

Exact diffusion equation for a model for superradiant emission

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The super-radiant master equation (SME) of Bonifacio *et al.* is analyzed using the coherent-atomic-state representation. We have succeeded in deriving an exact Fokker-Planck equation for the density function corresponding to the reduced atomic density operator in the diagonal atomic-state representation. A solution to the Fokker-Planck equation has been provided in an elementary fashion for arbitrary atomic states which are sufficiently removed from the state of complete inversion at time zero. The general solution for arbitrary initial conditions (including the initial state of complete inversion) has been obtained using the method of eigenfunction expansions and the final result expressed in terms of an integral over the initial density function. The moments of the collective atomic operators are also discussed.

I. INTRODUCTION

The cooperative emission of radiation (super-radiance) from a large collection of excited two-level systems was originally discussed by Dicke¹ in 1954 and has received renewed attention in the last few years from a large number of authors.²⁻²⁶ In his original paper Dicke pointed out many features of super-radiant emission, using a simple model for the active medium and applying first-order perturbation techniques to arrive at the emission rates. Many of the more recent contributions have attempted to improve on Dicke's calculations so as to eliminate the restrictions inherent in the first-order perturbation analysis. (The groups in Rochester, N. Y., and Milano, Italy, have been especially active along these lines.)

Picard and Willis²⁷ succeeded in deriving coupled equations for the density operators of the atoms and of the field in the framework of the theory of kinetic equations. Their treatment provides a very general description of the collective evolution which requires a limited amount of assumptions on the relative time scales of the interactions and on the spectral nature of the radiated field. As one might expect, the price one has to pay for the generality is an increased complexity in the resulting equations of motion. In contrast, the model of Bonifacio and Schwendimann¹⁰ and Bonifacio *et al.*,¹⁶ to be discussed shortly, assumes

on physical grounds that a single quasimode of the field is present at all times.

Agarwal¹¹ used the master-equation approach to arrive at a Fokker-Planck equation for a phase-space distribution function associated with the reduced atomic density operator. Here again, no *a priori* assumption was made on the spectral nature of the radiated field and the sample size. This theory provides an exact hierarchy of equations of motion for the moments of the atomic operators which cannot be solved exactly for a macroscopically large number of atoms unless some simplification is introduced.²¹ Agarwal also arrived at an operator master equation which is formally identical to the one analyzed in this paper by restricting to an active volume with linear dimensions much smaller than the radiation wavelength.

Of particular interest to the present work are the results of Bonifacio, Haake, and Schwendimann.¹⁶ These authors analyzed in detail the dynamics of the super-radiant emission from a low- Q pencil-shaped cavity containing the excited atoms in an arbitrary initial state, using a master equation for the reduced atomic density operator. The advantage of the geometrical shape considered by Bonifacio *et al.* (and originally proposed by Dicke²⁸) is that precise criteria can be established for the validity of the master equation. Under specified conditions (the so-called super-radiance limit) and in the Markovian approximation, the

master equation in question has the form

$$\dot{W}_A(t) = \frac{1}{2}[R^-, W_A(t)R^+] + \frac{1}{2}[R^-W_A(t), R^+]. \quad (1.1)$$

Here, $W_A(t)$ is the reduced atomic density operator and R^+ and R^- are the collective atomic operators which, with the collective energy operator R_3 , satisfy an angular momentum algebra. In Eq. (1.1) the time variable is scaled to the emission rate per atom in the diffraction solid angle of the end-fire mode.

The super-radiant master equation [Eq. (1.1)] contains complete statistical information on the evolution of the atomic system. In addition, Bonifacio *et al.*¹⁶ have demonstrated that it also provides information on the electromagnetic field statistics. Specifically, they have proved that, in the same super-radiant limit which defines the domain of validity of the master equation, the average of normal-ordered products of the electromagnetic field operators, a^+ and a , are related to the averages of ordered products of the atomic operators R^+ and R^- as follows:

$$\langle a^{+l}a^m(t) \rangle = (-ig/K)^{l+m}(-1)^l \langle R^{+l}R^{-m}(t) \rangle. \quad (1.2)$$

Here, g is the atom-field interaction constant, and K is the photon decay rate due to radiative losses out of the cavity.

Different schemes have been produced to arrive at solutions of the master equation [Eq. (1.1)]. Bonifacio *et al.*¹⁶ used numerical integration procedures to calculate the probability distribution $p(m, t)$ that the collective system be in a quantum state characterized by the eigenvalue m of the collective atomic operator R_3 . Their results were confirmed analytically by Degiorgio and Ghjelmetti²² using an ingenious heuristic procedure. Haake and Glauber²⁹ arrived at a solution using a quasi-probability-distribution representing the density operator of the atoms and derived explicit expressions for the moments of interest in the asymptotic limit in which the number of atoms (or more accurately the cooperation number) is very large.

Quasi-probability-distributions associated with density operators have in fact been used quite extensively in connection with the evolution of electromagnetic field operators. It is well known that, by using the Glauber coherent states³⁰ as a basis for a continuous representation, one can associate with the density operator $W(a, a^+, t)$, a quasi-probability-function $P(\alpha, t)$ such that the diagonal representation

$$W(a, a^+, t) = \int d^2\alpha P(\alpha, t) |\alpha\rangle\langle\alpha|$$

holds. After the density-operator equation is

transformed to anti-normal-ordered form, a simple rule of correspondence provides the differential equation for the function $P(\alpha, t)$. For most problems of interest (e.g., laser theory) the equation of motion for $P(\alpha, t)$ becomes a Fokker-Planck equation containing only a drift and a diffusion term and thus being, at most, of second order in the variables α and α^* .³¹ Very recently a number of authors³²⁻³⁴ have proposed continuous-basis representations to describe collections of two-level atoms. In particular, Arecchi, Courtens, Gilmore, and Thomas³² (ACGT) have introduced a continuous-basis representation analogous to the coherent-state representation for the electromagnetic field. The formal similarity between the coherent atomic states and the coherent Glauber states has prompted the present investigations.

Our calculations, which appear to be the first application of the ACGT formalism, lead to an exact Fokker-Planck diffusion equation for a quasiprobability distribution $P(\theta, \varphi, t)$ which is the diagonal weighting function of the density operator $W_A(t)$ in the ACGT representation. The Fokker-Planck equation can be solved in simple form for initial conditions such that the atomic system is not in the immediate neighborhood of the state of complete inversion. (This result, given in Sec. IV, follows from neglecting the diffusion term of the Fokker-Planck equation.) Furthermore, under the same conditions, we have demonstrated that the expectation values of normal-ordered products of atomic operators can be calculated by averaging powers of trigonometric functions according to the rule of correspondence,

$$\begin{aligned} R^{+l}R^{-l} &= (r \sin\theta)^{2l}, \\ R_3^n &= (-r \cos\theta)^n. \end{aligned} \quad (1.3)$$

Here r and θ are, respectively, the atomic cooperation number and the polar angle of the coherent atomic states. It follows from this that the arbitrary atomic moments $\langle R^{+l}R_3^nR^{-l} \rangle$ are completely specified by the set of reduced moments $\langle R_3^p \rangle$, a circumstance which might prove useful for computational purposes. Finally, we have obtained the general solution in integral form for arbitrary initial conditions by using the method of eigenfunction expansions. This solution can be used to describe the physically interesting case of initial complete inversion.

Our presentation is organized as follows: In Sec. II we review the physical model and the super-radiant master equation of Bonifacio *et al.* In Sec. III we introduce the ACGT states and summarize the properties which are relevant to the present work. In Sec. IV we give an outline of the derivation of the Fokker-Planck equation. (The

details of the calculation are presented in Appen-
dices A and B.) The solution of the Fokker-Planck
equation in the limit of negligible diffusion effects
is presented in Sec. V, together with the calcula-
tion of the atomic moments. In Sec. VI the proper-
ties of these atomic moments are studied further
and the hierarchy of time-dependent coupled equa-
tions which govern their evolution is constructed.
The explicit solutions given by Haake and Glauber
for the time-dependent atomic moments are shown
to satisfy our coupled equations. Finally, the
analysis and solution of the complete Fokker-
Planck equation which includes the diffusive term
is presented in Sec. VII using the eigenfunction
expansion method described by Titchmarsh.

II. SUPER-RADIANT MASTER EQUATION

The master equation (1.1) has been derived by
Bonifacio *et al.*¹⁶ from the consideration of the
radiative properties of a collection of N two-level
systems contained in a pencil-shaped cavity with
a very low Q . The atomic system prepared in an
arbitrary initial state is assumed to interact with
only one resonant axial eigenmode. Some ques-
tions have been raised in connection with the
validity of this assumption, since there seems to
be no reason *a priori* why the atomic system
placed in the vacuum state of the field should ra-
diate initially only in a specified mode. However,
it is proved in Ref. 16 that this ansatz can be made
consistent with the spectral properties of the ra-
diated pulse which, in the super-radiant limit,
has indeed a bandwidth which is much smaller than
the separation of adjacent eigenmodes of the
cavity. Radiation processes in other axial as well
as nonaxial modes of the cavity are not supported
by cooperative effects, so that they can be treated
as a noise background. In addition to the inco-
herent radiative losses associated with the low
cavity Q , the model includes decay mechanisms
for the two-level systems leading to the familiar
damping of the population (characterized by a
longitudinal relaxation time T_1) and of the trans-
verse polarization (with a relaxation time T_2).

The cooperative radiation process has been
shown to occur under the following conditions:
(i) The incoherent atomic decay times T_1 and T_2
are much longer than the cooperative radiation
time t_c (super-radiance coherence time); (ii) the
length of the active medium is smaller than a cer-
tain correlation length l_c , so that the envelope of
the radiated field is constant over the active vol-
ume. The mathematical conditions on t_c and l_c
required for the existence of superradiant decay
have been discussed by Arecchi and Courtens³⁵
using a self-consistent argument based almost

entirely on physical considerations. The mathe-
matical development adopted by Bonifacio *et al.*
consists of projecting the Liouville equation for the
total density operator on the subspace of the atomic
variables by means of the Zwanzig projector tech-
nique.³⁶ The resulting integrodifferential operator
equation is then reduced to a manageable form
by using the first Born approximation. The final
form of the equation of motion for the reduced
atomic density operator, in the Markovian approxi-
mation, is

$$\dot{W}_A(t) = \frac{1}{2} I_1 \{ [R^-, W_A(t) R^+] + [R^+ W_A(t), R^-] \}, \quad (2.1)$$

where R^\pm are the collective atomic raising and
lowering operators which together with a third
operator R_3 obey the angular momentum algebra

$$\begin{aligned} [R^+, R^-] &= 2R_3, \\ [R_3, R^\pm] &= 2R^\pm. \end{aligned} \quad (2.2)$$

The parameter I_1 representing the radiated inten-
sity of a single isolated atom into the diffraction
solid angle of the end-fire mode will be eliminated
in the following development by a suitable scaling
of the time variable.

Two properties of the master equation which are
important in connection with the continuous-basis
representation are

- (i) $\text{Tr} W_A(t) = 1$ [conservation of probability],
- (ii) $\langle R^2(t) \rangle = \text{Tr}(R^2 W_A(t)) = \langle R^2(0) \rangle$
[conservation of the cooperation number].

The first property offers a convenient check
against algebraic errors in the rather involved
derivation of the Fokker-Planck equation (Sec.
IV); the second and more fundamental property
allows the consideration of a continuous basis rep-
resentation in the angular momentum subspace
of a given cooperation number. Both properties
can be verified quite easily from the master equa-
tion (2.1).

As shown by Bonifacio *et al.* and confirmed by
a number of other authors, the behavior of the
physical variables of interest (population inver-
sion, polarization, field intensity, etc.) is essen-
tially classical for all initial preparations of the
atomic systems that are sufficiently removed
from the state of complete inversion. It is impor-
tant to emphasize that, while the classicallike
behavior is characterized by statistical fluctua-
tions of the collective parameters which decrease
as the radiated intensity grows, the quantum
fluctuations do not damp out if the atomic system
is initially prepared in a state of complete inver-

sion. From a mathematical standpoint, the classical behavior will be accurately described by a simplified version of the Fokker-Planck equation, while the evolution of the density function corresponding to the initial state of complete inversion appears to be describable only by the exact Fokker-Planck equation.

III. ATOMIC COHERENT STATES

Several attempts have been made recently to define continuous-basis representations for collections of identical two-level atoms.³²⁻³⁴ A very elegant and convenient representation was independently proposed by Radcliffe³³ and by ACGT³² in terms of what ACGT have called Bloch states.³⁷

The Bloch states, which represent fully symmetrized states of the two-level atoms, are defined in the subspace of cooperation number r as

$$\begin{aligned} |\theta, \varphi\rangle &= \sum_{m=-r}^r |r, m\rangle \langle r, m | \theta, \varphi\rangle \\ &= \sum_{m=-r}^r |r, m\rangle \binom{2r}{m+r}^{1/2} (\sin \frac{1}{2}\theta)^{r+m} \\ &\quad \times (\cos \frac{1}{2}\theta)^{r-m} e^{-i(r+m)\varphi}, \end{aligned} \quad (3.1)$$

where θ and φ are the angular variables in a spherical coordinate system, and where $\theta=0$ corresponds to the south pole of the sphere, Fig. 1. The quantum states $|r, m\rangle$ are the usual Dicke states satisfying the eigenvalue equations

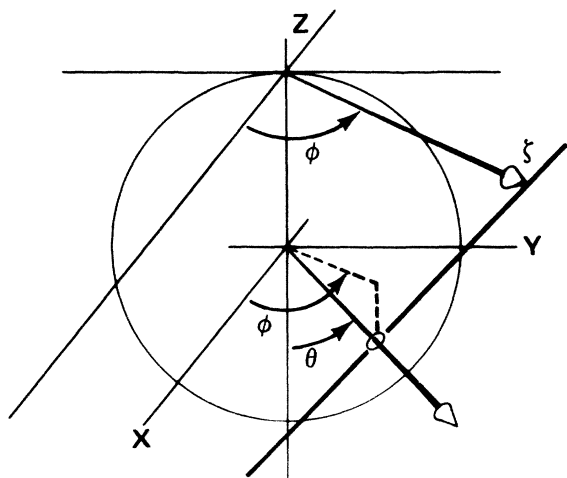


FIG. 1. Relation between a point on the Bloch sphere and the corresponding point in the complex plane. The complex plane is shown tangent to the north pole of the Bloch sphere, and ζ is the complex number defined by the intersection of the line, tangent to the sphere at the point (θ, ϕ) and passing through the Z -axis, with the plane.

$$R^2 |r, m\rangle = r(r+1) |r, m\rangle, \quad (3.2)$$

$$R_3 |r, m\rangle = m |r, m\rangle.$$

The cooperation number r ranges from 0 or $\frac{1}{2}$ (depending on the parity of the number N of atoms) to $\frac{1}{2}N$, while the energy eigenvalue m is restricted to the range $(-r, +r)$. In view of the conserved nature of the cooperation number during the super-radiant emission, it will be sufficient to restrict our considerations to $|\theta, \varphi\rangle$ states with a single value of r .

The atomic coherent states defined by Eq. (3.1) possess a striking similarity to the coherent states of the electromagnetic field.³⁸ They form a complete set in the sense that the identity operator in the $(2r+1)$ -dimensional space of the angular momentum \vec{R} can be resolved as follows:

$$\frac{(2r+1)}{4\pi} \int d\Omega |\theta, \varphi\rangle \langle \theta, \varphi| = \sum_{m=-r}^r |r, m\rangle \langle r, m| \equiv 1, \quad (3.3)$$

where $d\Omega = \sin\theta d\theta d\varphi$. Furthermore, they are normalized but not orthogonal; the inner product is given by

$$|\langle \theta, \varphi | \theta', \varphi' \rangle|^2 = (\cos \frac{1}{2}\Theta)^{4r},$$

where $\cos\Theta = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi')$.

Perhaps the most important property of the atomic coherent states in connection with the present work is the existence of the diagonal representation

$$G = \int d\Omega g(\theta, \varphi) |\theta, \varphi\rangle \langle \theta, \varphi| \quad (3.4)$$

for a wide class of operators G in terms of a c -number weighting function $g(\theta, \varphi)$. In particular, the representation (3.4) is always valid when G is the density operator, i.e.,

$$W_A(t) = \int d\Omega P(\theta, \varphi, t) |\theta, \varphi\rangle \langle \theta, \varphi|. \quad (3.5)$$

In this case the quasi-probability-distribution $P(\theta, \varphi, t)$, which in a sense defines the density of Bloch states on the surface of the sphere, is subject to the normalization condition

$$\text{Tr} W_A(t) = \int d\Omega P(\theta, \varphi, t) = 1. \quad (3.6)$$

Unlike the well-known P function associated with the density operator of the electromagnetic field, the $P(\theta, \varphi, t)$ function need not be singular (e.g., a tempered distribution), as it can always be expressed as a finite sum of at most $(2r+1)^2$ terms³² using as a convenient set of basis functions the spherical harmonics $Y_l^m(\theta, \varphi)$ with $l=0, 1, \dots, 2r$ and $|m| \leq l$. Besides the obvious significance of

the ACGT formulation from a purely formal point of view, the coherent atomic states provide a useful alternative for the calculation of arbitrary moments of the collective operators R^\pm and R_3 .

It follows from the representation (3.5) that the expectation values of ordered products of the form $R^{+l} R_3^n R^{-l'}$ are given by the integral

$$\langle R^{+l} R_3^n R^{-l'} \rangle = \int d\Omega P(\theta, \varphi, t) \langle \theta, \varphi | R^{+l} R_3^n R^{-l'} | \theta, \varphi \rangle. \quad (3.7)$$

The matrix elements $\langle \theta, \varphi | R^{+l} R_3^n R^{-l'} | \theta, \varphi \rangle$ have been calculated in Ref. 32. ACGT have defined the characteristic function

$$\chi(\alpha, \beta, \gamma) \equiv \langle \theta, \varphi | e^{\alpha R^+} e^{\beta R_3} e^{\gamma R^-} | \theta, \varphi \rangle \quad (3.8)$$

and, by using a powerful "disentangling theorem," have proved that $\chi(\alpha, \beta, \gamma)$ is given by

$$\chi(\alpha, \beta, \gamma) = [e^{\beta/2} \sin^2 \frac{1}{2} \theta + e^{-\beta/2} (\alpha e^{i\varphi} \sin \frac{1}{2} \theta + \cos \frac{1}{2} \theta) \times (\gamma e^{-i\varphi} \sin \frac{1}{2} \theta + \cos \frac{1}{2} \theta)]^{2r}. \quad (3.9)$$

The required moments can then be evaluated by direct differentiation of the function $\chi(\alpha, \beta, \gamma)$ as follows:

$$\langle \theta, \varphi | R^{+l} R_3^n R^{-l'} | \theta, \varphi \rangle = \left(\frac{\partial}{\partial \alpha} \right)^l \left(\frac{\partial}{\partial \beta} \right)^n \left(\frac{\partial}{\partial \gamma} \right)^{l'} \chi(\alpha, \beta, \gamma) \Big|_{\alpha=\beta=\gamma=0} \quad (3.10)$$

This result will prove extremely useful in dealing with the moments of the atomic operators for the super-radiant emission process, in which the number of atoms is intrinsically a very large number (Sec. V).

In Sec. IV the results summarized here will be applied to obtain the equation of motion for the quasi-probability-function $P(\theta, \varphi, t)$.

IV. DERIVATION OF c -NUMBER FOKKER-PLANCK EQUATION

In this section we derive the equation of motion for the quasi-probability-distribution $P(\theta, \varphi, t)$ which appears in Eq. (3.5). As we shall see, this will turn out to be a c -number Fokker-Planck equation on the Bloch sphere (Fig. 1). The solution of this equation with the appropriate initial conditions together with the results discussed in Sec. III allow one to calculate all the moments of interest for the atoms as well as the radiation field.

We find it convenient in this section to relabel the Dicke states which appear in the expansion of the Bloch states so that Eq. (3.1) is written

$$| \theta, \varphi \rangle = \sum_{p=0}^{2r} | r, p \rangle \langle r, p | \theta, \varphi \rangle = \sum_{p=0}^{2r} | r, p \rangle \binom{2r}{p}^{1/2} (\sin \frac{1}{2} \theta)^p (\cos \frac{1}{2} \theta)^{2r-p} e^{-i p \varphi}. \quad (4.1)$$

We observe that we now have the relations

$$R^+ | r, p \rangle = [(2r-p)(p+1)]^{1/2} | p+1 \rangle, \quad (4.2a)$$

$$R^- | r, p \rangle = [(2r-p+1)p]^{1/2} | p-1 \rangle, \quad (4.2b)$$

$$R_3 | r, p \rangle = (p-r) | r, p \rangle. \quad (4.2c)$$

If we use Eq. (4.1), the representation (3.5) of the density operator becomes

$$W_A(t) = \int_0^\pi d\theta \sin \theta P(\theta, t) \Lambda(\theta), \quad (4.3)$$

where

$$\Lambda(\theta) = \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2} \theta)^{2(2r-p)} (\sin \frac{1}{2} \theta)^{2p} | r, p \rangle \langle r, p |. \quad (4.4)$$

In Eq. (4.3) we have integrated over the variable φ under the physically plausible assumption that the initial phase of the collective atomic system is unspecified. From Eqs. (3.3), (4.1), and (4.4) we observe that

$$\text{Tr} \Lambda(\theta) = 1. \quad (4.5)$$

Next we replace $W_A(t)$ in the super-radiant master equation (SME), Eq. (2.1), with Eq. (4.3), and note that this results in the following terms on the right-hand side of the equation:

$$R^- W_A R^+ = \int_0^\pi d\theta \sin \theta P(\theta, t) R^- \Lambda(\theta) R^+, \quad (4.6a)$$

$$W_A(t) R^+ R^- = \int_0^\pi d\theta \sin \theta P(\theta, t) \Lambda(\theta) R^+ R^-, \quad (4.6b)$$

$$R^+ R^- W_A(t) = \int_0^\pi d\theta \sin \theta P(\theta, t) R^+ R^- \Lambda(\theta). \quad (4.6c)$$

The expansion of the above expressions in terms of Dicke states is given in Appendix A. When Eqs. (A2), (A3), and (A4) are used in Eq. (2.1) the SME becomes

$$\int_0^\pi d\theta \sin \theta \left[\Lambda(\theta) \frac{\partial P(\theta, t)}{\partial t} + P(\theta, t) \sum_{p=0}^{2r} \binom{2r}{p} \times (\cos \frac{1}{2} \theta)^{2(2r-p)} (\sin \frac{1}{2} \theta)^{2p} \times [(2r-p+1)p - (2r-p)^2] \times (\tan \frac{1}{2} \theta)^2 | r, p \rangle \langle r, p | \right] = 0. \quad (4.7)$$

Now we express the coefficients of P and P in Eq. (4.7) in terms of $\Lambda(\theta)$ and its derivatives. In order to accomplish this, the identities, Eqs. (A6)

and (A7), are used in Eq. (4.7); after collecting the coefficients of Λ and its first and second derivatives one obtains for the SME the form

$$\int_0^\pi d\theta \left(\Lambda(\theta) \frac{\partial Q}{\partial t} + \frac{1}{2} [(2r+1) + 2r \cos \theta] \frac{\sin \theta}{1 + \cos \theta} Q(\theta, t) \frac{\partial \Lambda(\theta)}{\partial \theta} - \frac{1}{2} (1 - \cos \theta) Q(\theta, t) \frac{\partial^2 \Lambda(\theta)}{\partial \theta^2} \right) = 0, \quad (4.8)$$

where we have written for convenience in the development which follows

$$Q(\theta, t) = \sin \theta P(\theta, t) = \sin \theta \int_0^{2\pi} d\varphi P(\theta, \varphi, t). \quad (4.9)$$

If we integrate by parts, Eq. (4.8) becomes

$$\int_0^\pi d\theta \Lambda(\theta) \left[\frac{\partial Q(\theta, t)}{\partial t} - \frac{1}{2} \frac{\partial}{\partial \theta} \left([(2r+1) + 2r \cos \theta] \frac{\sin \theta}{1 + \cos \theta} Q(\theta, t) \right) - \frac{1}{2} \frac{\partial^2}{\partial \theta^2} [(1 - \cos \theta) Q(\theta, t)] \right] + \Gamma_1 + \Gamma_2 - \Gamma_3 = 0, \quad (4.10)$$

where Γ_1 , Γ_2 , and Γ_3 are surface terms which are shown in Appendix B to add to zero. Thus, it is sufficient for Eq. (4.10) to be satisfied that the term in the integrand in large square brackets be identically zero. Therefore, the solution to the SME is equivalent to the solution of the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} Q(\theta, t) = & - \frac{\partial}{\partial \theta} \left[\left(-r \sin \theta - \frac{\sin \theta}{2(1 + \cos \theta)} \right) Q(\theta, t) \right] \\ & + \frac{\partial^2}{\partial \theta^2} \left(\frac{1 - \cos \theta}{2} Q(\theta, t) \right), \end{aligned} \quad (4.11)$$

with the appropriate initial conditions. This together with Eq. (4.9) determines the reduced density operator (4.3) for all time.

It is to be noted that Eq. (4.11) is a Fokker-Planck equation on the Bloch sphere which provides the equation of the quasi-probability-density $Q(\theta, t)$. The coefficient of $Q(\theta, t)$ in the first derivative term, the drift coefficient, describes the motion of the peak of the function on the sphere. It contains two terms, the first of which is consistent with the classical description of the evolution of the Bloch vector. The second term in the drift coefficient is, on the other hand, completely negligible for large values of the cooperation number r , except for values of θ very close to the state of complete inversion, i.e., $\theta \approx \pi$. We interpret this term as the source of spontaneous decay. The coefficient of the second derivative term, the diffusion coefficient, contributes to the spread of the function $Q(\theta, t)$ on the Bloch sphere³⁹ and has its maximum value for the state of complete inversion. It will be indicated in Sec. V that, for sufficiently large values of r , the second term

in the drift coefficient and the diffusion term contribute to the same order in the region where they are important, namely, in the neighborhood of $\theta = \pi$.

Equation (4.11), although quite elegant from the point of view of the physical interpretation, is not in a convenient form for the mathematical solution. For the condition where the diffusion term and the second term in the drift coefficient can be neglected, we have solved the reduced equation by the method of characteristics. This we treat in Sec. V along with the calculation and discussion of the relevant moments.

For the case where the system evolves from a state of complete inversion, the noise and diffusion coefficients cannot be neglected and we must solve Eq. (4.11). This can be best accomplished by performing the following transformation:

$$\tan^{\frac{1}{2}} \theta = 1/z, \quad (4.12a)$$

$$Q(\theta, t) = \frac{2^{2r+1} \sin \theta}{(1 - \cos \theta)^{2(r+1)}} \mathcal{P}(z(\theta, t)), \quad (4.12b)$$

which has the advantage of reducing the coefficients of the Fokker-Planck equation into simple polynomials in z .⁴⁰ The independent variable z is the modulus squared of the complex number obtained by mapping the Bloch sphere on the complex plane in the manner shown in Fig. 1. The factors appearing in Eq. (4.12b) are introduced for convenience and lead to a normalization condition for the weighting function $\mathcal{P}(z, t)$ on the plane of the form

$$\int_0^\infty dz \mathcal{P}(z, t) (1+z)^{2r} = 1. \quad (4.13)$$

By using Eqs. (4.12a) and (4.12b) in Eq. (4.11) we

find the transformed equation to be

$$\frac{\partial \mathcal{P}}{\partial t} = [1 + 2(\nu + 1)z] \frac{\partial \mathcal{P}}{\partial z} + z(1+z) \frac{\partial^2 \mathcal{P}}{\partial z^2}. \quad (4.14)$$

This equation is readily converted into canonical form; the solution by an eigenfunction expansion method will be presented in Sec. VII.

V. SMALL-FLUCTUATION LIMIT

As demonstrated in Sec. IV, the diffusion equation for the quasi-probability-distribution $Q(\theta, t)$ is a Fokker-Planck equation on the surface of the Bloch sphere. Beginning with a given initial condition, the function $Q(\theta, t)$ evolves in the direction of the south pole of the Bloch sphere while it is broadened (or narrowed) until, for very long times, the asymptotic distribution is essentially a δ function at $\theta = 0$ corresponding to the physical situation in which the system is in the ground state $|r, -r\rangle$.

The drift coefficient

$$A(\theta) = -r \sin \theta - \frac{\sin \theta}{2(1 + \cos \theta)}, \quad (5.1)$$

which governs the motion of the distribution function as a whole on the sphere, can be given an attractive physical interpretation in the limiting case of negligible fluctuations. In this case, the average value of the energy operator R_3 is essentially equal to

$$\langle R_3 \rangle = -r \cos \theta, \quad (5.2)$$

where $\theta = \theta(t)$ is the instantaneous angle between the Bloch vector and the south pole of the sphere. If fluctuations are neglected the evolution of the Bloch angle is given by the Langevin equation

$$\frac{d\theta}{dt} = -\left(r \sin \theta + \frac{1}{2} \frac{\sin \theta}{1 + \cos \theta} \right). \quad (5.3)$$

Using Eq. (5.2), one can easily arrive at the equation of motion for $\langle R_3 \rangle$,

$$\frac{d\langle R_3 \rangle}{dt} = -r^2(1 - \cos^2 \theta) - \frac{1}{2} r(1 - \cos \theta), \quad (5.4)$$

which in the limit of large cooperation numbers and upon identification of $\langle R_3 \rangle^2$ with $r^2 \cos^2 \theta$ becomes

$$\frac{d\langle R_3 \rangle}{dt} = -r^2 \left(1 - \frac{\langle R_3 \rangle^2}{r^2} \right). \quad (5.5)$$

The solution of Eq. (5.5) has the familiar form

$$\langle R_3 \rangle = -r \tanh r(t + \hat{t}), \quad (5.6)$$

where \hat{t} is defined in terms of the initial angle θ_0 of the Bloch vector as

$$r\hat{t} = \operatorname{arccosh}[(\sin \theta_0)^{-1}]. \quad (5.7)$$

This result, which strictly speaking applies only to the case in which fluctuations are negligible, actually holds for most initial conditions except those in the neighborhood of $\theta_0 = \pi$, provided the cooperation number is sufficiently large.

It is interesting to observe⁴¹ that in terms of the angle φ , defined by

$$\langle R_3 \rangle - \frac{1}{2} = (r + \frac{1}{2}) \cos \varphi, \quad (5.8)$$

the equation of motion (5.3) simply becomes

$$\frac{d\varphi}{dt} = -(r + \frac{1}{2}) \sin \varphi, \quad (5.9)$$

which is the semiclassical equation discussed by Bonifacio *et al.* Unlike the equation of motion (5.3) with the "noise term" removed, Eq. (5.9) evolves even when the initial atomic state is one of complete inversion ($\langle R_3(0) \rangle = r$) due to the fact that the initial value for φ ,

$$\varphi(0) = \arccos[(r - \frac{1}{2})/(r + \frac{1}{2})], \quad (5.10)$$

although quite small, is not identically zero. Still we must remember that Eq. (5.9) is an approximate result valid only when the atomic fluctuations which accompany the cooperative emission can be neglected.

On the other hand, if one assumes an initial distribution $Q(\theta, 0)$ which is sharply peaked around a value θ_0 sufficiently removed from π , it is obvious that for large cooperation numbers the dominant contribution to the evolution of $Q(\theta, t)$ comes from the drift term $-r \sin \theta$. In fact, it is easy to see from the short-time behavior of the averages of ordered atomic operators {which can be obtained by replacing $\partial Q / \partial t$ with the incremental ratio $[Q(\theta, t) - Q(\theta, 0)] / t$ } that for initial density functions peaked about $\theta_0 \neq \pi$ both the diffusion term and the noise contribution to the drift term are negligible. On the contrary, in the case $\theta_0 \approx \pi$, both terms contribute significantly to the evolution. In this section we analyze the behavior of the quasi-probability-function $Q(\theta, t)$ assuming that the initial distribution is sharply peaked around a value of θ_0 which is sufficiently removed from π .

In this case the Fokker-Planck equation (4.11) is, to order r^{-1} ,

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial \theta} [r \sin \theta Q(\theta, t)]. \quad (5.11)$$

A convenient procedure to solve Eq. (5.11) for an arbitrary initial condition,

$$Q(\theta, 0) = f(\theta), \quad (5.12)$$

is to use the method of characteristics.⁴² Accordingly, we replace Eq. (5.11) by the equivalent system of equations

$$\begin{aligned} dt/1 &= -d\theta/(r \sin\theta), \\ -d\theta/(r \sin\theta) &= dQ/(r \cos\theta Q). \end{aligned} \quad (5.13)$$

Integrating, we find

$$\begin{aligned} e^{rt} \tan^{\frac{1}{2}}\theta &= H, \\ \sin\theta Q(\theta, t) &= K, \end{aligned} \quad (5.14)$$

where H and K are constants of integration independent of θ and t . The objective is to determine the values of H and K such that for $t=0$ the prescribed initial condition (5.12) is satisfied. At time $t=0$ we have from Eqs. (5.14) and (5.12),

$$\begin{aligned} H &= \tan^{\frac{1}{2}}\theta, \\ K &= \sin(\theta)f(\theta). \end{aligned} \quad (5.15)$$

If we eliminate θ from the pair of Eqs. (5.15), we arrive at

$$K = [2H/(1+H^2)] f(\theta = 2 \arctan H), \quad (5.16)$$

which expresses the functional relation which H and K must obey in order that the initial condition is met. Since H and K are independent of t , Eq. (5.15) must be true at all times. From Eqs. (5.14) and (5.16) we finally arrive at

$$Q(\theta, t) = (\cosh rt - \cos\theta \sinh rt)^{-1} f(2 \tan^{-1}(e^{rt} \tan^{\frac{1}{2}}\theta)). \quad (5.17)$$

From the definition Eq. (4.9) of $Q(\theta, t)$ we see that the solution (5.17) must satisfy the normalization integral

$$\int_0^\pi d\theta Q(\theta, t) = 1.$$

This is indeed easily verified by means of the substitution $\tan^{\frac{1}{2}}\varphi = e^{rt} \tan^{\frac{1}{2}}\theta$ in the normalization integral. Furthermore if $f(\theta) \equiv Q(\theta, 0)$ is zero in a small neighborhood of $\theta = \pi$ at time $t=0$, then Eq. (5.17) shows that this remains true for all time. In particular, for an initial state of excitation corresponding to a given value $\theta = \theta_0$, we can conveniently choose for $f(\theta)$ the following distribution⁴³:

$$f(\theta) = \sin\theta \delta(\cos\theta - \cos\theta_0), \quad \pi - \theta_0 \geq \epsilon > 0. \quad (5.18)$$

The solution of the Fokker-Planck equation evolves on the sphere in such a way that the singularity satisfies the differential Eq. (5.3) without the small noise term. Specifically, we have

$$\begin{aligned} Q(\theta, t) &= \frac{\sin\theta}{(\cosh rt - \cos\theta \sinh rt)^2} \\ &\times \delta\left(\cos\theta_0 - \frac{\cos\theta \cosh rt - \sinh rt}{\cosh rt - \cos\theta \sinh rt}\right). \end{aligned} \quad (5.19)$$

To facilitate the integration involved in calculating the moments of the collective atomic operators, it is convenient to rewrite Eq. (5.19) in the form

$$Q(\theta, t) = \sin\theta \delta\left(\cos\theta - \frac{\cos\theta_0 \cosh rt + \sinh rt}{\cos\theta_0 \sinh rt + \cosh rt}\right). \quad (5.20)$$

As stated in Sec. III, the expectation value of any ordered product of operators can be reduced to an integral of the form (3.7), where the matrix element in the integrand can be calculated by repeated differentiation of the characteristic function (3.9). The multiple differentiation of $\chi(\alpha, \beta, \gamma)$ leads to a trigonometric polynomial upon setting α, β , and γ equal to zero. It is easy to see that each term of the polynomial contains successively lower powers of r . In particular, in the limit of large cooperation numbers, and for $\theta \neq \pi$, the leading contribution to the matrix elements of interest is of the form

$$\langle \theta, \varphi | R^{+l} R_3^n R^{-l} | \theta, \varphi \rangle \xrightarrow{r \rightarrow \infty} r^{2l+n} \sin^{2l}(-1)^n \cos^n \theta. \quad (5.21)$$

Matrix elements involving different powers of raising and lowering operators R^+ and R^- can be reduced to a similar asymptotic expression, but their weighted average vanishes on account of the ϕ independence of the distribution function. Carrying out the integral in Eq. (3.7) using Eq. (5.20), we arrive at the required result

$$\langle R^{+l} R_3^n R^{-l} \rangle = (r \operatorname{sech} r(t + \hat{t}))^{2l} (-r \tanh r(t + \hat{t}))^n. \quad (5.22)$$

The same expression has been obtained by Haake and Glauber using an entirely different procedure. In Sec. VI we shall demonstrate that the result, Eq. (5.22), can also be derived from the hierarchy of coupled moment equations which in the asymptotic limit $r \rightarrow \infty$ becomes exactly soluble in the small-fluctuation limit.

VI. MOMENTS OF COLLECTIVE ATOMIC OPERATORS

In Sec. IV we stated that the expectation values of the ordered products $R^{+l} R_3^n R^{-l'}$ can be reduced to an integral of the form

$$\langle R^{+l} R_3^n R^{-l'} \rangle = \int d\Omega P(\theta, \varphi, t) \langle \theta, \varphi | R^{+l} R_3^n R^{-l'} | \theta, \varphi \rangle, \quad (6.1)$$

where the matrix element in the integral of Eq. (6.1) can be obtained by repeated differentiation of a suitable generating function [Eq. (3.9)]. For large values of r and not too large values of l, l' ,

and n , the cumbersome differentiation can be avoided if we confine our attention to the leading term of the trigonometric polynomial $\langle \theta, \varphi | \cdots | \theta, \varphi \rangle$. In particular, if the density function $P(\theta, \varphi, t)$ is independent of φ , the moments involving $l \neq l'$ are identically zero. We shall be interested in the moments of the collective operators where $l = l'$ and where n and l are sufficiently small compared to r . A simple calculation reveals that the trigonometric polynomial $\langle \theta, \varphi | \cdots | \theta, \varphi \rangle$ contains decreasing powers of the cooperation number r . The leading term is of the form of Eq. (5.21) provided $\theta \neq \pi$. We thus find the attractive rule of correspondence,

$$R^{+l} R^{-l} \rightarrow (r \sin \theta)^{2l} \quad (6.2)$$

and

$$R_3^n \rightarrow (-r \cos \theta)^n. \quad (6.3)$$

More precisely, the expectation values in Eq. (6.1) can be reduced to weighted averages of the classical functions (6.3) as follows:

$$\langle R^{+l} R_3^n R^{-l} \rangle = \int d\Omega P(\theta, \varphi, t) (r \sin \theta)^{2l} (-r \cos \theta)^n. \quad (6.4)$$

If $\theta \approx \pi$ Eq. (6.2) breaks down in view of the presence of the factor $(\sin \theta)^{2l}$. If, however, the distribution function $P(\theta, \varphi, t)$ is *not* sharply peaked in the neighborhood of $\theta = \pi$, the result (6.4) is still valid if we can ensure that the major contribution to the integral arises from a region of the Bloch sphere which is outside a small circle surrounding the north pole. If one is interested in expectation values of operators of the form R_3^n the rule of correspondence (6.3) holds without restrictions.

The usefulness of this representation has been demonstrated in Sec. V in calculating the moments of the collective atomic operators corresponding to an arbitrary initial state which is sufficiently removed from the state of complete inversion. For the initial condition discussed in Sec. V [Eq. (5.18)] the moments (6.4) factorize into products of the form $\langle R^+ R^- \rangle^l \langle R_3 \rangle^n$. This factorization is clearly a consequence of the sharpness of the assumed initial distribution. It is easy to see that in general the expectation values (6.4) will not factorize, although for most initial situations of physical interest it appears that a quasi-probability-distribution of the form given by Eq. (5.19) will be quite adequate to represent the evolution of the atomic system from an initial state sufficiently removed from the state of complete inversion.

At this time, the description of the moments of the collective atomic operators corresponding

to an initial state of complete inversion is still an open question, at least in connection with the representation (6.1). If, however, we make the simplifying assumption that the initial distribution is not vanishingly small outside a small area surrounding $\theta = \pi$, some interesting general results can be derived from Eq. (6.4). In particular, we see that the arbitrary ordered moments $\langle R^{+l} R_3^n R^{-l} \rangle$ can be constructed from the hierarchy of moments of the energy operator R_3 . This result can be shown as follows: We define the scaled moments $\mathcal{R}_{2l,n}(t)$ as

$$\begin{aligned} \mathcal{R}_{2l,n}(t) &= \frac{1}{r^{2l+n}} \langle R^{+l} R_3^n R^{-l} \rangle \\ &= \int_0^\pi d\theta Q(\theta, t) (\sin^2 \theta)^l (-\cos \theta)^n, \end{aligned} \quad (6.5)$$

where $Q(\theta, t)$ is the solution of the Fokker-Planck equation on the Bloch sphere. We can also express Eq. (6.5) in the form

$$\begin{aligned} \mathcal{R}_{2l,n}(t) &= \int_0^\pi d\theta Q(\theta, t) (1 - \cos^2 \theta)^l (-\cos \theta)^n \\ &= \sum_{i=0}^l \binom{l}{i} \mathcal{R}_{0,2i+n} (-1)^i, \end{aligned} \quad (6.6)$$

where $\mathcal{R}_{0,2i+n}$ is the scaled expectation value of R_3^{2i+n} . The range of validity of Eq. (6.6) is limited to values of l and n which are much smaller than the cooperation number r and to initial quasi-probability-distributions which satisfy the restrictions discussed above. With these limitations, Eq. (6.6) can be used to derive all the relevant statistical information about the super-radiant decay.⁴⁴

Another interesting consequence of Eq. (6.5) is the existence of a hierarchy of coupled equations of motion for the scaled moments $\mathcal{R}_{2l,n}(t)$. If we differentiate both sides of Eq. (6.5) with respect to t , replace $\partial Q / \partial t$ with the right-hand side of the Fokker-Planck equation, and integrate by parts until each term reduces to a weighted average of powers of $\sin \theta$ and $\cos \theta$, we find, in the limit of large cooperation numbers,

$$\frac{\partial}{\partial t} \mathcal{R}_{2l,n}(t) = 2lr \mathcal{R}_{2l,n+1} - rn \mathcal{R}_{2l+2,n-1}. \quad (6.7)$$

We have summarized a few pertinent comments regarding the derivation of Eq. (6.7) in Appendix C. Presently, we focus on some of the consequences of the hierarchy of Eqs. (6.7). In view of the restrictions imposed on the indices l and n , Eq. (6.7) becomes inaccurate when l and n are comparable to or larger than r [in fact, when $l > r$, $\mathcal{R}_{2l,n}(t) \equiv 0$]. If however, as expected, the scaled higher-order moments become small as n and l increase, it is possible to define a generating

function $W(x, y, t)$ such that

$$W(x, y, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathcal{R}_{2i,j} \frac{x^i}{i!} \frac{y^j}{j!}. \quad (6.8)$$

In terms of $W(x, y, t)$ the reduced moments can be calculated as follows:

$$\mathcal{R}_{2i,n} = \left(\frac{\partial}{\partial x} \right)^i \left(\frac{\partial}{\partial y} \right)^n W(x, y, t) \Big|_{x,y=0}. \quad (6.9)$$

It is easy to verify from Eqs. (6.7) and (6.8) that the generating function $W(x, y, t)$ satisfies the partial differential equation

$$\frac{\partial W}{\partial t} = 2rx \frac{\partial^2 W}{\partial x \partial y} - ry \frac{\partial W}{\partial x}. \quad (6.10)$$

In particular, corresponding to the initial conditions

$$\mathcal{R}_{2i,n}(0) = (\sin \theta_0)^{2i} (-\cos \theta_0)^n \quad (6.11)$$

and

$$W(x, y, 0) = e^{(\sin^2 \theta_0)x} e^{(-\cos \theta_0)y}, \quad (6.12)$$

it is rather easy to verify (e.g., by Laplace-transform techniques) that the solution of Eq. (6.10) corresponding to $W(x, y, 0)$ given in (6.12) is

$$W(x, y, t) = \exp[x \operatorname{sech}^2 r(t + \hat{t})] \exp[-y \tanh r(t + \hat{t})], \quad (6.13)$$

where $r\hat{t}$ is defined as in Eq. (5.7). Hence, in the limit of negligible fluctuations we recover the moments of the ordered atomic operators calculated in Sec. V from the explicit solution of the Fokker-Planck equation.

It is interesting to observe that the scaled moments derived by Haake and Glauber [Eq. (6.9) of Ref. 29] for the case of an initial state of "complete inversion" satisfy our Eq. (6.7) identically. The proof of this statement, which involves a few mathematical manipulations, is presented in Appendix D. Here we mention that, since our moment equation is equivalent to the Fokker-Planck equation in the limit of negligible fluctuations, one would expect (as is indeed easily verified) that an initial condition of complete inversion ($\mathcal{R}_{2i,n}(0) = \delta_{i,0}$) would correspond to an unstable equilibrium point for the hierarchy of coupled moment equations. An analysis of the Haake and Glauber scaled moments reveals that corresponding to their initial state of "complete inversion" [$\nu=0$ in Eq. (6.9) of Ref. 29] the initial value of $\mathcal{R}_{2i,n}$ is unity for $t=0$ only within a correction of order $1/r$. This correction is the significant feature which enables the moments of Haake and Glauber to evolve in time. In terms of our probability function $Q(\theta, t)$ this is equivalent to choosing an initial condition such that $Q(\theta, 0)$ is sharply

peaked near $\theta = \pi$ but is not identically zero outside a small region surrounding the north pole of the Bloch sphere.

A more detailed analysis of the connection between the results of Refs. 22 and 29 and our moment equation will be presented in a separate paper.

VII. SOLUTION OF FOKKER-PLANCK EQUATION

We can derive the general solution of Eq. (4.14) for a specified initial condition at time zero by separating variables and solving the resulting eigenvalue problem. Setting

$$\mathcal{P}(z, t) = \phi(z) e^{-\lambda t} \quad (7.1)$$

in Eq. (4.14) leads to the eigenvalue equation

$$z(1+z)\phi'' + [1 + (2r+2)z]\phi' + \lambda\phi = 0. \quad (7.2)$$

We must now solve the eigenfunction expansion problem for Eq. (7.2). Once this is done the solution of Eq. (4.14) for an arbitrary initial condition can immediately be written down.

Equation (7.2) is the hypergeometric equation for parameters

$$(a, b; c) = (r + \frac{1}{2} + [(r + \frac{1}{2})^2 - \lambda]^{1/2}, r + \frac{1}{2} - [(r + \frac{1}{2})^2 - \lambda]^{1/2}; 1)$$

and variable $-z$.⁴⁵ However, since $c=1$ is a singular case involving a second solution with a logarithmic term, and since the variable z is of interest for $0 \leq z < \infty$, and not just for the range $(0, 1)$ in which the hypergeometric series converges, it is convenient to express the solution of Eq. (7.2), not in terms of hypergeometric functions $F(a, b; c; u)$ of the arguments above, but rather in the form

$$\begin{aligned} \phi_{\pm}(z, \lambda) &= \left(\frac{1}{1+z} \right)^{r+1/2 \pm \rho} \\ &\times F\left(r + \frac{1}{2} \pm \rho, -r + \frac{1}{2} \pm \rho; 1 \pm 2\rho; \frac{1}{1+z} \right), \end{aligned} \quad (7.3)$$

where

$$\rho = [(r + \frac{1}{2})^2 - \lambda]^{1/2}. \quad (7.4)$$

For future use we note that the square root defining ρ in Eq. (7.4) will be made unique by cutting the complex λ plane from $(r + \frac{1}{2})^2$ to ∞ along the positive real axis and requiring ρ to be positive when $(r + \frac{1}{2})^2 - \lambda$ is real and positive. It follows that $\operatorname{Re} \rho > 0$ when $\operatorname{Im} \lambda \neq 0$.

We can now derive the relation for the expansion of a function $g(z)$ in an appropriate linear

combination of the functions (7.3) by employing essentially the method described by Titchmarsh.⁴⁶ We first transform Eq. (7.3) to the standard form

$$\frac{d^2 \zeta(x, \lambda)}{dx^2} + [\lambda - q(x)] \zeta(x, \lambda) = 0 \quad (7.5)$$

by setting

$$z = \sinh^2 \frac{1}{2} x \quad (7.6)$$

and

$$\zeta(x, \lambda) = \sinh^{1/2}(\frac{1}{2}x) \cosh^{2r+1/2}(\frac{1}{2}x) \phi(z(x), \lambda). \quad (7.7)$$

Explicitly, we find

$$q(x) = \frac{(4r+2)^2 \sinh^4(\frac{1}{2}x) + 4(4r+1) \sinh^2(\frac{1}{2}x) - 1}{16 \sinh^2(\frac{1}{2}x) \cosh^2(\frac{1}{2}x)}. \quad (7.8)$$

The form of the eigenfunction expansion formula for Eq. (7.5) will now be obtained, and reversing the transformations (7.6) and (7.7) will then yield the solution of the eigenfunction expansion problem for Eq. (7.2). It is important to note in this connection that transformations (7.6) and (7.7) applied to *any* function $\phi(z)$ [not necessarily a solution of (7.2)] which is analytic at $z=0$ will lead to a function $\zeta(x)$ of the form

$$x^{1/2} \times \text{function analytic in } x \text{ at } x=0.$$

This fact will be important in determining the appropriate solution of Eq. (7.2) [or Eq. (7.5)] to be used in the eigenfunction expansion.

From Eq. (7.3) we see that linearly independent solutions of (7.5) [with $q(x)$ given by Eq. (7.8)] are

$$\zeta_{\pm}(x, \lambda) = \sinh^{1/2}(\frac{1}{2}x) \cosh^{-1/2 \mp 2\rho}(\frac{1}{2}x) \times F\left(r + \frac{1}{2} \pm \rho, -r + \frac{1}{2} \pm \rho; 1 \pm 2\rho; \frac{1}{\cosh^2(\frac{1}{2}x)}\right). \quad (7.9)$$

We must now construct a Green's function of the form

$$G(x, y, \lambda) = \begin{cases} \frac{\psi_1(x, \lambda) \psi_2(y, \lambda)}{W(\psi_1, \psi_2)}, & x \leq y \\ \frac{\psi_1(y, \lambda) \psi_2(x, \lambda)}{W(\psi_1, \psi_2)}, & x \geq y \end{cases} \quad (7.10)$$

where ψ_1 and ψ_2 are linear combinations of the solutions (7.9) satisfying appropriate boundary conditions at 0 and ∞ , respectively, and $W(\psi_1, \psi_2) \equiv \psi_1 \psi_2' - \psi_1' \psi_2$ is their (x -independent) Wronskian. The appropriate boundary condition at ∞ is square integrability, which for $\text{Im} \lambda \neq 0$ leads to

$$\psi_2(x, \lambda) = \zeta_+(x, \lambda). \quad (7.11)$$

The boundary condition at the origin requires more attention. Both solutions (7.9) are square integrable at the singular point $x=0$, so one has Titchmarsh's "limit-circle" case there.⁴⁷ Specifically, by utilizing the representation⁴⁸ for the hypergeometric functions near the singular point $\cosh^2(\frac{1}{2}x) = 1$, we find

$$\zeta_{\pm}(x, \lambda) = a_{\pm}(\lambda) x^{1/2} \ln x + b_{\pm}(\lambda) x^{1/2} + O(x^{5/2} \ln x), \quad x \rightarrow 0 \quad (7.12)$$

where

$$a_{\pm}(\lambda) = -2^{1/2} \frac{\Gamma(1 \pm 2\rho)}{\Gamma(r + \frac{1}{2} \pm \rho) \Gamma(-r + \frac{1}{2} \pm \rho)}. \quad (7.13)$$

The correct boundary condition to be imposed at the origin in this case is that the Wronskian of ψ_1 with the function $g(x)$ to be expanded in eigenfunctions must vanish at $x=0$. As noted earlier, it is expected that at the origin $g(x)$ will satisfy

$$g(x) = cx^{1/2} + O(x^{3/2}). \quad (7.14)$$

We can easily verify that a *necessary* condition for

$$W(g, \psi_1)|_{x=0} = 0,$$

if the $O(x^{1/2})$ term is actually nonzero, and a *sufficient* condition in the other case, is that $\psi_1(x, \lambda)$ contain no $x^{1/2} \ln x$ term. We may then assume that

$$\psi_1(x, \lambda) = \frac{\Gamma(r + \frac{1}{2} + \rho) \Gamma(-r + \frac{1}{2} + \rho)}{\Gamma(1 + 2\rho)} \zeta_+(x, \lambda) - \frac{\Gamma(r + \frac{1}{2} - \rho) \Gamma(-r + \frac{1}{2} - \rho)}{\Gamma(1 - 2\rho)} \zeta_-(x, \lambda). \quad (7.15)$$

By looking at the large- x limit we find

$$W(\zeta_-, \zeta_+) = -2\rho,$$

so the Wronskian appearing in Eq. (7.10) is

$$W(\psi_1, \psi_2) = \frac{2\rho \Gamma(r + \frac{1}{2} - \rho) \Gamma(-r + \frac{1}{2} - \rho)}{\Gamma(1 - 2\rho)}. \quad (7.16)$$

This completes the specification of the Green's function.

Now let $g(x)$ be a function satisfying (7.14) and the condition that both g and g'' are square integrable at infinity, and define the function

$$\begin{aligned} \Phi(x, \lambda, g) &= \int_0^{\infty} G(x, y, \lambda) g(y) dy \\ &= \frac{\psi_2(x, \lambda)}{W(\psi_1, \psi_2)} \int_0^x \psi_1(y, \lambda) g(y) dy \\ &\quad + \frac{\psi_1(x, \lambda)}{W(\psi_1, \psi_2)} \int_x^{\infty} \psi_2(y, \lambda) g(y) dy. \end{aligned} \quad (7.17)$$

By using (7.5) to replace $\psi_i(y, \lambda)$ in Eq. (7.17) by $\lambda^{-1}[q\psi_i - \psi_i'']$ and integrating twice by parts (using the properties of g and the ψ_i to verify that all integrals converge), we find

$$\Phi(x, \lambda, g) = \lambda^{-1}[g(x) + \Phi(x, \lambda, qg - g'')], \quad (7.18)$$

where $\Phi(x, \lambda, qg - g'')$ is calculated from the function $qg - g''$ in the same way $\Phi(x, \lambda, g)$ is calculated from g .

The next step is to integrate Eq. (7.18) with respect to λ over the path C_R comprised by the straight line segment from $-R - i\epsilon$ to $R - i\epsilon$ plus the straight line segment from $R + i\epsilon$ to $-R + i\epsilon$ (R and ϵ positive real numbers). Since we can easily verify that, as a function of λ , $G(x, y, \lambda)$ has no singularities in the finite λ plane except on the real axis, the path of integration in the $\Phi(x, \lambda, qg - g'')$ term can be distorted into the path C'_R comprised by the positively directed semicircle of radius R in the lower half-plane beginning at $-R - i\epsilon$ and ending at $+R - i\epsilon$ plus the complex conjugate of this semicircle (in the upper half-plane) also traversed in the positive direction. By slight modification of the proofs of Titchmarsh's Lemmas⁴⁹ 2.8 and 2.14 we conclude that for fixed $x > 0$ and large $|\lambda|$

$$\Phi(x, \lambda, qg - g'') = O(|\lambda|^{1/2} |\text{Im}\lambda|^{-1}). \quad (7.19)$$

It follows that in the limit as $R \rightarrow \infty$,

$$\lim_{R \rightarrow \infty} \int_{C'_R} \lambda^{-1} \Phi(x, \lambda, qg - g'') d\lambda = 0,$$

and, in this same limit, the integral of Eq. (7.18) becomes

$$2\pi i g(x) = \left(\int_{-\infty - i\epsilon}^{\infty - i\epsilon} - \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \right) \Phi(x, \lambda, g) d\lambda. \quad (7.20)$$

Examination of the analytic properties of $\Phi(x, \lambda, g)$

$$f(z) = \frac{1}{2\pi} \int_0^\infty d\sigma \frac{1}{i} [e^{i\theta(\sigma)} \phi_+(z, \sigma) - e^{-i\theta(\sigma)} \phi_-(z, \sigma)] \\ \times \int_0^\infty \frac{1}{i} [e^{i\theta(\sigma)} \phi_+(z', \sigma) - e^{-i\theta(\sigma)} \phi_-(z', \sigma)] f(z') (1+z')^{2r} dz' \quad (7.25)$$

with [cf. Eq. (7.3)]

$$\phi_\pm(z, \sigma) = (1+z)^{-r-1/2 \pm i\sigma} F(r + \frac{1}{2} \mp i\sigma, -r + \frac{1}{2} \mp i\sigma; 1 \mp 2i\sigma; (1+z)^{-1}) \quad (7.26)$$

and $e^{i\theta(\sigma)}$ determined by Eq. (7.23). By using the linear transformation formula⁴⁸

$$F(a, b; c; z) = (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\ + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F\left(b, c-a; b-a+1; \frac{1}{1-z}\right)$$

as a function of λ for fixed x shows that it has no singularities in the finite λ plane except for the cut along the positive real λ axis from $(r + \frac{1}{2})^2$ to ∞ arising from the cut in the definition of ρ , and that along this cut the limit of $\Phi(x, \lambda, g)$ for $\text{Im}\lambda \rightarrow 0+$ is the complex conjugate of the value for $\text{Im}\lambda \rightarrow 0-$. By inserting Eq. (7.17) into Eq. (7.20) and transforming to the integration variable

$$\sigma = [\lambda - (r + \frac{1}{2})^2]^{1/2}, \quad \lambda \geq r + \frac{1}{2} \quad (7.21)$$

we obtain the relation

$$g(x) = \frac{1}{2\pi} \int_0^\infty d\sigma \frac{1}{i} [e^{i\theta(\sigma)} \eta_+(x, \sigma) - e^{-i\theta(\sigma)} \eta_-(x, \sigma)] \\ \times \int_0^\infty \frac{1}{i} [e^{i\theta(\sigma)} \eta_+(y, \sigma) - e^{-i\theta(\sigma)} \eta_-(y, \sigma)] g(y) dy, \quad (7.22)$$

where

$$e^{2i\theta(\sigma)} = \frac{\Gamma(1+2i\sigma)\Gamma(r+\frac{1}{2}-i\sigma)\Gamma(-r+\frac{1}{2}-i\sigma)}{\Gamma(1-2i\sigma)\Gamma(r+\frac{1}{2}+i\sigma)\Gamma(-r+\frac{1}{2}+i\sigma)} \quad (7.23)$$

and

$$\eta_\pm(x, \sigma) = \sinh^{1/2}(\frac{1}{2}x) \cosh^{-1/2 \pm 2i\sigma}(\frac{1}{2}x) \\ \times F(r + \frac{1}{2} \mp i\sigma, -r + \frac{1}{2} \mp i\sigma; 1 \mp 2i\sigma; \cosh^{-2}(\frac{1}{2}x)). \quad (7.24)$$

Equations (7.21)–(7.24) give the solution to the eigenfunction expansion problem for the differential equation specified by Eqs. (7.5) and (7.8) for all functions $g(x)$ satisfying (7.14) and the condition that g and g'' be $L^2(0, \infty)$. To derive from this the solution to the eigenvalue expansion problem for Eq. (7.2) we only need to invert the transformations of dependent and independent variables given by Eqs. (7.6) and (7.7). We find

together with the Γ -function properties⁵⁰

$$\Gamma(\frac{1}{2} + iy)\Gamma(\frac{1}{2} - iy) = \pi / \cosh \pi y, \quad \Gamma(1 + iy)\Gamma(1 - iy) = \pi y / \sinh \pi y,$$

we may express Eq. (7.25) in the simpler-appearing form

$$f(z) = 2 \int_0^\infty \sigma d\sigma \left\{ \frac{\tanh \pi \sigma}{\coth \pi \sigma} \right\} F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z) \int_0^\infty F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z') f(z') (1+z')^{2r} dz', \tag{7.27}$$

where the upper function in curly brackets is to be chosen for r integral and the lower one for r half-integral. The initial restriction on $f(z)$ in Eq. (7.25) or (7.27), obtained from the properties of $g(x)$ in Eq. (7.22), are that $f(z)$ and the quantity

$$z(1+z)f'' + [(2r+2)z+1]f' + \frac{(4r+2)^2 z^2 + 4(4r+1)z - 1}{16z(1+z)} f \tag{7.28}$$

must be $L^2(0, \infty)$ with respect to the measure $(1+z)^{2r} dz$ and that

$$f(z) = \text{const} + O(z), \quad z \rightarrow 0. \tag{7.29}$$

Square integrability of (7.28) at $z=0$ in fact requires

$$f(z) = O(z), \quad z \rightarrow 0.$$

By appropriate limiting procedures we can now expand the class of functions f for which the eigenvalue expansion (7.25) or (7.27) is valid. In particular, we obtain the formal “ δ -function decomposition”

$$\delta(z-z') = 2(1+z')^{2r} \int_0^\infty \sigma d\sigma \left\{ \frac{\tanh \pi \sigma}{\coth \pi \sigma} \right\} F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z) \times F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z'), \left\{ \begin{array}{l} r \text{ integral} \\ r \text{ half-integral} \end{array} \right\}. \tag{7.30}$$

Solution of Eq. (4.14) subject to a given initial value of $P(z, t)$ at time 0 is now straightforward. From Eqs. (7.2), (7.21), and (7.27) one finds

$$\mathcal{P}(z, t) = 2 \int_0^\infty \sigma d\sigma \left\{ \frac{\tanh \pi \sigma}{\coth \pi \sigma} \right\} F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z) \exp\{-[\sigma^2 + (r + \frac{1}{2})^2]t\} \times \int_0^\infty F(r + \frac{1}{2} + i\sigma, r + \frac{1}{2} - i\sigma; 1; -z') \mathcal{P}(z', 0) (1+z')^{2r} dz' \tag{7.31}$$

as the general solution to the initial-value problem for those $\mathcal{P}(z', 0)$ of physical interest [i.e., those which are $L^2(0, \infty)$ with respect to the measure $(1+z)^{2r} dz$, and which can be obtained as appropriate limits of sequences of functions which are $L^2(0, \infty)$ with respect to this measure].

VIII. CONCLUSIONS

The continuous-basis representation of the coherent atomic states introduced by Arecchi *et al.* affords a very appealing description of the super-radiant decay in terms of a diffusion process on the surface of the Bloch sphere. Several authors have previously used suitable c -number representations to map the SME into a differential equation.^{11, 24, 29} These attempts have resulted in non-

local differential equations (i.e., containing derivatives of arbitrarily high order) which had to be properly truncated to yield any useful information. Our Fokker-Planck equation, on the contrary, is exact and provides the same information as the operator master equation. Under most conditions of interest the solution of the Fokker-Planck equation can be readily derived in the limit of large cooperation numbers. When the diffusion term can be ignored, the noise contribution to the drift term can also be safely ignored and the evolution of the density function reduces to a simple drift motion in a dispersive medium. For arbitrary initial distributions corresponding to atomic systems which are sufficiently removed from the state of complete inversion the drift motion is accompanied by a

general increase in the width of the initial distribution since different portions of the density function propagate with a different drift velocity. The mathematical specification of the initial state of complete inversion corresponds to an initial density function which is identically zero outside an arbitrarily small neighborhood of the north pole of the Bloch sphere. This type of initial condition results in a very complicated evolution. We have succeeded in formally solving the Fokker-Planck equation by the method of eigenfunction expansions, but unfortunately we have not been able to provide a simple representation of the final result [Eq. (7.31)] for this initial condition, even in the limit of very large cooperation numbers. Although the δ -function initial condition corresponding to the state of complete inversion is likely to be over-restrictive for practical physical systems, where a finite spread of the distribution is expected even at $t=0$, the problem of finding a simple representation of our Eq. (7.31) for a δ -function initial condition is nonetheless an interesting challenge from a mathematical point of view. We hope to return to this problem in a later paper.

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APPENDIX A

We wish to expand the quantities $R^{-1}\Lambda(\theta)R^+$, $\Lambda(\theta)R^{-1}R^+$, and $R^+R^{-1}\Lambda(\theta)$ in terms of the Dicke

$$\begin{aligned} \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2}\theta)^{2(2r-p)} (\sin \frac{1}{2}\theta)^{2p} p^2 |r, p\rangle \langle r, p| &= \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \frac{\partial}{\partial \theta} \left(\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \frac{\partial \Lambda(\theta)}{\partial \theta} \right) \\ &+ 2r \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \frac{\partial}{\partial \theta} \left(\sin^2 \frac{1}{2}\theta \Lambda(\theta) \right) \\ &+ 2r \sin^2 \frac{1}{2}\theta \left(\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \frac{\partial \Lambda}{\partial \theta} + 2r \sin^2 \frac{1}{2}\theta \Lambda(\theta) \right). \end{aligned} \quad (A7)$$

APPENDIX B

We examine the surface terms Γ_1 , Γ_2 , and Γ_3 which arise from the integration by parts of Eq. (4.8). We have

states defined in Eq. (4.1). We use Eqs. (4.2) and (4.4) to get

$$\begin{aligned} R^{-1}\Lambda(\theta)R^+ &= \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2}\theta)^{2(2r-p)} (\sin \frac{1}{2}\theta)^{2p} \\ &\times [p(2r-p+1)] |p-1\rangle \langle p-1|, \end{aligned} \quad (A1)$$

or upon replacing $p-1$ by p

$$\begin{aligned} R^{-1}\Lambda(\theta)R^+ &= \sum_{p=0}^{2r} \binom{2r}{p} (2r-p)^2 \cos^2(2r-p-1) \\ &\times (\sin \frac{1}{2}\theta)^{2(p+1)} |r, p\rangle \langle r, p|. \end{aligned} \quad (A2)$$

Similarly, we have

$$\begin{aligned} \Lambda(\theta)R^+R^- &= \sum_{p=0}^{2r} \binom{2r}{p} p(2r-p+1) (\cos \frac{1}{2}\theta)^{2(2r-p)} \\ &\times (\sin \frac{1}{2}\theta)^{2p} |r, p\rangle \langle r, p|, \end{aligned} \quad (A3)$$

and

$$R^+R^{-1}\Lambda(\theta) = \Lambda(\theta)R^+R^-. \quad (A4)$$

In order to rewrite the SME, Eq. (4.7), in terms of $\Lambda(\theta)$ and its derivatives, we examine the derivative of Eq. (4.4):

$$\begin{aligned} \frac{\partial \Lambda(\theta)}{\partial \theta} &= (\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta)^{-1} \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2}\theta)^{2(2r-p)} \\ &\times (\sin \frac{1}{2}\theta)^{2p} (p-2r \sin^2 \frac{1}{2}\theta) |r, p\rangle \langle r, p|. \end{aligned} \quad (A5)$$

From Eq. (A5) we easily establish the identity

$$\begin{aligned} \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2}\theta)^{2(2r-p)} (\sin \frac{1}{2}\theta)^{2p} p |r, p\rangle \langle r, p| \\ = \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \frac{\partial \Lambda(\theta)}{\partial \theta} + 2r \sin^2 \frac{1}{2}\theta \Lambda(\theta). \end{aligned} \quad (A6)$$

On differentiating Eq. (A6) with respect to θ and rearranging terms, we have

$$\Gamma_1 = [(2r+1) + 2r \cos \theta] \frac{\sin \theta}{(1 + \cos \theta)} \Lambda(\theta) Q(\theta, t) \Big|_0^\pi, \quad (B1)$$

$$\Gamma_2 = (1 - \cos\theta)Q(\theta, t) \frac{\partial \Lambda(\theta)}{\partial \theta} \Big|_0^\pi, \quad (\text{B2})$$

and

$$\Gamma_3 = \Lambda(\theta) \frac{\partial}{\partial \theta} (1 - \cos\theta)Q(\theta, t) \Big|_0^\pi. \quad (\text{B3})$$

We assume in what follows that $P(\theta, t)$ is no more singular than a δ function at the end points. From Eq. (4.4) one observes that

$$\Lambda(\pi) = |r, 2r\rangle \langle r, 2r| \quad (\text{B4})$$

and

$$\Lambda(0) = |r, 0\rangle \langle r, 0|. \quad (\text{B5})$$

So, Eq. (B1) becomes

$$\begin{aligned} \Gamma_1 &= |r, 2r\rangle \langle r, 2r| P(\pi, t) \lim_{\theta \rightarrow \pi} \frac{\sin^2 \theta}{1 + \cos \theta} \\ &= -2|r, 2r\rangle \langle r, 2r| P(\pi, t), \end{aligned} \quad (\text{B6})$$

where we have used Eq. (4.9). Similarly, Eq. (B3) becomes

$$\Gamma_3 = -2|r, 2r\rangle \langle r, 2r| P(\pi, t). \quad (\text{B7})$$

For Eq. (B2) we have, using Eq. (A5),

$$\begin{aligned} \Gamma_2 &= P(\theta, t) \sin\theta (1 - \cos\theta) \sum_{p=0}^{2r} \binom{2r}{p} (\cos \frac{1}{2} \theta)^{2(2r-p)-1} \\ &\quad \times (\sin \frac{1}{2} \theta)^{2p-1} [p - 2r \sin^2 \frac{1}{2} \theta] |r, p\rangle \langle r, p| \Big|_0^\pi. \end{aligned} \quad (\text{B8})$$

Thus,

$$\Gamma_2 = 0. \quad (\text{B9})$$

Now, combining Eqs. (B6), (B7), and (B9) in Eq. (4.10), the algebraic sum of the surface terms in the latter equation are readily seen to vanish.

APPENDIX C

The coupled equations of motion for the scaled moments $\mathcal{R}_{2l, n}$ defined by Eq. (6.4) can be derived as follows. From the definition

$$\mathcal{R}_{2l, n}(t) = \int_0^\pi d\theta Q(\theta, t) (\sin\theta)^{2l} (-\cos\theta)^n, \quad (\text{C1})$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{R}_{2l, n}(t) &= \int_0^\pi d\theta (\sin\theta)^{2l} (-\cos\theta)^n \left[\frac{\partial}{\partial \theta} [r \sin\theta Q(\theta, t)] + \frac{\partial}{\partial \theta} \left(\frac{\sin\theta}{2(1 + \cos\theta)} Q(\theta, t) \right) \right. \\ &\quad \left. + \frac{\partial^2}{\partial \theta^2} \left\{ \left[\frac{1}{2}(1 - \cos\theta) \right] Q(\theta, t) \right\} \right]. \end{aligned} \quad (\text{C2})$$

We consider only the first contribution on the right-hand side of Eq. (C2); the other terms are treated in the same way. Integrating by parts we have

$$\begin{aligned} \int_0^\pi d\theta (\sin\theta)^{2l} (-\cos\theta)^n \frac{\partial}{\partial \theta} [r \sin\theta Q(\theta, t)] &= r (\sin\theta)^{2l+1} (-\cos\theta)^n Q(\theta, t) \Big|_0^\pi - \int_0^\pi d\theta r \sin\theta Q \frac{\partial}{\partial \theta} [(\sin\theta)^{2l} (-\cos\theta)^n] \\ &= - \int_0^\pi d\theta r \sin\theta Q [-2l(\sin\theta)^{2l-1} (-\cos\theta)^{n+1} + n(\sin\theta)^{2l+1} (-\cos\theta)^{n-1}]. \end{aligned} \quad (\text{C3})$$

According to Eq. (6.4) we finally arrive at

$$\int_0^\pi d\theta (\sin\theta)^{2l} (-\cos\theta)^n \frac{\partial}{\partial \theta} [r \sin\theta Q(\theta, t)] = r 2l \mathcal{R}_{2l, n+1} - r n \mathcal{R}_{2l+2, n-1}. \quad (\text{C4})$$

Although we do not use the complete moment equation in view of the fact that the contribution (C4) is the dominant one for the initial distribution functions of interest, it might be useful to reproduce the final result of our calculation:

$$\frac{\partial}{\partial t} \mathcal{R}_{2l, n} = r 2l \mathcal{R}_{2l, n+1} - r n \mathcal{R}_{2l+2, n-1} \quad (\text{C5a})$$

$$+ l \mathcal{R}_{2l-2, n+1} + l \mathcal{R}_{2l-2, n+2} - \frac{1}{2} n \mathcal{R}_{2l, n-1} - \frac{1}{2} n \mathcal{R}_{2l, n} \quad (\text{C5b})$$

$$\left. \begin{aligned} &+ l(2l-1) \mathcal{R}_{2l-2, n+2} - l(n+1) \mathcal{R}_{2l, n} - \frac{1}{2} n(2l+1) \mathcal{R}_{2l, n} \\ &+ \frac{1}{2} n(n-1) \mathcal{R}_{2l+2, n-2} + l(2l-1) \mathcal{R}_{2l-2, n+3} - l(n+1) \mathcal{R}_{2l, n+1} \\ &- \frac{1}{2} n(2l+1) \mathcal{R}_{2l, n+1} + \frac{1}{2} n(n-1) \mathcal{R}_{2l+2, n-1}. \end{aligned} \right\} \quad (\text{C5c})$$

Equation (C5) has been separated into three parts: Equation (C5b) derives from the noise contribution to the drift term of the Fokker-Planck equation and Eq. (C5c) is obtained from the diffusion term.

APPENDIX D

Using Haake and Glauber's time variable $z = 2\tau e^{-2\tau t}$ we can write Eq. (6.9) of Ref. 29 in the form

$$\mathcal{R}_{2l,k}(z) = (4z)^l \int_0^\infty d\eta \frac{\eta^l (z - \eta)^k}{(z + \eta)^{2l+k}} e^{-\eta}, \quad (\text{D1})$$

where we have set $\eta = y - z$. Our moment equation [Eq. (6.7)] in terms of the time variable z becomes

$$z \frac{\partial}{\partial z} \mathcal{R}_{2l,k}(z) = -l \mathcal{R}_{2l,k+1} + \frac{1}{2} k \mathcal{R}_{2l+2,k-1}. \quad (\text{D2})$$

Substituting Eq. (D1) in (D2) we can verify that the scaled moments $\mathcal{R}_{2l,k}(z)$ are indeed solutions of the moment equation. The same result follows for values of $\nu \neq 0$, where ν is defined in Ref. 29 as the degree of departure from an initial state of complete inversion.

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