PHYSICAL REVIEW A VOLUME 9, NUMBER 1 JANUARY 1974

Penetration of the electric and magnetic velocity fields of a nonrelativistic point charge into a conducting plane

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(Received 9 July 1973)

The electric and magnetic fields of a point charge moving with constant velocity outside and parallel to a conducting wall of finite conductivity are studied within classical electrodynamics for a nonrelativistic particle and a good conductor. Results are obtained through first order in the particle velocity for a wall with permeability $\mu = 1$, dielectric constant $\epsilon = 1$, and resistivity η. Calculations show the following: (i) There is no skin-depth behavior for the fields $\vec{E}$ or $\vec{B}$ or for the currents $\vec{J}$ inside the conductor. (ii) The fields $\vec{B}$ and $\vec{E}$ fall off with distance as r^{-2} and r^{-3}, respectively, inside the conductor. (iii) The magnetic field $\vec{B}$ and the current density $\vec{J}$ inside the conductor are independent of the resistivity η, although the electric field $\vec{E}$ inside the conductor is suppressed by a factor of η. (iv) In the limit of many point charges moving so as to form a steady current outside the conductor, the magnetic field $\vec{B}$ penetrates the conductor as though it were not present while the electric field $\vec{E}$ and current density $\vec{J}$ vanish inside the conductor. The results obtained here are quite different from those of the familiar calculations involving radiation fields where the penetration depth and the size of the fields inside the conductor are governed by the resistivity. The new results run contrary to the expectations of some physicists and contradict some earlier work in the literature. The calculations arose in connection with the Aharonov-Bohm effect where electrons moving with approximately constant velocity pass very close to a conducting solenoid.

I. INTRODUCTION

A. The problem and conclusion

A static magnetic field penetrates into a good conductor without hindrance. However, a plane electromagnetic wave incident upon a conductor penetrates only a surface layer and is screened out of the body of the conductor. This contrast in behavior raises the question as to what is the penetration of the magnetic velocity field of a charged particle into a conductor of finite conductivity.

In this paper we consider the electromagnetic fields caused by a point charge moving with a small constant velocity outside and parallel to a plane conducting surface of finite conductivity. We conclude that the penetration of the electric and magnetic velocity fields of a nonrelativistic particle is of a totally different character from that of the radiation fields. The velocity fields $\vec{B}$ and $\vec{E}$ fall off as r^{-2} and r^{-3}, respectively, inside a good conductor, rather than being exponentially damped in a skin depth. The electric field inside the conductor is decreased by a factor of the resistivity, but the magnetic field and the currents inside the conductor are independent of the conductivity of the material.

B. Motivation for the analysis

The penetration problem for the electromagnetic velocity fields seems rarely treated in the literature. In general the velocity fields are of no concern in electromagnetic shielding questions because charged particles are located far from the region of interest where only the radiation fields remain relatively large. However, in the tests of the Aharonov-Bohm effect[1] involving the passage of electrons close to microsolenoids or magnetic whiskers, the velocity fields are the relevant ones for treating the classical electromagnetic interactions between the electrons and the solenoids. The energy changes associated with the magnetic velocity fields are comparable to contributions from terms in the Hamiltonian on which the quantum-mechanical explanation of the effect is usually based.

Indeed the author has suggested[2] that contrary to the presently accepted views, the interactions between a charged particle and a solenoid leading to the Aharonov-Bohm effect may involve classical electromagnetic forces. This view has been rejected by the experimentalists at Tübingen who have done the careful experiments[3] verifying the Aharonov-Bohm interference pattern shift. The reason given by the experimentalists for rejecting any explanation based on classical electromagnetic interactions involves precisely the question of the penetration depth of magnetic velocity fields into a conductor of finite conductivity. The experimentalists refer to an (erroneous) analysis by Kasper[4] who purports to show that the magnetic velocity fields are screened out at the surface of a conductor. The relevance of the penetration-depth problem to the Aharonov-Bohm effect will be considered in detail in another publication.[5] Here we restrict our attention to the purely classical electromagnetic question of the penetration of the electric and magnetic velocity fields into a conductor of finite conductivity.

C. Outline of the paper

The basic analysis of this paper is contained in Sec. II. Some limiting cases are presented as corollaries in Sec. III. In Sec. II A, we note the velocity fields of a point charge in free space, and introduce the coordinate system used in our calculations for a moving charge and conducting wall. Then we touch on three qualitative aspects where the interactions of the velocity fields with a conductor differ from the interactions of the radiation fields. These aspects involve the relative orientation of the fields $\vec{E}$ and $\vec{B}$, the possibility of going to a magnetostatic situation, and the availability of a velocity parameter for perturbation theory. Although in principle the problem is soluble exactly, the relativistic behavior may involve considerable complications. Hence, next we start the actual analysis by outlining our nonrelativistic approximations for a good conductor. This shows that the surface charge plays a crucial role. We go on to evaluate the surface charge, the electric field, the currents in the conductor, and finally the magnetic field in all space. The results are valid through first order in the velocity with a well-defined meaning as to what constitutes a good conductor.

Section III considers three limiting cases obtained by combining the point-charge results of Sec. II. We first check that our results agree with the well-known magnetostatic limit for a steady current. Then we treat a line charge moving perpendicular to its axis outside a conductor. This is the form in which the penetration-depth problem for the velocity fields was attempted by Kasper.[4] Finally, we consider the limit of a steady current sheet outside a conducting wall.

Section IV emphasizes the restricted nature of our analysis for the penetration of the velocity fields, and then gives a closing summary.

II. BASIC CALCULATIONS: PENETRATION OF THE VELOCITY FIELDS OF A POINT CHARGE

A. Statement of the problem

The velocity fields $\vec{E}_{e,\beta}(\vec{r},t)$ and $\vec{B}_{e,\beta}(\vec{r},t)$ of a point charge e moving in free space with constant velocity $\vec{v}=c\beta\hat{i}$ and x coordinate $x=c\beta t$ along the line $y=0$, $z=d$ are well known[6] as

$$\vec{E}_{e,\beta}(x,y,z,t)=e\gamma\frac{\hat{i}(x-vt)+\hat{j}y+\hat{k}(z-d)}{[\gamma^2(x-vt)^2+y^2+(z-d)^2]^{3/2}} =e\frac{\hat{i}(x-vt)+\hat{j}y+\hat{k}(z-d)}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}}+O(\beta^2), \tag{1}$$

$$\vec{B}_{e,\beta}(x,y,z,t)=\vec{\beta}\times\vec{E}_{e,\beta}(\vec{r},t) =e\beta\gamma\frac{-\hat{j}(z-d)+\hat{k}y}{[\gamma^2(x-vt)^2+y^2+(z-d)^2]^{3/2}} =e\beta\frac{-\hat{j}(z-d)+\hat{k}y}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}}+O(\beta^3), \tag{2}$$

where $\vec{\beta}=\beta\hat{i}$ and in the usual notation $\gamma=(1-\beta^2)^{-1/2}$. Suppose now that conductor of resistivity η, and, for convenience, permeability $\mu=1$, dielectric constant $\epsilon=1$, filled the half-space $z\leq 0$. Then the presence of the conductor allows image charges and currents which alter the electric and magnetic fields in all space. Our problem is to find the new electric and magnetic fields when the point charge e moves with constant velocity exactly as described initially in the free-space situation.

B. Failure of the traditional skin-depth analysis

When considering a problem of changing electromagnetic fields falling on a conductor, one thinks first of the traditional text-book analysis[7] for the penetration of a plane electromagnetic wave of frequency ω into a conductor of finite resistivity characterized by a permeability μ, a dielectric constant ϵ, and a resistivity η. In this case, the component of the electric field normal to the conducting surface ends on surface charge, penetrating a negligible distance. The remaining electric and magnetic fields are exponentially damped inside the conductor with a characteristic skin depth δ given by

$$\delta=\frac{c}{(\mu\epsilon)^{1/2}\omega}\left(\frac{2}{[1+(4\pi/\omega\epsilon\eta)^2]^{1/2}-1}\right)^{1/2},$$

becoming

$$\delta\cong c(\eta/2\pi\mu\omega)^{1/2}$$

for a good conductor when

$$4\pi/\omega\epsilon\eta\gg 1.$$

Now our problem concerns the appropriate analysis not for a plane electromagnetic wave but for the velocity fields of a charged particle outside a conducting surface. It has been argued by Kasper[4] that the plane-wave analysis can be applied to this latter case where ω^{-1} is taken as the time required for the charged particle to traverse some characteristic distance—such as the distance d from the particle to the conducting wall. However, this suggestion seems totally misdirected. Here we will give three qualitative distinctions between the penetration behavior of the electromagnetic radiation fields and of the velocity fields. We believe that the third distinction involving the passage to a magnetostatic limit provides an irrefutable argument against the applicability of the same penetration-depth analysis to both sets of fields.

The contrast in the physical behavior for the radiation as compared to the velocity fields can be noted immediately in the relative orientations of the electric and magnetic fields. For a plane wave incident normally on a conducting surface, the electric field causes an oscillating current flow. This current in turn generates electric and magnetic fields which inside the conductor tend to cancel the fields of the incident wave and which outside the conductor provide the reflected wave. In the case of non-normal incidence, this basic situation still holds. However, for a charged particle moving with constant velocity outside and parallel to a conducting surface, the orientation of the (undistorted) electric field is such that it has no tendency to cause a current cancelling the magnetic field inside the conductor. The fields required to cancel the penetration of the magnetic field into the conductor can be found immediately by considering the Lorentz transformation of the electrostatic situation of a point charge at rest relative to a conductor. In the transformed situation involving a moving charge, the fields $\vec{E}$ and $\vec{B}$ still vanish inside the (moving) conductor. We see that what is required is a surface current corresponding to the motion of the electrostatic surface charge with the particle velocity $\vec{v}$. However, our problem involves the conductor at rest with respect to the observer while the point charge is moving. If the conductor obeys Ohm's law, then no surface currents are possible unless the resistivity $\eta \to 0$ or the electric field $\vec{E} \to \infty$ at the surface. Moreover the orientation of the free-particle electric field is such that it does not even tend to produce the required currents near the surface.

Next we note that the velocity of a point charge plays a role in the electromagnetic velocity fields which has no counterpoint for a plane wave. The radiation fields necessarily travel at the speed of light in the medium, and the electric and magnetic fields are comparable in size. On the other hand, a point charge may travel at any velocity with $|\beta| < 1$. Thus β forms a continuous parameter available in the description of the system, and it seems natural to expect that physical situations which differ by small changes in β will have only slightly different physical interactions. In particular, it seems appropriate that for low particle velocities $\beta \ll 1$, the physical situation can be regarded as a perturbation of the electrostatic situation $\beta = 0$.

Finally, we wish to consider a limit which seems to indicate conclusively that the electromagnetic velocity fields cannot be exponentially damped inside a conductor as are the radiation fields. The results involving a point charge moving with uniform speed outside and parallel to a conducting surface can be combined so as to give the description for a series of point charges all moving with constant velocity so as to form a constant current. This magnetostatic situation forms a natural limit involving the velocity fields, but there is no nonvanishing static limit involving the radiation fields. Also, the behavior of the fields in this magnetostatic limit is well known so that it can be used as a rough test for the validity of the results obtained in the single-particle situation. In particular, for a steady current, the magnetic field penetrates into a conductor with $\mu = 1$ just as if the conductor were not present. We note, however, that if the fields of a single point charge were indeed exponentially damped within the conductor, then no linear superposition of point charges to form a steady current would remove this exponential damping.

These qualitative considerations contrasting the behavior of the electromagnetic velocity and radiation fields emphasize the need for a separate analysis of the penetration of the velocity fields of a charged particle into a conducting surface of finite conductivity. In the work to follow, we will make use of the velocity parameter β to work from the electrostatic situation as the unperturbed limit. In Sec. III A, we will show that our new results indeed go over to the appropriate solution in the magnetostatic limit. The earlier and erroneous results of Kasper[4] do not.

C. Constant velocity pattern

The charged particle in our problem moves with constant velocity parallel to the conducting surface $z = 0$. Hence whatever the combination of currents, charges, and fields associated with the particle and conducting wall, we expect the entire pattern to move at constant velocity with the particle. In particular, all partial time derivatives of a quantity f can be related to a partial space derivative in the direction of motion

$$\frac{\partial}{\partial t} f(\vec{r}, t) = -c\beta \frac{\partial}{\partial x} f(\vec{r}, t). \tag{3}$$

An example of this functional behavior is seen in Eqs. (1) and (2).

D. Perturbation calculation in particle velocity

Although the solution to the complete penetration problem posed here may be complicated, we will concern ourselves with only an approximate solution holding at nonrelativistic particle velocities for a good conductor. This will be sufficient to indicate that the penetration problem for the velocity fields is totally different from the skin-depth considerations involving plane waves. It will also indicate the erroneous character of the one penetration-depth calculation in the literature[4] of which the present author is aware.

Here we consider the dependence of the electric and magnetic fields upon the velocity of the moving point charge. We expect that the fields $\vec{E}_{e,\beta}(\vec{r}, t)$ and $\vec{B}_{e,\beta}(\vec{r}, t)$ at the field point $\vec{r}$ due to the charged particle moving with constant velocity and located instantaneously at the source point $\vec{\xi}$ can be expressed as a power series in the velocity ratio $\beta = v/c$. Thus when the charged particle is stationary at $\vec{\xi}$, the fields $\vec{E}_e(\vec{r}; \vec{\xi})$ and $\vec{B}_e(\vec{r}; \vec{\xi})$ are known from the electrostatic solutions. When the particle is moving with a small velocity, we expect changes in $\vec{E}_{e,\beta}(\vec{r}, t)$ and $\vec{B}_{e,\beta}(\vec{r}, t)$ which are proportional to β.

In the low-velocity limit, the Coulomb gauge is particularly convenient. In this case the scalar potential $\Phi(\vec{r}, t)$ is given by the instantaneous position of the charges as

$$\Phi(\vec{r}, t) = \int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3r' , \tag{4}$$

while the vector potential involves the retarded time and transverse current,

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{[\vec{J}_\perp(\vec{r}',t')]_{\text{ret}}}{|\vec{r} - \vec{r}'|} d^3r' , \tag{5}$$

$$\vec{J}_\perp(\vec{r}, t) = \frac{1}{4\pi} \nabla\times\nabla\times \int \frac{\vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3r' . \tag{6}$$

The fields are then derived from the potentials as

$$\vec{E}(\vec{r}, t) = -\nabla\Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}, t), \tag{7}$$

$$\vec{B}(\vec{r}, t) = \nabla\times\vec{A}(\vec{r}, t). \tag{8}$$

Furthermore, for the situation considered here, the time derivative may be converted into a space derivative as in (3) giving

$$\vec{E}(\vec{r}, t) = -\nabla\Phi + \beta\frac{\partial}{\partial x}\vec{A}(\vec{r}, t). \tag{9}$$

Now the current density for a point charge at $\vec{\xi}(t)$ is

$$\vec{J}(\vec{r}, t) = e\vec{v}\delta^3(\vec{r} - \vec{\xi}(t)),$$

contributing to first order in the particle velocity, and, to lowest order, we expect the same velocity dependence for the currents $\vec{J}$ inside the conductor. Then from Eqs. (5) and (6), we see that $\vec{A}$ is first order in $\beta = v/c$. Hence from (8), the magnetic field $\vec{B}$ is first order in β, and from (9), the correction to the electric field $\vec{E}$ beyond the electrostatic field is second order in β. Hence we conclude that if we can obtain the charge distribution correct through order β, then a calculation using (4) treating the sources as static will give the electric field $\vec{E}$ correct through order β. If we can obtain all currents $\vec{J}$ through order β, then to lowest order in β we may drop the retardation in (5) and so derive $\vec{A}$ and hence $\vec{B}$ through order β as if they were due to steady currents. This approximation procedure is followed in the remainder of our analysis.

E. Physical assumptions: currents and surface charges

Our first physical assumption is the natural one that currents inside the conductor are given by Ohm's law

$$\vec{J} = (1/\eta)\vec{E}, \tag{10}$$

where η is the resistivity of the conductor. It follows from this, coupled with Maxwell's equation and the continuity equation for charge,

$$0 = \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J}, \tag{11}$$

that

$$0 = \frac{\partial\rho}{\partial t} + \nabla\cdot\left(\frac{1}{\eta}\vec{E}\right) = \frac{\partial\rho}{\partial t} + \frac{4\pi}{\eta}\rho. \tag{12}$$

This last differential equation requires that any charge density inside the conductor decreases exponentially with time. Hence we will assume that the only charges involved in our problem are surface charges on the surface of the conductor.

F. Surface charge distribution through order β

Combining several aspects of the argument above, we see that if we can determine the surface charge distribution through order β, then by an electrostatic calculation we can find $\vec{E}$ through order β, from this find $\vec{J}$ as in (10), and then by a magnetostatic calculation determine $\vec{B}$ through order β. Accordingly, we first consider the surface charge distribution.

We expect that at sufficiently low velocity, the surface charge present on the conductor differs by only a small correction from that present in the electrostatic situation when $\beta = 0$. However, the small correction to the electrostatic surface charge determines the magnitude of the electric field inside the conductor, and this field in turn determines the rate of change of the surface charge through Eqs. (10) and (11). It turns out that this interconnection between the change in the surface charge and the electric field causing a change in the surface charge uniquely determines the surface charge through first order in β. The surface charge will lead to a discontinuity in the normal component of the electric field. Integrating Maxwell's equation

$$\nabla\cdot\vec{E}=4\pi\rho \tag{13}$$

over a flat box with surfaces just inside and outside the surface of the metal at $z=0$,

$$E_{zo}-E_{zi}=4\pi\sigma, \tag{14}$$

where E_{zi} and E_{zo} refer to the fields just inside and outside the conducting surface. As the charge particle moves along parallel to the conducting surface, the surface charge distribution must move along with it. However, it is a crucial observation that since the currents in the conductor are given by Ohm's law and the tangential electric field is finite at the surface, there can be no surface currents. The surface charges cannot move as a whole at velocity $\vec{v}$. Rather the currents inside the conductor must bring up to the surface of the conductor the correct amount of charge to keep the distribution of surface charge moving at velocity $\vec{v}$ along with the passing particle. Integrating the continuity equation (11) over a flat box with sides just inside and outside the conducting surface, we require

$$0=\frac{\partial\sigma}{\partial t}-J_{zi} \tag{15}$$

or

$$0=-c\beta\frac{\partial\sigma}{\partial x}-\frac{1}{\eta}E_{zi}\,, \tag{16}$$

where the current J_{zi} and field E_{zi} are evaluated just inside the conducting surface.

The Eqs. (14) and (16) hold to all orders in β. However, we can now analyze these expressions as power series in β. To zero order in β, the surface charge $\sigma^{(0)}$ is known from the electrostatic situation where

$$E_{zi}^{(0)}=0,\quad E_{zo}^{(0)}=4\pi\sigma^{(0)}. \tag{17}$$

The zero-order surface charge $\sigma_{e,\beta}^{(0)}(\vec{r},t)$ for our problem is just the electrostatic surface charge[8] $\sigma_e(\vec{r};\vec{\xi})$ on the plane $z=0$ due to a point charge e located at $\vec{\xi}$, where here $\xi_x=vt$, $\xi_y=0$, $\xi_z=d$,

$$\sigma_{e,\beta}^{(0)}(x,y,z,t)=\sigma_e(\vec{r};\vec{\xi})=\frac{-ed}{2\pi[(x-vt)^2+y^2+d^2]^{3/2}}\,. \tag{18}$$

But then from Eq. (16), we have $E_{zi}^{(1)}$ in first order as

$$E_{zi}^{(1)}=-c\eta\beta\frac{\partial\sigma^{(0)}}{\partial x}\,. \tag{19}$$

Since the surface charge is confined to a plane surface, symmetry requires that the electrostatic fields due to the surface charge be symmetric on opposite sides of the plane. The first-order electric fields are due solely to the charge on the wall and thus are related as

$$E_{zo}^{(1)}=-E_{zi}^{(1)}\,. \tag{20}$$

The first-order terms from Eq. (14) give accordingly,

$$\sigma^{(1)}=-\frac{1}{2\pi}E_{zi}^{(1)}=\frac{c\eta}{2\pi}\beta\frac{\partial\sigma^{(0)}}{\partial x}, \tag{21}$$

$$\sigma_{e,\beta}^{(1)}(x,y,z,t)=\frac{c\eta}{2\pi}\beta\frac{\partial}{\partial x}\sigma_e(\vec{r};\vec{\xi})=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{-ed}{2\pi[(x-vt)^2+y^2+d^2]^{3/2}}\right)\cdot \tag{22}$$

But this gives us just what is wanted. Now the first-order correction to the surface charge is known in terms of the zero-order electrostatic surface charge distribution. The full charge distribution through first order is

$$\sigma_{e,\beta}(x,y,z,t)=\frac{-ed}{2\pi[(x-vt)^2+y^2+d^2]^{3/2}}+\frac{c\eta\beta}{2\pi}\left(\frac{3e(x-vt)d}{2\pi[(x-vt)^2+y^2+d^2]^{5/2}}\right). \tag{23}$$

G. Condition for a good conductor

The validity of our perturbation-theory analysis requires that the first-order correction $\sigma^{(1)}$ to the surface charge σ should be small compared to the zero-order term $\sigma^{(0)}$. From Eq. (23), we see that this holds provided

$$c\eta\beta/d\ll 1. \tag{24}$$

This gives us the requirement for a good conductor; the resistivity η must be sufficiently small that this condition (23) holds.

The requirements of nonrelativistic particle velocities $\beta\ll 1$ is implied throughout our analysis. However, for fixed resistivity η, our perturbation solution is the low-velocity result. If β is made small enough, the condition (24) can always be satisfied for fixed η and d.

H. Electric field inside and outside the conductor

As indicated in Sec. II D, the electric field inside the conductor, and also outside, can be obtained through order β by integrating over the instantaneous surface charge density and adding the electrostatic field $\vec{E}_e(\vec{r};\vec{\xi})$ due to the passing charged particle at $\xi_x = vt,\ \xi_y = 0,\ \xi_z = d$,

$$\vec{E}_{e,\beta}(\vec{r},t) = \vec{E}_e(\vec{r};\vec{\xi}) + \int da' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \times [\sigma^{(0)}_{e,\beta}(\vec{r}',t) + \sigma^{(1)}_{e,\beta}(\vec{r}',t)].$$

Now the electrostatic field due to the zero-order surface charge is familiar from the use of image charges in elementary electrostatics. It looks exactly like that due to a point charge $-e$ located at the position $\vec{\xi}$ of the charged particle or at its position reflected through the plane $R\vec{\xi}$, $(R\vec{\xi})_x = vt$, $(R\vec{\xi})_y = 0$, $(R\vec{\xi})_z = -d$,

$$\vec{E}_{\sigma(0)}(\vec{r},t) = \int da' \frac{(\vec{r}-\vec{r}')\sigma_e(\vec{r}';\vec{\xi})}{|\vec{r}-\vec{r}'|^3} = \begin{cases} \vec{E}_{-e}(\vec{r};\vec{\xi}), & z<0 \\ \vec{E}_{-e}(\vec{r};R\vec{\xi}), & z>0. \end{cases} \tag{25}$$

Thus inside the conductor $z<0$, the total zero-order electric fields cancel

$$\vec{E}^{(0)}_{e,\beta}(\vec{r},t) = \vec{E}_e(\vec{r};\vec{\xi}) + \vec{E}_{-e}(\vec{r};\vec{\xi}) = 0, \quad \text{for } z<0, \tag{26}$$

whereas outside the conductor, we find the familiar electrostatic image charge solution

$$\vec{E}^{(0)}_{e,\beta}(\vec{r},t) = \vec{E}_e(\vec{r};\vec{\xi}) + \vec{E}_{-e}(\vec{r};R\vec{\xi}), \quad \text{for } z>0. \tag{27}$$

The first-order contribution to the electric field is also easy to evaluate from the form of the charge distribution $\sigma^{(1)}_{e,\beta}(\vec{r},t)$ in Eq. (21),

$$\sigma^{(1)}_{e,\beta}(\vec{r},t) = \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\sigma_e(\vec{r};\vec{\xi}). \tag{28}$$

Since $\sigma_e(\vec{r};\vec{\xi})$ is a function of $x-\xi_x$, this may be rewritten as

$$\sigma^{(1)}_{e,\beta}(\vec{r},t) = -\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial \xi_x}\sigma_e(\vec{r};\vec{\xi}). \tag{29}$$

However, then we can pull the derivative with respect to ξ_x outside the integral:

$$\begin{aligned}\vec{E}^{(1)}_{e,\beta}(\vec{r},t) &= \vec{E}_{\sigma(1)}(\vec{r},t) \\ &= \int da' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\sigma^{(1)}_{e,\beta}(\vec{r}',t) \\ &= -\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial\xi_x}\int da' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\sigma_e(\vec{r}',\vec{\xi}) \\ &= \begin{cases} -\dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial\xi_x}\vec{E}_{-e}(\vec{r};\vec{\xi}), & z<0 \\ -\dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial\xi_x}\vec{E}_{-e}(\vec{r};R\vec{\xi}), & z>0 \end{cases} \\ &= \begin{cases} \dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\vec{E}_{-e}(\vec{r};\vec{\xi}), & z<0 \\ \dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\vec{E}_{-e}(\vec{r};R\vec{\xi}), & z>0. \end{cases}\end{aligned} \tag{30}$$

The derivatives clearly give dipole electric fields as if due to point electric dipoles of moment $c\eta\beta e/2\pi$, oriented along the x axis and located at the position $\vec{\xi}$ of the passing point charge or of its mirror image $R\vec{\xi}$ through the conducting-plane boundary.

The total electric field in space through first order in β is thus

$$\vec{E}_{e,\beta}(x,y,z,t) = e\frac{\hat{i}(x-vt)+\hat{j}y+\hat{k}(z-d)}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}} - e\frac{\hat{i}(x-vt)+\hat{j}y+\hat{k}(z+d)}{[(x-vt)^2+y^2+(z+d)^2]^{3/2}} - \frac{c\eta\beta e}{2\pi}\left(\frac{\hat{i}}{[(x-vt)^2+y^2+(z+d)^2]^{3/2}} - \frac{3(x-vt)[\hat{i}(x-vt)+\hat{j}y+\hat{k}(z+d)]}{[(x-vt)^2+y^2+(z+d)^2]^{5/2}}\right), \quad \text{for } z>0, \tag{31}$$

$$\vec{E}_{e,\beta}(x,y,z,t) = -\frac{c\eta\beta e}{2\pi}\left(\frac{\hat{i}}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}} - \frac{3(x-vt)[\hat{i}(x-vt)+\hat{j}y+\hat{k}(z-d)]}{[(x-vt)^2+y^2+(z-d)^2]^{5/2}}\right), \quad \text{for } z<0. \tag{32}$$

I. Procedure for obtaining the magnetic field

Once we have obtained the electric field through first order in β, we can find the currents in all of space to this same order in β. Thus the charged particle contributes a current

$$\vec{J}_e(\vec{r},t) = ec\vec{\beta}\delta^3(\vec{r}-\vec{\xi}), \quad \vec{\beta} = \beta\hat{i} \tag{33}$$

while the currents inside the conductor are given by Ohm's law and the result in (30) and (32),

$$\vec{J}(\vec{r},t) = (1/\eta)\vec{E}_{e,\beta}(\vec{r},t) = \frac{c\beta}{2\pi}\frac{\partial}{\partial x}\vec{E}_{-e}(\vec{r};\vec{\xi}), \quad \text{for } z<0. \tag{34}$$

The magnetic field through first order in β follows as the integral in the Biot-Savart law:

$$\vec{B}_{e,\beta}(\vec{r},t) = \frac{1}{c}\int d^3r' \frac{\vec{J}(\vec{r}',t)\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}. \tag{35}$$

Substituting the total currents above in (33) and (34),

$$\vec{B}_{e,\beta}(\vec{r}, t) = \vec{\beta}\times\vec{E}_e(\vec{r}, \vec{\xi}) + \frac{\beta}{2\pi}\int d^3r' \frac{[(\partial/\partial x')\vec{E}_{-e}(\vec{r}'; \vec{\xi})]\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}. \tag{36}$$

Although sophisticated vector manipulations may indeed allow a simple evaluation of this last integral, the author has not discovered them and hence has proceeded to the solution on a more round-about route. Our treatment works first with the equation giving curl$\vec{B}$ in terms of current densities and the time derivative of the electric field. The source contribution can be recognized in terms of familiar patterns due to moving charges. Once we obtain $\vec{B}$ due to these familiar sources, we then check all of Maxwell's equations. We discover a further correction is needed to meet the boundary conditions on $\vec{B}$ at the surface of the conductor. We then add the required correction terms obtaining fields satisfying all of Maxwell's equations and the boundary conditions through first order in the velocity of the passing charged particle.

J. Contributions appearing in the curl equation for the magnetic field

In order to obtain the magnetic field through first order in the velocity of the passing charged particle, we turn to Maxwell's equations. The equation

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{37}$$

which for our special case becomes exactly

$$\nabla\times\vec{E} = \beta\frac{\partial\vec{B}}{\partial x}, \tag{38}$$

is not useful here. Since $\vec{B}$ is already first order in β, the right-hand side is second order in β, telling us what we remarked upon before, that up through first order, $\vec{E}$ behaves like an electrostatic field

$$\nabla\times\vec{E} = 0. \tag{39}$$

However, the equation

$$\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \tag{40}$$

will allow us to determine curl$\vec{B}$ in terms of the known electric fields. Throughout this work, we are assuming for convenience that $\mu = 1$, $\epsilon = 1$ inside the conductor.

For a free particle moving with uniform velocity in free space, Eq. (40) becomes to first order

$$\nabla\times\vec{B} = 4\pi e\vec{\beta}\delta^3(\vec{r}-\vec{\xi}) - \beta\frac{\partial}{\partial x}\vec{E}_e(\vec{r}; \vec{\xi}), \tag{41}$$

where as usual $\vec{\xi}$ stands for $\xi_x = vt$, $\xi_y = 0$, $\xi_z = d$, $\vec{\beta} = \beta\hat{i}$, and $\vec{E}_e(\vec{r}; \vec{\xi})$ is the electrostatic field of the charged particle located at $\vec{\xi}$. To first order, the magnetic field $\vec{B}$ has the familiary form

$$\vec{B}_{e,\beta}(\vec{r}; \vec{\xi}) = \vec{\beta}\times\vec{E}_e(\vec{r}; \vec{\xi}). \tag{42}$$

Now in the situation we are considering of a particle and a conductor, the same basic form of field reappears. Thus outside the conductor, the current density $\vec{J}$ becomes simply that due to the passing charged particle, and the electric field is needed only to zero order so that from (27),

$$\nabla\times\vec{B} = 4\pi e\vec{\beta}\delta^3(\vec{r}-\vec{\xi}) - \beta\frac{\partial}{\partial x}[\vec{E}_e(\vec{r}; \vec{\xi}) + \vec{E}_{-e}(\vec{r}; R\vec{\xi})]. \tag{43}$$

This looks just like the fields due to point charges e at $\vec{\xi}$ and $-e$ at $R\vec{\xi}$, both moving with velocity $c\beta$. Hence outside the conductor

$$\vec{B}_{e,\beta}(\vec{r}, t) = \vec{\beta}\times\vec{E}_e(\vec{r}; \vec{\xi}) + \vec{\beta}\times\vec{E}_{-e}(\vec{r}; R\vec{\xi}) - \nabla\phi, \quad \text{for } z > 0. \tag{44}$$

The unknown term $-\nabla\phi$ has been included to meet the boundary conditions; it does not contribute in $\nabla\times\vec{B}$ since $\nabla\times\nabla\phi = 0$. Inside the conductor, the electric field vanishes to zero order in β and hence only the current density contributes to Maxwell's equation in first order. From Eq. (34),

$$\vec{J}(\vec{r}, t) = \frac{c\beta}{2\pi}\frac{\partial}{\partial x}\vec{E}_{-e}(\vec{r}; \vec{\xi}) = -\frac{c\beta}{2\pi}\frac{\partial}{\partial x}\vec{E}_e(\vec{r}; \vec{\xi}), \tag{45}$$

$$\nabla\times\vec{B} = -\beta\frac{\partial}{\partial x}[2\vec{E}_e(\vec{r}; \vec{\xi})]. \tag{46}$$

This corresponds to a particle of charge $2e$ moving with velocity $c\beta$ at the position $\vec{\xi}$. Thus inside the conductor to first order

$$\vec{B}_{e,\beta}(\vec{r}, t) = \vec{\beta}\times 2\vec{E}_e(\vec{r}; \vec{\xi}) - \nabla\phi, \quad \text{for } z < 0, \tag{47}$$

again including the possibility of a gradient term not contributing to the curl equation for $\vec{B}$.

K. Correction to satisfy boundary conditions on the magnetic field

The remaining condition which must be imposed upon the magnetic field is that given by Maxwell's equation:

$$\nabla\cdot\vec{B} = 0. \tag{48}$$

The form of the magnetic field in (44) and (47) in terms of the familiar magnetic fields of moving point charges immediately allows verification of this equation for $z < 0$ and $z > 0$ provided the additional scalar function ϕ satisfies Laplace's equation:

$$\nabla^2\phi = 0 \quad \text{for } z<0 \text{ and } z>0. \tag{49}$$

At the surface boundary $z=0$, the usual volume integral of $\nabla\cdot\vec{B}$ over a box with flat sides just inside and outside the conductor provides the boundary condition

$$B_{zi} = B_{zo}. \tag{50}$$

This does not hold for Eqs. (44) and (47) when $\phi = 0$. Rather this boundary condition implies

$$-\left.\frac{\partial\phi}{\partial z}\right|_{z=0_-} + \hat{k}\cdot[\vec{\beta}\times 2\vec{E}_e(\vec{r};\vec{\xi})]_{z=0} = -\left.\frac{\partial\phi}{\partial z}\right|_{z=0_+}. \tag{51}$$

The physical conditions also require that ϕ should fall to zero at spatial infinity at least as fast as r^{-1}.

The requirements on ϕ given in Eqs. (49) and (51) make it natural to view the problem of determining ϕ as analogous to one in electrostatics involving the potential ϕ due to a surface charge density σ

$$\sigma = (1/4\pi)\,\hat{k}\cdot[\vec{\beta}\times 2\vec{E}_e(\vec{r};\vec{\xi})]_{z=0}. \tag{52}$$

L. Electrostatic analog problem

What we need is the evaluation of the electrostatic potential ϕ, or rather the electrostatic field $-\nabla\phi$, due to the surface charge density in the xy plane given by Eq. (52),

$$\sigma = \frac{1}{4\pi}\frac{2ey\beta}{(x^2+y^2+d^2)^{3/2}}, \tag{53}$$

where here we have chosen the position $\vec{\xi}$ of the particle on the z axis, $\xi_x=\xi_y=0$, $\xi_z=d$. The solution is, of course

$$\vec{E}(x,y,z) = -\nabla\phi = \int dx'\,dy'\left(\frac{1}{4\pi}\right)\frac{2ey'\beta}{(x'^2+y'^2+d^2)^{3/2}} \times\frac{[\hat{i}(x-x')+\hat{j}(y-y')+\hat{k}z]}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{54}$$

It may be perfectly feasible to carry out this integral by direct integration. However, again we have decided to approach the problem indirectly in order to avoid an apparently difficult integral. What we will do is to find a charge distribution in the region $z<0$ which would induce the surface charge density (53) on a perfectly conducting surface in the xy plane. Then we know that in the region $z>0$, the field due to the surface charge is exactly equal in magnitude and opposite in sign from that due to the charge distribution located at $z<0$. We will find it easier to evaluate the electrostatic field for $z>0$ by integrating over the charge sources suggested at $z<0$.

First of all, a point charge q located at $x'=y'=0$, z' will induce a surface charge distribution on a perfectly conducting surface in the xy plane:

$$\sigma_q(x,y,0;0,0,z') = \frac{q}{4\pi}\frac{-2|z'|}{(x^2+y^2+z'^2)^{3/2}}. \tag{55}$$

Then it follows that a line charge of charge λ per unit length along the negative z axis from $-\infty$ to $z'=-d$ induces a surface charge

$$\sigma_\lambda(x,y,0;-d) = \int_{-\infty}^{z'=-d}\frac{(\lambda\,dz')}{4\pi}\frac{+2z'}{(x^2+y^2+z'^2)^{3/2}} = \frac{-\lambda}{4\pi}\frac{2}{(x^2+y^2+z'^2)^{1/2}}\Bigg|_{-\infty}^{z'=-d} = \frac{-\lambda}{4\pi}\frac{2}{(x^2+y^2+d^2)^{1/2}}. \tag{56}$$

Further, a line of dipoles of moment $p\hat{j}$ per unit length along the negative z axis from $-\infty$ to $z'=-d$ induces a surface charge

$$\sigma_{p\hat{j}}(x,y,0;-d) \cong \frac{-\lambda}{4\pi}\frac{2}{(x^2+y^2+d^2)^{1/2}} + \frac{\lambda}{4\pi}\frac{2}{[x^2+(y+\Delta y)^2+d^2]^{1/2}} \cong +\frac{\lambda\Delta y}{4\pi}\frac{\partial}{\partial y}\left(\frac{2}{(x^2+y^2+d^2)^{1/2}}\right), \tag{57}$$

$$\sigma_{p\hat{j}}(x,y,0;-d) = \frac{-p}{4\pi}\frac{2y}{(x^2+y^2+d^2)^{3/2}}. \tag{58}$$

This surface charge density agrees with that required by Eq. (53), provided we identify the dipole moment per unit length p with $-e\beta$

$$p = -e\beta. \tag{59}$$

As we remarked above, the electric field due to the surface charge on the xy plane may be evaluated as the negative of that due to the suggested charges on the other side of the plane. Thus here we can obtain the require electric field $\vec{E}$ in (54) as the field of a line of dipoles along the z axis magnitude $e\beta$ per unit length located on the opposite side of the xy plane from the field point.

When the field point is in the region $z>0$, the line of dipoles stretches along the negative z axis from $-\infty$ to $z'=-d$. The electric field will be evaluated in stages analogous to those used in analyzing the surface charge. A point charge q at $x'=y'=0$, z' gives a field

$$\vec{E}_q(x,y,z;0,0,z') = q\frac{\hat{i}x+\hat{j}y+\hat{k}(z-z')}{[x^2+y^2+(z-z')^2]^{3/2}}. \tag{60}$$

A line charge λ per unit length along the negative

z axis from $-\infty$ to $z'=-d$ causes an electrostatic field

$$\vec{E}_\lambda(x,y,z;-d)=\int_{-\infty}^{z'=-d}(\lambda\,dz')\frac{\hat{i}x+\hat{j}y+\hat{k}(z-z')}{[x^2+y^2+(z-z')^2]^{3/2}}$$
$$=\left[\lambda\frac{(\hat{i}x+\hat{j}y)(z'-z)}{(x^2+y^2)[x^2+y^2+(z'-z)^2]^{1/2}}+\lambda\frac{\hat{k}}{[x^2+y^2+(z'-z)^2]^{1/2}}\right]_{-\infty}^{z'=-d}$$
$$=\lambda\frac{-(\hat{i}x+\hat{j}y)(z+d)+\hat{k}(x^2+y^2)}{(x^2+y^2)[x^2+y^2+(z+d)^2]^{1/2}}+\lambda\frac{\hat{i}x+\hat{j}y}{x^2+y^2}. \tag{61}$$

A line of dipoles along the negative z axis with dipole moment $e\beta\hat{j}$ per unit length and stretching from $-\infty$ to $z'=-d$ gives an electrostatic field

$$\vec{E}_{(e\beta)}(x,y,z;-d)\cong\vec{E}_\lambda(x,y,z;-d)+\vec{E}_{-\lambda}(x,y+\Delta y,z;-d)$$
$$\cong-(\lambda\Delta y)\frac{\partial}{\partial y}\left[\frac{1}{\lambda}\vec{E}_\lambda(x,y,z;-d)\right]. \tag{62}$$

Identifying the dipole moment per unit length $\lambda\Delta y=e\beta$, the electrostatic field for $z>0$ is

$$\vec{E}_{(e\beta)}(x,y,z;-d)=-e\beta\frac{\partial}{\partial y}\left(\frac{-(\hat{i}x+\hat{j}y)(z+d)+\hat{k}(x^2+y^2)}{(x^2+y^2)[x^2+y^2+(z+d)^2]^{1/2}}+\frac{\hat{i}x+\hat{j}y}{x^2+y^2}\right). \tag{63}$$

By symmetry, the electrostatic field due to the surface charge is found for $z<0$ by the substitutions $z\to -z$, $\hat{k}\to-\hat{k}$.

M. Result for the magnetic field

Having completed the calculation for the electrostatic analogy, we substitute from (63) into $-\nabla\phi$ required in Eqs. (44) and (47) to obtain the magnetic field due to a particle of charge e moving with velocity $\vec{v}=c\beta\hat{i}$ parallel to the surface of a conductor in the half-plane $z<0$. Through first order in the velocity,

$$\vec{B}_{e,\beta}(x,y,z,t)=e\beta\left(\frac{-\hat{j}(z-d)+\hat{k}y}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}}+\frac{\hat{j}(z+d)}{[(x-vt)^2+y^2+(z+d)^2]^{3/2}}\right.$$
$$-\frac{\hat{i}(x-vt)y[3(x-vt)^2+3y^2+2(z+d)^2](z+d)}{[(x-vt)^2+y^2][(x-vt)^2+y^2+(z+d)^2]^{3/2}}+\frac{2\hat{i}(x-vt)y}{[(x-vt)^2+y^2]^2}$$
$$-\frac{\hat{j}[-(x-vt)^2+y^2](z+d)}{[(x-vt)^2+y^2]^2[(x-vt)^2+y^2+(z+d)^2]^{1/2}}-\frac{\hat{j}y^2(z+d)}{[(x-vt)^2+y^2][(x-vt)^2+y^2+(z+d)^2]^{3/2}}$$
$$\left.-\frac{\hat{j}[(x-vt)^2-y^2]}{[(x-vt)^2+y^2]^2}\right)\quad\text{for } z>0, \tag{64}$$

$$\vec{B}_{e,\beta}(x,y,z,t)=e\beta\left(\frac{-\hat{j}2(z-d)+\hat{k}y}{[(x-vt)^2+y^2+(z-d)^2]^{3/2}}+\frac{\hat{i}(x-vt)y[3(x-vt)^2+3y^2+2(z-d)^2](z-d)}{[(x-vt)^2+y^2]^2[(x-vt)^2+y^2+(z-d)^2]^{3/2}}+\frac{2\hat{i}(x-vt)y}{[(x-vt)^2+y^2]^2}\right.$$
$$+\frac{\hat{j}[-(x-vt)^2+y^2](z-d)}{[(x-vt)^2+y^2]^2[(x-vt)^2+y^2+(z-d)^2]^{1/2}}+\frac{\hat{j}y^2(z-d)}{[(x-vt)^2+y^2][(x-vt)^2+y^2+(z-d)^2]^{3/2}}$$
$$\left.-\frac{\hat{j}[(x-vt)^2-y^2]}{[(x-vt)^2+y^2]^2}\right)\quad\text{for } z<0. \tag{65}$$

We can check this result for the magnetic field through first order in β by explicit calculation in Maxwell's equations and boundary conditions at $z=0$.

N. Comment on the electromagnetic fields

The results (31), (32), (64) and (65) for the electric and magnetic fields $\vec{E}_{e,\beta}(\vec{r},t)$ and $\vec{B}_{e,\beta}(\vec{r},t)$ inside the conductor show that the character of the penetration is totally different from the skin-depth behavior familiar for a plane electromagnetic wave inside a good conductor. For nonrelativistic charges, there is no exponential damping of the velocity fields of a charged particle. Rather the electric and magnetic fields are modified by the presence of the conducting wall but fall off as r^{-3}

and r^{-2} inside the conductor. Moreover from Eqs. (64) and (65) we see that the modification of the magnetic velocity fields both inside and outside the conductor through first order in the velocity is independent of the resistivity of the conductor. Provided the conductor is a good conductor as in (24) so that the new surface charge is a small perturbation on the electrostatic surface charge, the resistivity is immaterial. Indeed the electric field inside the conductor is smaller if the conductivity is large, but the electric currents inside the conductor and hence the magnetic fields in all space are independent of the resistivity. In the limit that a conductor of finite conductivity becomes a perfect conductor, the magnetic velocity fields still penetrate into the conductor.

O. Energy loss for a charged particle passing a conductor

It is interesting to look at our results from the view of energy conservation. We have assumed that the point charge moves with constant velocity, and clearly external forces are needed to hold the particle on the straight line path. The electric field back at the point charge is from Eq. (31),

$$\vec{E}_{e,\beta}(vt, 0, d, t) = -\hat{k}\,\frac{e}{(2d)^2} - \hat{i}\,\frac{c\eta e\beta}{2\pi(2d)^3}, \tag{66}$$

and the force on the charge is just

$$\vec{F}_{em} = e\vec{E}.$$

The force in the $-\hat{k}$ direction represents the electrostatic attraction of the charge towards its image charge in the conductor. However, the second term in the $-\hat{i}$ direction is a retarding force.

In order to maintain the charge moving at constant velocity against this retarding force, an external force is required which is equal to the negative of the retarding force,

$$\vec{F}_{\rm ext} = \frac{\hat{i}\,c\eta e^2\beta}{2\pi(2d)^3}.$$

Moreover, since the particle moves at velocity $v = c\beta$, this external force must do work at the rate

$$P_{\rm ext} = \vec{F}_{\rm ext}\cdot\vec{v} = \frac{c^2\eta e^2\beta^2}{2\pi(2d)^3}. \tag{67}$$

We expect this work to be expended in Joule heating inside the conducting wall.

Now the presence of currents the Ohmic conductor leads to energy loss due to Joule heating at a rate

$$\begin{aligned} P_{\rm Joule} &= \int d^3r\,\vec{J}\cdot\vec{E} = \int_{z<0} d^3r\,\frac{1}{\eta}\,\vec{E}^2_{e,\beta}(\vec{r}, t) \\ &= \int_{z<0} d^3r\,\frac{1}{\eta}\left(\frac{c\eta\beta}{2\pi}\,\frac{\partial}{\partial x}\vec{E}_{-e}(\vec{r};\vec{\xi})\right)^2 \\ &= \frac{c^2\eta e^2\beta^2}{4\pi^2 d^3}\int_{z<0} d^3r' \\ &\quad\times\left[\frac{\partial}{\partial x'}\left(\frac{\hat{i}x' + \hat{j}y' + \hat{k}(z'-1)}{\{x'^2 + y'^2 + (z'-1)^2\}^{3/2}}\right)\right]^2. \end{aligned} \tag{68}$$

Here we have used Eq. (32) for the electric field inside the conductor through first order in β, and then have changed variable of integration so as to separate off a dimensionless integral. The integral can be evaluated directly by successive integrations in x, y, and z. It has the value $\frac{1}{4}\pi$. Thus indeed

$$P_{\rm Joule} = \frac{c^2\eta^2 e^2\beta^2}{16\pi d^3} = P_{\rm ext}, \tag{69}$$

and through lowest order in β there is agreement between the power expended by the external force and the power lost in Joule heating.

III. VELOCITY FIELDS IN SOME LIMITING CONFIGURATIONS

A. Limit of a steady current

The results in Sec. II for the penetration of the electric and magnetic velocity fields of a point charge are the basic part of our analysis. Here in Sec. III we wish to carry out various integrals of the point-charge results so as to obtain the penetration into a conductor for the velocity fields in various limiting configurations.

A crude but convenient test of the point-charge results consists in superimposing the fields for a sequence of point charges all moving with constant velocity so as to form a steady current. In this limit of a steady current passing a conductor of finite conductivity with $\mu = 1$, $\epsilon = 1$, the penetration of the velocity fields is quite familiar. The magnetic field is not screened but penetrates the conductor as though the conductor were not present. On the other hand, the electric field is completely excluded from the conductor corresponding to an electrostatic field situation.

The arrangement corresponding to a steady current along the line $y = 0$, $z = d$ is obtained by displacing the point-charge situation repeatedly in x. Thus in Eqs. (23), (31), (32), (64), and (65), we replace the source point vt by x', replace the charge e by $\lambda\,dx'$, and integrate from $-\infty$ to ∞. All of the integrands vanish at least as fast as $(x')^{-2}$ so that there are no difficulties involved in evaluating the integrals at $\pm\infty$. Here and throughout Sec. III, the integrals required can be found as indefinite integrals in standard elementary lists.

The surface charge density from (23), through first order in β, is

$$\sigma_{\lambda,\beta}(x,y,0,t)=\int_{-\infty}^{\infty}(\lambda\,dx')\left[\frac{-d}{2\pi[(x-x')^2+y^2+d^2]^{3/2}}-\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{d}{2\pi[(x-x')^2+y^2+d^2]^{3/2}}\right)\right]$$
$$=\frac{-2\lambda d}{2\pi(y^2+d^2)},\tag{70}$$

with the entire contribution coming from the zero-order charge density and none from the first-order term. The surface charge corresponds to that from a static line charge λ along $y=0$, $z=d$.

The electric field behaves similarly. From Eq. (31),

$$\vec{E}_{\lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\lambda\,dx')\left[\frac{\hat{i}(x-x')+\hat{j}y+\hat{k}(z-d)}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}-\frac{\hat{i}(x-x')+\hat{j}y+\hat{k}(z+d)}{[(x-x')^2+y^2+(z+d)^2]^{3/2}}\right.$$
$$\left.-\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-x')+\hat{j}y+\hat{k}(z-d)}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}\right)\right]$$
$$=2\lambda\frac{\hat{j}y+\hat{k}(z-d)}{y^2+(z-d)^2}-2\lambda\frac{\hat{j}y+\hat{k}(z+d)}{y^2+(z+d)^2}\quad\text{for } z>0,\tag{71}$$

and from Eq. (32)

$$\vec{E}_{\lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\lambda\,dx')\left(-\frac{c\eta\beta}{2\pi}\right)\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-x')+\hat{j}y+\hat{k}(z-d)}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}\right)=0\text{ for } z<0.\tag{72}$$

Thus inside the conductor, the electric field vanishes, while outside it corresponds to the line charge λ at $z=d$ and an image charge $-\lambda$ at $z=-d$.

The magnetic field contribution follows from Eqs. (64) and (65). Here outside the conductor, the integral

$$\vec{B}_{\lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\lambda\,dx')\beta\left[\left(\frac{-\hat{j}(z-d)+\hat{k}y}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}+\frac{\hat{j}(z+d)}{[(x-x')^2+y^2+(z+d)^2]^{3/2}}\right)\right.$$
$$-\frac{\hat{i}(x-x')y[3(x-x')^2+3y^2+2(z+d)^2](z+d)}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z+d)^2]^{3/2}}+\frac{2\hat{i}(x-x')y}{[(x-x')^2+y^2]^2}$$
$$-\frac{\hat{j}[-(x-x')^2+y^2](z+d)}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z+d)^2]^{1/2}}-\frac{\hat{j}y^2}{[(x-x')^2+y^2][(x-x')^2+y^2+(z+d)^2]^{3/2}}$$
$$\left.-\frac{\hat{j}[(x-x')^2-y^2]}{[(x-x')^2+y^2]^2}\right].\tag{73}$$

The variable of integration can be changed from x' to $x'-x$, and the component along the x axis is seen to vanish by symmetry. Then still denoting the variable of integration by x',

$$\vec{B}_{\lambda,\beta}(x,y,z,t)=\lambda\beta\left[\left(\frac{[-\hat{j}(z-d)+\hat{k}y]\,x'}{[y^2+(z-d)^2][x'^2+y^2+(z-d)^2]^{1/2}}+\frac{\hat{j}(z+d)x'}{[y^2+(z+d)^2][x'^2+y^2+(z+d)^2]^{1/2}}\right)\right.$$
$$\left.-\frac{\hat{j}(z+d)x'}{(x'^2+y^2)[x'^2+y^2+(z+d)^2]^{1/2}}-\frac{\hat{j}(z+d)x'}{[y^2+(z+d)^2][x'^2+y^2+(z+d)^2]^{1/2}}+\frac{\hat{j}x'}{x'^2+y^2}\right]_{x'=-\infty}^{x'=\infty}$$
$$=2\lambda\beta\,\frac{-\hat{j}(z-d)+\hat{k}y}{y^2+(z-d)^2}\text{ for } z>0.\tag{74}$$

Inside the conductor for $z<0$,

$$\vec{B}_{\lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\lambda\,dx')\beta\left(\frac{-\hat{j}2(z-d)+\hat{k}y}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}+\frac{\hat{i}(x-x')y[3(x-x')^2+3y^2+2(z-d)^2](z-d)}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z-d)^2]^{3/2}}\right.$$
$$+\frac{2\hat{i}(x-x')y}{[(x-x')^2+y^2]^2}+\frac{\hat{j}[-(x-x')^2+y^2](z-d)}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z-d)^2]^{1/2}}$$
$$\left.+\frac{\hat{j}y^2(z-d)}{[(x-x')^2+y^2][(x-x')^2+y^2+(z-d)^2]^{3/2}}-\frac{\hat{j}[(x-x')^2+y^2]}{[(x-x')^2+y^2]^2}\right).\tag{75}$$

Again it is convenient to take $x' - x$ as the variable of integration, and again the x components vanish. Now

$$\vec{B}_{\lambda\beta}(x,y,z,t) = \lambda\beta\Bigg[\frac{[-\hat{j}\,2(z-d)+\hat{k}y]x'}{[y^2+(z-d)^2][x'^2+y^2+(z-d)^2]^{1/2}} + \frac{\hat{j}(z-d)x'}{(x'^2+y^2)[x'^2+y^2+(z-d)^2]^{1/2}} + \frac{\hat{j}(z-d)x'}{[y^2+(z-d)^2][x'^2+y^2+(z-d)^2]^{1/2}} + \frac{\hat{j}x'}{x'^2+y^2}\Bigg]_{x'=-\infty}^{x'=\infty}$$

$$= 2\lambda\beta\,\frac{-\hat{j}(z-d)+\hat{k}y}{y^2+(z-d)^2} \quad \text{for } z<0. \tag{76}$$

The results inside and outside the conductor yield the same expression for the magnetic field. The field pattern corresponds to that from a line charge λ moving along its length with constant velocity β giving a steady current $I=\lambda\beta$. The conducting wall has no effect on the magnetic field. The steady-current limit derived from our point-charge results of Sec. II is in precise agreement with the expected behavior.

B. Line charge moving perpendicular to its axis

In 1966 Kasper[4] attempted to treat the penetration of the electromagnetic velocity fields into a conductor of finite conductivity. This is the only such attempt which has come to the author's attention. Kasper considered a line charge rather than a point charge so as to reduce the number of spatial dimensions in the problem. The line charge moved with constant velocity perpendicular to its length outside and parallel to a conducting surface. Kasper assumed that the penetration of the fields would show a skin-depth behavior, and introduced approximations on the size of derivatives of the fields which would be appropriate for such a situation. However, our results for a point charge in Sec. II show that there is no skin-depth behavior for the velocity fields, and hence that Kasper's approximations are in error. Here we provide the solution to Kasper's problem through first order in the velocity of the line charge. The results are completely different from those found by Kasper.

In order to treat a line charge of charge Λ per unit length oriented parallel to the y axis and moving with constant velocity $\vec{v}=c\beta\hat{i}$ in the plane $z=d$, we imagine displacing the point-charge situation of Sec. II along the y axis. Thus in Eqs. (23), (31), (32), (64), and (65), we replace the field coordinate y by the displacement $y-y'$ of the field point from the source point, replace the charge e by $\Lambda\,dy'$, and integrate from $-\infty$ to ∞.

The charge density follows from Eq. (23) through first order,

$$\sigma_{\Lambda,\beta}(x,y,z,t) = \int_{-\infty}^{\infty}(\Lambda\,dy')\left[\frac{-d}{2\pi[(x-vt)^2+(y-y')^2+d^2]^{3/2}} + \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{-d}{2\pi[(x-vt)^2+(y-y')^2+d^2]^{3/2}}\right)\right]$$

$$= \frac{-\Lambda d}{\pi[(x-vt)^2+d^2]} + \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{-\Lambda d}{\pi[(x-vt)^2+d^2]}\right). \tag{77}$$

The electric field through first order in β is obtained from Eqs. (31) and (32),

$$\vec{E}_{\Lambda,\beta}(x,y,z,t) = \int_{-\infty}^{\infty}(\Lambda\,dy')\Bigg[\frac{\hat{i}(x-vt)+\hat{j}(y-y')+\hat{k}(z-d)}{[(x-vt)^2+(y-y')^2+(z-d)^2]^{3/2}} - \frac{\hat{i}(x-vt)+\hat{j}(y-y')+\hat{k}(z+d)}{[(x-vt)^2+(y-y')^2+(z+d)^2]^{3/2}} - \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-vt)+\hat{j}(y-y')+\hat{k}(z+d)}{[(x-vt)^2+(y-y')^2+(z+d)^2]^{3/2}}\right)\Bigg]$$

$$= 2\Lambda\,\frac{\hat{i}(x-vt)+\hat{k}(z-d)}{(x-vt)^2+(z-d)^2} - 2\Lambda\,\frac{\hat{i}(x-vt)+\hat{k}(z+d)}{(x-vt)^2+(z+d)^2} - \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{2\Lambda[\hat{i}(x-vt)+\hat{k}(z+d)]}{(x-vt)^2+(z+d)^2}\right) \quad \text{for } z>0 \tag{78}$$

and

$$\vec{E}_{\Lambda,\beta}(x,y,z,t) = \int_{-\infty}^{\infty}(\Lambda\,dy')\left(-\frac{c\eta\beta}{2\pi}\right)\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-vt)+\hat{j}(y-y')+\hat{k}(z-d)}{[(x-vt)^2+(y-y')^2+(z-d)^2]^{3/2}}\right)$$

$$= -\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{2\Lambda[\hat{i}(x-vt)+\hat{k}(z-d)]}{(x-vt)^2+(z-d)^2}\right) \quad \text{for } z<0. \tag{79}$$

The current inside the conductor again follows from Ohm's law. The magnetic field through first order in

β follows from Eqs. (64) and (65). However, since the integration is in the y direction, it is worthwhile noting the y derivative arising in Eq. (63). Thus outside the conductor

$$\vec{B}_{\Lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\Lambda\,dy')\beta\left[\frac{-\hat{j}(z-d)+\hat{k}(y-y')}{[(x-vt)^2+(y-y')^2+(z-d)^2]^{3/2}}+\frac{\hat{j}(z+d)-\hat{k}(y-y')}{[(x-vt)^2+(y-y')^2+(z+d)^2]^{3/2}}\right.$$
$$\left.-\frac{\partial}{\partial y}\left(\frac{-[\hat{i}(x-vt)+\hat{j}(y-y')](z+d)+\hat{k}[(x-vt)^2+(y-y')^2]}{[(x-vt)^2+(y-y')^2][(x-vt)^2+(y-y')^2+(z+d)^2]^{1/2}}+\frac{\hat{i}(x-vt)+\hat{j}(y-y')}{(x-vt)^2+(y-y')^2}\right)\right]$$
$$=2\Lambda\beta\hat{j}\left(\frac{-(z-d)}{(x-vt)^2+(z-d)^2}+\frac{z+d}{(x-vt)^2+(z+d)^2}\right)\quad\text{for } z>0. \tag{80}$$

Inside the conductor,

$$\vec{B}_{\Lambda,\beta}(x,y,z,t)=\int_{-\infty}^{\infty}(\Lambda\,dy')\beta\left[\frac{-\hat{j}2(z-d)+\hat{k}2(y-y')}{[(x-vt)^2+(y-y')^2+(z-d)^2]^{3/2}}\right.$$
$$\left.-\frac{\partial}{\partial y}\left(\frac{-[\hat{i}(x-vt)+\hat{j}(y-y')](z-d)-\hat{k}[(x-vt)^2+(y-y')^2]}{[(x-vt)^2+(y-y')^2][(x-vt)^2+(y-y')^2+(z-d)^2]^{1/2}}+\frac{\hat{i}(x-vt)+\hat{j}(y-y')}{(x-vt)^2+(y-y')^2}\right)\right]$$
$$=2\Lambda\beta\hat{j}\left(\frac{-2(z-d)}{(x-vt)^2+(z-d)^2}\right)\quad\text{for } z<0. \tag{81}$$

The expressions for the magnetic fields correspond to simple physical situations. Outside the conductor, the magnetic field is that due to the moving line charge Λ plus an image line charge $-\Lambda$ inside the conductor also moving with velocity $c\beta$. Inside the conductor, the magnetic field is that due to a line charge of strength 2Λ moving at the position and with the velocity of the actual line charge Λ.

The results for the fields, charges, and currents through first order in β found in Eqs. (77)–(81) can be checked by direct substitution into Maxwell's equations with attention to the appropriate boundary conditions. The verification calculations for the line charge are somewhat easier than for the point-charge situation of Sec. II.

C. Limit of a current sheet

A further simple limiting case involves a sheet of steady current running parallel to a conducting wall. Such a current sheet can be formed by combining the point-charge situation in various ways. Here we will consider two alternatives—forming the current sheet first by adding together the steady line currents of Sec. III A, and second by adding the moving line charges of Sec. III B. The magnetic fields for the two limiting situations are different. This example reminds one of the delicate nature of physical situations involving charges and currents placed at spatial infinity.

The limit of a steady current sheet in the plane $z=d$ may be obtained from the familiar constant-line-current results of Eqs. (70), (71), (72), (74), and (76). We replace the field point y by the displacement $y-y'$ of the field point from the source point, replace λ by $\Sigma dy'$, and integrate from $-\infty$ to ∞. The calculations are elementary. The surface charge density is

$$\sigma_{\Sigma,\beta}(x,y,0,t)=\int_{-\infty}^{\infty}(\Sigma\,dy')\frac{-2d}{2\pi[(y-y')^2+d^2]}=-\Sigma; \tag{82}$$

the electric field for $z>0$ is

$$\vec{E}_{\Sigma,\beta}(x,y,z,t)=2\int_{-\infty}^{\infty}(\Sigma\,dy')\left(\frac{\hat{j}(y-y')+\hat{k}(z-d)}{(y-y')^2+(z-d)^2}-\frac{\hat{j}(y-y')+\hat{k}(z+d)}{(y-y')^2+(z+d)^2}\right)$$
$$=\begin{cases}-4\pi\Sigma\hat{k}, & z<d\\ 0\;, & z>d,\end{cases} \tag{83}$$

while for $z<0$

$$\vec{E}_{\Sigma,\beta}(x,y,z,t)=0, \tag{84}$$

and for all z the magnetic field is

$$\vec{B}_{\Sigma,\beta}(x,y,z,t)=2\beta\int_{-\infty}^{\infty}(\Sigma\,dy')\frac{-\hat{j}(z-d)+\hat{k}(y-y')}{(y-y')^2+(z-d)^2}=\begin{cases}-2\pi\Sigma\beta\hat{j}, & z>d\\ 2\pi\Sigma\beta\hat{j}, & z<d.\end{cases} \tag{85}$$

These results are as expected. The sheet of moving charge causes a magnetic field spreading through all space. The image charge on the surface of the conductor is not moving; it leads to a cancellation of the electric field inside the conductor, but does not contribute to any magnetic field.

We may also arrive at a constant current sheet by adding line charges moving perpendicular to their axes. Here we turn to Eqs. (77)–(81), replace the line charge position vt by x', replace the charge per unit length Λ by $\Sigma dx'$, and integrate from $-\infty$ to ∞. The surface charge becomes

$$\sigma_{\Sigma,\beta}(x, y, 0, t) = \int_{-\infty}^{\infty} (\Sigma dx')\left[\frac{-d}{\pi[(x-x')^2+d^2]} + \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{-d}{\pi[(x-x')^2+d^2]}\right)\right] = -\Sigma. \tag{86}$$

The electric field for $z>0$ is

$$\vec{E}_{\Sigma,\beta}(x, y, z, t) = 2\int_{-\infty}^{\infty} (\Sigma dx')\left[\frac{\hat{i}(x-x')+\hat{k}(z-d)}{(x-x')^2+(z-d)^2} - \frac{\hat{i}(x-x')+\hat{k}(z+d)}{(x-x')^2+(z+d)^2} - \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-x')+\hat{k}(z+d)}{(x-x')+(z+d)^2}\right)\right]$$

$$= \begin{cases} -4\pi\Sigma\hat{k}, & z<d \\ 0, & z>d, \end{cases} \tag{87}$$

and for $z<0$

$$\vec{E}_{\Sigma,\beta}(x, y, z, t) = 2\int_{-\infty}^{\infty} (\Sigma dx')\left(\frac{-c\eta\beta}{2\pi}\right)\frac{\partial}{\partial x}\left(\frac{\hat{i}(x-x')+\hat{k}(z-d)}{(x-x')^2+(z-d)^2}\right) = 0. \tag{88}$$

The magnetic field for $z>0$ is

$$\vec{B}_{\Sigma,\beta}(x, y, z, t) = 2\beta\int_{-\infty}^{\infty} (\Sigma dx')\left(\frac{-\hat{j}(z-d)}{(x-x')^2+(z-d)^2} + \frac{\hat{j}(z+d)}{(x-x')^2+(z+d)^2}\right)$$

$$= \begin{cases} 0, & z>d \\ 4\pi\Sigma\beta\hat{j}, & z<d, \end{cases} \tag{89}$$

while inside the conductor $z<0$

$$\vec{B}_{\Sigma,\beta}(x, y, z, t) = 2\beta\int_{-\infty}^{\infty} (\Sigma dx')\,\frac{-2\hat{j}(z-d)}{(x-x')^2+(z-d)^2} = 4\pi\Sigma\beta\hat{j}. \tag{90}$$

The results here for the surface charge and electric field agree between the two procedures; however, the magnetic fields are different. Both results are valid solutions of Maxwell's equations satisfying the boundary conditions at $z=0$ and $z=d$. The two solutions differ in the boundary conditions on the currents at spatial infinity.

IV. CONCLUSION

A. Further developments

Although this paper has analyzed part of the penetration problem for the electromagnetic velocity fields, there is clearly much unfinished work. Even for the simple situation treated here of a point charge moving parallel to a conducting plane with $\mu=1$ and $\epsilon=1$, the solutions are obtained only through first order in the particle velocity. The full calculation to all orders remains to be done. Moreover, the analysis assumed that the conductor filled a half-plane. The form of the electric and magnetic fields on the far side of a conducting wall of finite thickness is untreated, and clearly this latter situation is the realistic one allowing experimental measurement of the penetration. Finally the calculations for other geometries of conductors and charged-particle motions remain untouched.

B. Closing summary

Although the skin-depth calculations for the penetration of plane-wave radiation into a conductor of finite conductivity are reproduced universally in the textbook literature and are the results of principal interests in electromagnetic screening, there is a remaining problem in the penetration of the electric and magnetic velocity fields. Indeed in recent years the question of the penetration of the velocity fields has been of interest in connection with experiments on the Aharonov-Bohm effect. Kasper attacked the problem in 1966 but introduced approximations which invalidate his calculations.

In this paper, we have treated a problem whose simplicity of statement makes it seem appropriate for a student home-work exercise. Nevertheless, the solution seems unexpected, contradicting Kasper's earlier work and also the suggestions of some researchers on the Aharonov-Bohm effect. We consider a point charge moving with uniform velocity outside and parallel to a conducting plane. The conductor occupies a half-space and is characterized by resistivity η, permeability $\mu = 1$, and dielectric constant $\epsilon = 1$. For this simple situation, what are the electric and magnetic fields in all space?

Our calculations are valid only in the nonrelativistic low-velocity domain for a good conductor. They indicate that there is no skin-depth phenomenon for the velocity fields, but rather these fields fall off as r^{-2} or r^{-3} inside the conductor. Moreover the magnetic fields in space and inside the conductor are independent of the conductivity of the wall. These results stand in striking contrast with the more familiar penetration characteristics of electromagnetic radiation fields. For the radiation fields, the conductivity determines the depth of penetration and the size of the electromagnetic fields inside the conductor. The penetration-depth problem for the electromagnetic velocity fields is merely begun in this paper; the first results indicate some of the surprising variety to be found in classical electromagnetism.

ACKNOWLEDGMENT

The author wishes to thank Judith Penney Soukup for checking the calculations and for evaluating the integral in Sec. II O.

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