N+(E,F^N)]+\sum_{N=1}^{\infty}\int d\Gamma(N,W(\vec{Q}))(-\beta H^{N\prime})\times\exp[-\beta H^N+\alpha N+(E,F^N)]-\left\{\sum_{N=1}^{\infty}\int(h^3N)^{-1}\sum_{l=1}^{N}d\Gamma(N-1,W(\vec{Q}))d^3P_l\,d^3Q_l\,\delta(W(\vec{Q}_l))\nabla W(\vec{Q}_l)\cdot\vec{g}(\vec{Q}_l)\times\exp[-\beta H^N+\alpha N+(E,F^N)]\right\}. \qquad (8)$$

The term in curly brackets on the rhs of Eq.8 is a sum of terms in each of which the lth particle is confined to the surface $W=0$. An argument which makes essential use of the local character of both the energy density and $F(\vec{z})$ shows us that in the infinite-volume limit the last term becomes a constant times $\mathfrak{z}(E)$. Another argument which appeals to the invariance of Ξ under canonical transformations allows us to identify the constant with $-\beta\langle H'\rangle$. Hence we may display $\mathfrak{z}'(E)$ as follows:

$$\mathfrak{z}'(E)=\sum_{N=0}^{\infty}\int d\Gamma(N,W(Q))\times[-\beta\Delta H^{N\prime}(Q,P)+(E,F^{N\prime}-J'F^N)]\times\exp[-\beta H^N+\alpha N+(E,F^N)], \qquad (9a)$$

$$\Delta H^{N\prime}=H^{N\prime}-\langle H'\rangle. \qquad (9b)$$

After obtaining this expression for $\mathfrak{z}'(E)$, it is a simple matter to show that Eq. (10b) follows from Eq. (10a):

$$\sum_{i=1}^{m}\vec{g}'(\vec{x}_i)\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}F_{a_j}(\vec{x}_j)F_{a_i}(\vec{x}_i)\prod_{j=i+1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c=\prod_{j=1}^{m}\frac{\delta}{\delta E_{a_j}(\vec{x}_j)}\left(\frac{\mathfrak{z}'(E)}{\mathfrak{z}(E)}\right) \qquad (10a)$$

$$\sum_{i=1}^{m}\vec{g}'(\vec{x}_i)\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}F_{a_j}(\vec{x}_j)F_{a_i}(\vec{x}_i)\prod_{j=i+1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c = -\beta\left\langle H'\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c+\sum_{i=1}^{m}\left\langle\prod_{j=1}^{i-1}F_{a_j}(\vec{x}_j)[F'_{a_i}(\vec{x}_i)-J'(\vec{x}_i)F_{a_i}(\vec{x}_i)]\prod_{j=i+1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c. \qquad (10b)$$

Equation (10b) is the analog of the Ward-Takahashi identity in field theory. It equates the infinitesimal response of the cumulant average to changes in its arguments engendered by a group transformation to the sum of two terms. The first of these depends explicitly on the Hamiltonian while the second reflects how the fluctuating quantities themselves transform under the group.

IV. ILLUSTRATIONS

In order to make our identity more familiar, we will discuss the forms it takes in several applications involving connected two-point functions. The groups of interest are rotations, translations, dilations, and the special conformal transformations.

First, consider rotations about a fixed but arbitrary axis $\hat{e}$, $x_a \to R_{ab}(\lambda\hat{e})x_b$, where λ is the angle of rotation.[11] In this case $\vec{g}'(\vec{x})\cdot\nabla = (\vec{x}\times\hat{e})\cdot\nabla$ and $J'(\vec{x})=0$. If H is a scalar then, of course, $H'=0$. When the two fluctuating quantities are taken to be different components of the matter current, $\vec{J}(\vec{x})$, Eq. (10b) implies that

$$\hat{e}\cdot(\vec{x}_1\times\nabla_1+\vec{x}_2\times\nabla_2)\langle J_a(\vec{x}_1)J_b(\vec{x}_2)\rangle_c = \langle J_a'(\vec{x}_1)J_b(\vec{x}_2)+J_a(\vec{x}_1)J_b'(\vec{x}_2)\rangle_c. \quad (11)$$

By consulting Eqs. (5b) and (7b), we learn that $\vec{J}'=\hat{e}\times\vec{J}$. If we now assume translation invariance,

$$\langle J_a(\vec{x}_1)J_b(\vec{x}_2)\rangle_c = \langle J_a(\vec{x}_1-\vec{x}_2)J_b(0)\rangle_c,$$

and recall that $\hat{e}$ is arbitrary, we see that Eq. (11) implies that

$$\langle J_a(\vec{x}_1)J_b(\vec{x}_2)\rangle_c = \mathcal{J}(|\vec{x}_1-\vec{x}_2|)(\vec{x}_1-\vec{x}_2)_a(\vec{x}_1-\vec{x}_2)_b,$$

where $\mathcal{J}(|\vec{x}_1-\vec{x}_2|)$ is of course undetermined.

Let us also consider translations in an arbitrary direction which we again denote by $\hat{e}$. Since the transformation is $\vec{x}\to\vec{T}(\vec{x},\lambda\hat{e})=\vec{x}+\lambda\hat{e}$ we find $\vec{g}'(\vec{x})\cdot\nabla=\hat{e}\cdot\nabla$ and $J'(\vec{x})=0$. Rather than assume H^N is invariant, we suppose it has the symmetry-breaking term $\sum_{n=1}^{N} m\vec{g}\cdot\hat{e}$, where $|\vec{g}|=980\ \mathrm{cm\,sec}^{-2}$ which implies that $H^{N\prime}=Nm\vec{g}\cdot\hat{e}$. If we choose each fluctuating quantity to be a density, $\rho(\vec{x})$, then Eq. (7b) implies $\rho'(\vec{x})=0$. Recalling that $\hat{e}$ is arbitrary, one finds that our identity takes the form

$$\nabla_R\langle\rho(\vec{R}+\tfrac{1}{2}\vec{r})\rho(\vec{R}-\tfrac{1}{2}\vec{r})\rangle_c = -m\vec{g}\beta\int d^3z\,\langle\rho(\vec{z})\rho(\vec{R}+\tfrac{1}{2}\vec{r})\rho(\vec{R}-\tfrac{1}{2}\vec{r})\rangle_c, \quad (12)$$

which shows us that the dependence of the pair correlation on $\vec{R}$ is only through its component parallel to $\vec{g}$ and is the generalization of the barometer law. The replacement of $H^{N\prime}=Nm\vec{g}\cdot\hat{e}$ by $m\vec{g}\cdot\hat{e}\int d^3z\,\rho^N(\vec{z})$ foreshadows a technique we will use extensively in Sec. V.

We turn to groups which enjoy greater topical interest by considering dilations and special conformal transformations. The group of dilations is defined by the transformation $\vec{x}\to\vec{D}(\vec{x},\lambda)=e^{\lambda}\vec{x}$. Consequently we find $\vec{g}'(\vec{x})\cdot\nabla=\vec{x}\cdot\nabla$ and $J'(\vec{x})=3$. The identities for the energy-density–energy-density and the density–density correlation functions are Eqs. (13) and (14):

$$(\vec{x}_1\cdot\nabla_1+\vec{x}_2\cdot\nabla_2)\langle e(\vec{x}_1)e(\vec{x}_2)\rangle_c = -\beta\langle H_D' e(\vec{x}_1)e(\vec{x}_2)\rangle_c - 6\langle e(\vec{x}_1)e(\vec{x}_2)\rangle_c + \langle e_D'(\vec{x}_1)e(\vec{x}_2)+e(\vec{x}_1)e_D'(\vec{x}_2)\rangle_c, \quad (13)$$

$$(\vec{x}_1\cdot\nabla_1+\vec{x}_2\cdot\nabla_2)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c = -\beta\langle H_D'\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c, \quad (14)$$

where

$$H_D^{N\prime}\equiv\sum_n(-2)\frac{\vec{p}_n\cdot\vec{p}_n}{2m}+\tfrac{1}{2}\sum_{n\neq l}(\vec{q}_n\cdot\nabla_n+\vec{q}_l\cdot\nabla_l)V(\vec{q}_n,\vec{q}_l). \quad (15)$$

Note that $\int d^3z\, e_D^{N\prime}(\vec{z})=H_D^{N\prime}$.

We may also consider the identities, analogous to those just given, associated with the special conformal transformations,

$$\vec{x}\to\vec{C}(\vec{x},\lambda\hat{e})=\frac{\vec{x}+\lambda\hat{e}x^2}{1+2\lambda\hat{e}\cdot\vec{x}+\lambda^2x^2},$$

which for infinitesimal λ may be regarded, locally, as position-dependent dilations.[12] For this group we find

$$\vec{g}'(\vec{x})\cdot\nabla=(x^2\hat{e}-2\vec{x}\cdot\hat{e}\vec{x})\cdot\nabla,\quad J'(\vec{x})=-6\hat{e}\cdot\vec{x}.$$

Hence, the analogs of Eqs. (13)–(15) are

$$[(x_1^2\hat{e}-2\vec{x}_1\cdot\hat{e}\vec{x}_1)\cdot\nabla_1+(x_2^2\hat{e}-2\vec{x}_2\cdot\hat{e}\vec{x}_2)\cdot\nabla_2]\langle e(\vec{x}_1)e(\vec{x}_2)\rangle_c = -\beta\langle H_C' e(\vec{x}_1)e(\vec{x}_2)\rangle_c+(6\hat{e}\cdot\vec{x}_1+6\hat{e}\cdot\vec{x}_2)\langle e(\vec{x}_1)e(\vec{x}_2)\rangle_c + \langle e_C'(\vec{x}_1)e(\vec{x}_2)+e(\vec{x}_1)e_C'(\vec{x}_2)\rangle_c. \quad (16)$$

$$[(x_1^2\hat{e}-2\vec{x}_1\cdot\hat{e}\vec{x}_1)\cdot\nabla_1+(x_2^2\hat{e}-2\vec{x}_2\cdot\hat{e}\vec{x}_2)\cdot\nabla_2]\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c = -\beta\langle H_C'\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c+(6\hat{e}\cdot\vec{x}_1+6\hat{e}\cdot\vec{x}_2)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c \quad (17)$$

$$H_C^{N\prime}\equiv\sum_n 4\hat{e}\cdot\vec{q}_n\left(\frac{\vec{p}_n\cdot\vec{p}_n}{2m}\right)+\frac{1}{2}\sum_{n\neq l}[(q_n^2\hat{e}-2\vec{q}_n\cdot\hat{e}\vec{q}_n)\cdot\nabla_n+(q_l^2\hat{e}-2\vec{q}_l\cdot\hat{e}\vec{q}_l)\cdot\nabla_l]V(\vec{q}_l,\vec{q}_n). \quad (18)$$

Once again, note that

$$\int d^3z\, e_C^{N\prime}(\vec{z})=H_C^{N\prime}.$$

We now make the point, which is almost trivial, that if $\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c$ is homogeneous function of $|\vec{x}_1-\vec{x}_2|$ of degree $-(1+\eta)$, it is also conformally covariant, which to say that if the rhs of Eq. (14) is $-(1+\eta)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c$, then the rhs of Eq. (17)

is $(1+\eta)(\hat{e}\cdot\vec{x}_1+\hat{e}\cdot\vec{x}_2)\langle\rho(\vec{x}_1)\rho(\vec{x}_2)\rangle_c$. The proof consists of simply assuming homogeneity and then calculating the left-hand side (lhs) of Eq. (17). Similar reasoning will *not* establish such a relation if more than two points are considered. Nonetheless a more subtle argument to be given in Sec. V will establish the intimate connection between the identities for dilations and those for the conformal group. Similar remarks hold for Eqs. (13) and (16).

V. OPERATOR ALGEBRA, CONFORMAL AND DILATION COVARIANCE AND EXPONENTS

In this section we will establish the connection we alluded to between our identities for dilations and special conformal transformations for connected m-point functions and then discuss the specific forms of these identities for the case of the cumulant averaged product of m densities at the critical point.

Let us consider $H_D^{N\prime}$ and $H_C^{N\prime}$. We suppose that the potential term in H arises from the sum of two-body central forces. If this is so, Eqs. (15) and (18) become

$$H_D^{N\prime}=\sum_{n=1}^{N}-2\frac{\vec{p}_n\cdot\vec{p}_n}{2m}+\tfrac{1}{2}\sum_{n\neq l}r_{nl}V'(r_{nl}), \tag{19a}$$

$$H_C^{N\prime}=\sum_{n=1}^{N}4\hat{e}\cdot\vec{q}_n\frac{\vec{p}_n\cdot\vec{p}_n}{2m}+\tfrac{1}{2}\sum_{n\neq l}(-2\hat{e}\cdot\vec{R}_{nl})r_{nl}V'(r_{nl}), \tag{20a}$$

where

$$\vec{R}_{nl}\equiv\tfrac{1}{2}(\vec{q}_n+\vec{q}_l),\quad r_{nl}\equiv|\vec{q}_n-\vec{q}_l|.$$

Now we express $H_C^{N\prime}$ and $H_D^{N\prime}$ as integrals of local quantities; thus

$$H_D^{N\prime}=-\int d^3z\,(3)\left\{\tfrac{1}{3}\left[2T^N(\vec{z})-\tfrac{1}{2}\int d^3r\times rV'(r)\rho_2^N(\vec{z},\vec{r})\right]\right\}, \tag{19b}$$

$$H_C^{N\prime}=-\int d^3z\,(-6\hat{e}\cdot\vec{z})\left\{\tfrac{1}{3}\left[2T^N(\vec{z})-\tfrac{1}{2}\int d^3r\,rV'(r)\rho_2^N(\vec{z},\vec{r})\right]\right\} \tag{20b}$$

where

$$\rho_2^N(\vec{z},\vec{r})\equiv\rho^N(\vec{z}+\tfrac{1}{2}\vec{r})\rho^N(\vec{z}-\tfrac{1}{2}\vec{r})-\delta^3(\vec{r})\rho^N(\vec{z}).$$

The quantities in parentheses on the rhs of Eqs. (19b) and (20b) are the derivatives of the Jacobians of the respective transformations while the quantities in the curly brackets are the same for both transformations. In fact, this quantity is the locally fluctuating pressure $P(\vec{z})$, or $\frac{1}{3}$ of the trace of the locally fluctuating stress tensor. Thus for dilations the first term on the rhs of Eq. (10b) becomes

$$+\beta\int d^3z\,(3)\left\langle P(\vec{z})\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c,$$

and for special conformal transformations it becomes

$$+\beta\int d^3z\,(-6\hat{e}\cdot\vec{z})\left\langle P(\vec{z})\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c.$$

We will see below that circumstances may arise in which Eq. (21) holds:

$$3\beta\left\langle P(\vec{z})\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c=\left(\sum_{j=1}^{m}x_{a_j}\delta^3(\vec{z}-\vec{x}_j)\right)\times\left\langle\prod_{j=1}^{m}F_{a_j}(\vec{x}_j)\right\rangle_c. \tag{21}$$

When this happens our identities demand both scale and conformal covariance.

The methods of the operator algebra can be used to find situations in which Eq. (21) holds and to give expressions for the constants x_2. For simplicity, we now take $F_{a_j}(\vec{x}_j)=\rho(\vec{x}_j)$ and note that our two identities are then

$$\sum_{i=1}^{m}\vec{x}_i\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}\rho(\vec{x}_j)\rho(\vec{x}_i)\prod_{j=i+1}^{m}\rho(\vec{x}_j)\right\rangle_c=\beta\int d^3z\,(3)\left\langle P(\vec{z})\prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c-3m\left\langle\prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c \tag{22}$$

and

$$\sum_{i=1}^{m}(x_i^2\hat{e}-2\vec{x}_i\cdot\hat{e}\vec{x}_i)\cdot\nabla_i\left\langle\prod_{j=1}^{i-1}\rho(\vec{x}_j)\rho(\vec{x}_i)\prod_{j=i+1}^{m}\rho(\vec{x}_j)\right\rangle_c$$
$$=\beta\int d^3z\,(-6\hat{e}\cdot\vec{z})\left\langle P(\vec{z})\prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c-\left(\sum_{j=1}^{m}-6\hat{e}\cdot\vec{x}_j\right)\left\langle\prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c. \tag{23}$$

We recall that according to the operator algebra the product of locally fluctuating quantities can be represented as a linear combination of certain particular locally fluctuating quantities. Because we are concerned only with density fluctuations at the critical point, it is sufficient to take into account only the largest of them, the local density fluctuation at constant temperature. When $\vec{z}$ is far on a microscopic scale from each of the $\vec{x}_j$'s, we represent $P(\vec{z})$ itself by

$$\langle P(\vec{z})\rangle + \frac{\partial P}{\partial \rho}\bigg|_{T_c} (\rho(\vec{z}) - \langle \rho(\vec{z})\rangle).$$

We then have

$$\left\langle P(\vec{z}) \prod_{j=1}^{m} \rho(\vec{x}_j)\right\rangle_c = \frac{\partial P}{\partial \rho}\bigg|_{T_c} \left\langle \rho(\vec{z}) \prod_{j=1}^{m} \rho(\vec{x}_j)\right\rangle_c .$$

However, $(\partial P/\partial \rho)|_T$ vanishes at the critical point. Thus we need only concern ourselves with the case where $\vec{z}$ is near at least one $\vec{x}_j$. Since homogeneity is only expected when each $\vec{x}_j$ is far from all the rest, we further restrict ourselves to this case. It follows that $\vec{z}$ may be near at most one $\vec{x}_j$, call it $\vec{x}_j^*$. We then regard $P(\vec{z})\rho(\vec{x}_j^*)$ as a single fluctuating quantity and write

$$P(\vec{z})\rho(\vec{x}_j^*) = \langle P(\vec{z})\rho(\vec{x}_j^*)\rangle + \frac{\partial}{\partial \rho}\langle P(\vec{z})\rho(\vec{x}_j^*)\rangle|_{T_c} \times (\rho(\vec{y}) - \langle \rho(\vec{y})\rangle)$$

where $\vec{y}$ is in the neighborhood of both $\vec{z}$ and $\vec{x}_j^*$. Henceforth we identify $\vec{y}$ with $\vec{x}_j^*$. Let us define the quantity x by Eq. (24),

$$x \equiv 3\beta_c \int d^3z \frac{\partial}{\partial \rho}\langle P(\vec{z})\rho(\vec{x}_j^*)\rangle|_{T_c}, \tag{24}$$

and note that it is entirely reasonable to write

$$\int d^3z(-6\hat{e}\cdot\vec{z})\frac{\partial}{\partial \rho}\langle P(\vec{z})\rho(\vec{x}_j^*)\rangle|_{T_c} = -2\hat{e}\cdot\vec{x}_j^*\,\beta_c^{-1}x. \tag{25}$$

We conclude that at the critical point and as long as each $\vec{x}_j$ is far from all the rest we may write

$$\beta_c\left\langle P(\vec{z}) \prod_{j=1}^{m} \rho(\vec{x}_j)\right\rangle_c = \tfrac{1}{3}x\left(\sum_{j=1}^{m} \delta^3(\vec{z}-\vec{x}_j)\right) \times \left\langle \prod_{j=1}^{m} \rho(\vec{x}_j)\right\rangle_c . \tag{26}$$

The identities, Eqs. (21) and (22), become

$$\sum_{i=1}^{m} \vec{x}_i\cdot\nabla_i\left\langle \prod_{j=1}^{i-1}\rho(\vec{x}_j)\rho(\vec{x}_i)\prod_{j=i+1}^{m}\rho(\vec{x}_j)\right\rangle_c = (x-3)m\left\langle \prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c , \tag{27}$$

and

$$\sum_{i=1}^{m} (x_i^2\hat{e} - 2\vec{x}_i\cdot\hat{e}\vec{x}_i)\cdot\nabla_i\left\langle \prod_{j=1}^{i-1}\rho(\vec{x}_j)\rho(\vec{x}_i)\prod_{j=i+1}^{m}\rho(\vec{x}_j)\right\rangle_c = (x-3)\left(-\sum_{j=1}^{m} 2\hat{e}\cdot\vec{x}_j\right)\left\langle \prod_{j=1}^{m}\rho(\vec{x}_j)\right\rangle_c . \tag{28}$$

Equation (27) simply states that the cumulant average product, $\langle\prod_{j=1}^{m}\rho(\vec{x}_j)\rangle_c$, is a homogeneous function of degree $(x-3)m$ of its m arguments. Equation (28) is a much more stringent restriction whose consequences have been investigated by Polykov.[9] We note that the expression given for x in Eq. (24) is precisely that given by Green and Gunton for the exponent $\frac{1}{2}(5-\eta)$.[13]

VI. DISCUSSION AND SUMMARY

We have developed for classical statistical mechanics an analog of the Ward-Takahashi identities of quantum field theory which give the infinitesimal response of vacuum expectation values to both internal and space-time transformation. Rather than these transformations we have discussed arbitrary continuous transformation groups of three-dimensional space. Instead of local-operator-valued distributions, we have local fluctuating quantities, that is to say generalized functions which only depend on those particles which are near the arguments of these functions, and rather than averages in the vacuum state, we considered thermal averages in the grand canonical ensemble. A generating functional, from which our identity may be obtained by functional differentiation, was constructed. The lhs of our identity is a first-order differential operator, which characterizes the group, acting on the cumulant average of the product of the fluctuating quantities of interest, and the rhs is the sum of two terms, one of which reflects the action of the group on the local fluctuating quantities themselves while the other contains a new fluctuating quantity H', which results from the action of the group on the Hamiltonian.

In order to domesticate our identity, we exhibited the forms it takes for a variety of connected two-point functions and for translations and rotations as well as dilations and special conformal transformations. We then considered in detail the possible covariance of connected m-point functions under dilations and conformal transformations which have a special relevance for critical phenomena. The most significant result to emerge from these considerations was that for both groups, H' is equal to an integral of the negative of the product of J' and the trace of the local fluctuating stress tensor which is onethird of the local fluctuating pressure. It follows from this that there is an intimate connection between covariance under dilations and covariance under the special conformal transformations.

We then used the methods of the operator algebra to show that at the critical point our identity implies both conformal and dilation covariance for $\langle\prod_{j=1}^{m}\rho(\vec{x}_j)\rangle_c$. The essential reason for this covariance was shown to be the fact that the additional fluctuating variable which appears on the rhs of the identities for these groups is the local pressure. In the course of this analysis we gave another derivation of the expression for the exponent $x=\frac{1}{2}(5-\eta)$ given by Green and Gunton[13] and confirmed Kadanoff length scaling.[8,14] We also derived without any appeal to the Migdal-Polyakov bootstrap the additional restrictions required by conformal covariance due to Polyakov.[9]

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[10]Throughout this paper $\vec{z}$ and $\vec{x}$ (with or without subscripts) denote points in space while $\vec{q}_k$ denotes the location of the kth particle, q denotes the location of the mechanical system in $3N$-dimensional configuration space, and (q,p) denotes the system's location in $6N$-dimensional phase space. The subscript a labels external fields and their conjugate densities. It is to be distinguished from the subscripts a and b which label the Cartesian components of vectors.

[11]The subscripts in italics which are used in Eqs. (5a) and (5b) and in the beginning of Sec. IV run from 1 to 3. We use the convention in which $g_{a,b}(\vec{x})\equiv(\partial/\partial x_b)\times[g_a(\vec{x})]$.

[12]The precise meaning of this phrase is that for all $\hat{r}$,

$$\tfrac{1}{2}[\vec{C}(\vec{R}+\rho\hat{r},\lambda)+\vec{C}(\vec{R}-\rho\hat{r},\lambda)] = \vec{R}+\lambda\,(R^2\hat{e}-2\vec{R}\cdot\hat{e}\vec{R})+\lambda\rho^2(\tfrac{1}{2}\hat{e}-2\hat{r}\cdot\hat{e}\hat{r})+O(\lambda^2)$$

and

$$\|\vec{C}(\vec{R}+\rho\hat{r},\lambda)-\vec{C}(\vec{R}-\rho\hat{r},\lambda)\|=2\rho-4\lambda\rho\hat{e}\cdot\vec{R}+O(\lambda^2).$$

From these two results we see that, when terms proportional to λ^2 and $\lambda\rho^2$ are ignored, a sphere centered at $\vec{R}$ with radius ρ is transformed into a sphere centered at $\vec{R}+\lambda(R^2\hat{e}-2\vec{R}\cdot\hat{e}\vec{R})$ with radius $\rho-2\lambda\rho\hat{e}\cdot\vec{R}$. Dilations transform a sphere centered at $\vec{R}$ with the radius ρ into one centered at $e^{\lambda}\vec{R}$ with radius $e^{\lambda}\rho$.

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