

Free-path model for the resonances in the Scott effect

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A free-path model is set forth for the phenomenon discovered by Scott whereby a gas exerts a torque on a heated cylinder when the gas is subjected to a magnetic field parallel to the cylinder axis. The model is particularly applicable to the phenomenon of torque resonance found by Smith and Scott when they modulated the magnetic field. The free-path model is not intended to compete with Chapman-Enskog-type theory in providing a precise description of the Scott effect, but is intended rather as an aid to physical intuition. Within the limits of its purpose, the model appears to be quite successful. Namely, it provides an intuitive picture of why there is a torque and it enables one to understand how the resonances come about. Moreover, it gives the same formula for torque resonance as do the systematic treatments. The model has an interesting sidelight: In its simple form the model yields Lorentzian-type resonances with molecular speed as a parameter. When the speed is averaged over a typical distribution, the Lorentzian resonances are converted into functions of similar shape involving the incomplete Γ function. These latter functions seem to be natural generalizations of the Lorentzian.

I. INTRODUCTION

The Scott effect (also referred to as the thermomagnetic gas torque effect) was originally observed¹ as a torque exerted by a dilute gas on a heated torsion pendulum when the gas was subjected to a static magnetic field directed along the axis of the pendulum. The arrangement of the apparatus is illustrated in Fig. 1. For the majority of the gases studied^{2,3} it has been found that when the mean free path in the gas is much less than the distance between the heated cylinder and the vacuum chamber, the magnitude of the static torque is proportional to a function of the gas pressure p , and the magnetic field strength H similar to the function

$$\frac{1}{p} \left(\frac{H/p}{C + H^2/p^2} \right),$$

where C is constant for any given gas. A typical example of this behavior is illustrated in Fig. 2.

Further interesting experimental results were obtained by Smith and Scott⁴ as a result of Smith's suggestion that it would be worthwhile to look for resonances in the Scott effect. The experimental arrangement was changed by the addition of an alternating magnetic field (referred to as the modulating field) also directed along the axis of the pendulum. The results of these "dynamic" experiments are conveniently expressed by plots of $\Delta N/N$ vs ν , where ΔN denotes the change in torque caused by turning on the modulating field, N denotes the static torque (i.e., the value of the torque with the modulation turned off), and ν is the frequency of modulation. Resonances were indeed found, as illustrated by the experimental results shown in Fig. 3. There α is the ratio of

the static field strength to the field strength which would give maximum static torque at the given pressure, and M (denoted by Smith and Scott as β) is the ratio of the amplitude of the modulating field to the static field strength.

The Scott effect is a recently discovered example of the fact that a magnetic field can affect the transport behavior of a gas of electrically neutral molecules. This fact itself was discovered forty years ago by Senftleben⁵ in experiments on the influence of a magnetic field on heat conduction in paramagnetic gases (such as oxygen and nitric oxide). As a result of Senftleben's discoveries there has been some effort over a number of years at developing a theory for describing this sort of phenomenon. The theory has come in two principal stages, the first having been based on a free-path-type model⁶ and the second having been based on use of the systematic method of Chapman and Enskog to solve (a generalized form of) the Boltzmann equation for the situation at hand.⁷ The application of the systematic method to the Senftleben effect was approaching maturity when the Scott effect was discovered, and this method began to be applied to the Scott effect shortly thereafter.⁸

While the Chapman-Enskog-type approach is the proper one for obtaining a quantitative description of the Scott effect, it has the disadvantages of being quite complicated and of offering little help toward intuitive understanding of why the Scott effect occurs. The present paper uses a free-path approach to develop a physical picture of one kind of molecular motion contributing to the Scott effect.

The physical picture has two aspects. The first aspect is that molecules traveling in any particular

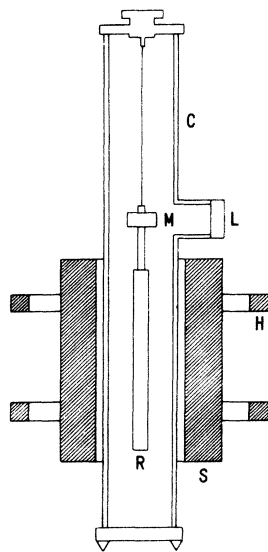


FIG. 1. Schematic drawing of apparatus used to measure the thermomagnetic gas torque. C: glass chamber, inside diam. = 7.6 cm, length = 61 cm; M: mirror; L: lens; H: Helmholtz coil used for fields up to 120 Oe, diam. 30 cm; S: solenoid used for fields above 120 Oe, inside diam. = 9.5 cm, outside diam. = 19 cm, length = 30 cm; R: heated cylindrical torque detector, diam. = 1.9 cm, length = 20 cm. (Figure and caption are from Ref. 2.)

direction and having their rotation axes polarized in a suitable manner will be deflected systematically to one side when they undergo collision. The second aspect is that molecules which are deflected laterally will contribute to the shear stress in the gas (we will be interested specifically in the τ, ϕ component of the stress tensor) and that if the lateral deflection depends in a suitable way on direction of molecular travel, the sum of the various contributions will be nonzero.

The first aspect of the picture is developed via a model for the deflection of a molecule at the end of a representative free flight. The model does not lend itself to a determination of the numerical magnitude of the deflection, but it does provide an expression giving the dependence of the deflection on the magnetic field. Thus, the expression to be obtained for the torque itself will give only the form of its dependence on the magnetic field and not its magnitude.

The second aspect of the picture is developed just sufficiently to indicate the spatial symmetries which are required in order for the net shear stress not to vanish.

Two principal results are obtained from the model. One is the form of the static torque as a function of H/p . This form is fairly close to the one which is obtained experimentally. The other

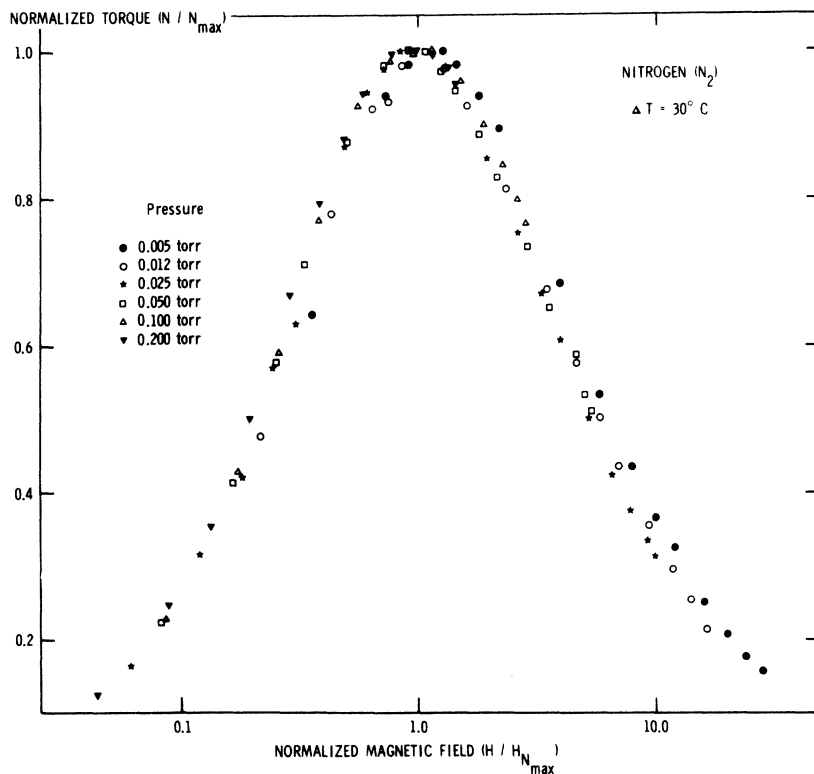


FIG. 2. Typical behavior of Scott torque when field is static. Data points are for N₂ and may be represented by curves which are functions of H are proportional to $H/H_{N_{\max}}/[1 + (H/H_{N_{\max}})^2]$ where $H_{N_{\max}}$, the field strength for which (at a given pressure) the torque is largest, is found experimentally to be proportional to the pressure. (Figure is from Ref. 2.)

result is a formula for $\Delta N/N$ due to modulation. This formula has the virtue that it specifies the magnitude of $\Delta N/N$ as well as the functional form. The results are the same as those obtained through systematic treatments⁸ and agree with experiment as to the presence and location of resonances in $\Delta N/N$ (as a function of ν) but differ somewhat as to the specific magnitude of $\Delta N/N$.

One interesting conclusion which can be drawn from this work is that it is easier to understand the occurrences of resonances in the Scott effect than it is to understand the details of the underlying Scott effect itself. More generally, one may expect to find a sequence of resonances of this sort in any effect which depends on a circular motion, assuming that the circular frequency can be modulated and that the relative change in the effect can be observed as a function of the frequency of modulation. The model presented here offers a simplified way of seeing how these resonances arise in the case of the Scott effect. If the form of such a resonance is accurately understood, then it will afford a means for precise measurement of the underlying circular frequency.

II. MODEL FOR DEFLECTION AT END OF FREE FLIGHT

We consider a gas of diatomic molecules. (See Ref. 9.) One representative molecule is considered to rotate as a classical dumbbell and attention is paid to the orientation of its rotation axis.

When the representative molecule collides with one of the other molecules, the latter is regarded as being spherical. When the representative molecule is traveling through the gas, the rate at which it undergoes collisions may be expressed as

$$v(A + B \cos 2\xi),$$

where v is the speed of the molecule, ξ is the angle between its rotation axis and its velocity vector, and A and B are functions of molecular speed which satisfy the relation

$$B \ll A.$$

(A is roughly the reciprocal of the mean free path.) Attention will be limited to representative molecules whose velocity vectors and rotation axes lie in the r, ϕ plane, as shown in Fig. 4(a). This limitation is introduced for the sake of simplicity. It is appropriate for the model to be presented here but would not be appropriate for study of one mechanism proposed by Waldmann.¹⁰

The magnetic field is given as a function of time by

$$H(t) = H_{dc} (1 + M \cos \omega_m t),$$

where $\omega_m = 2\pi\nu$. Thus, as the molecule precesses, the azimuthal coordinate of its rotation axis will vary as

$$\xi(t) = \omega_p t + (\omega_p M / \omega_m) \sin(\omega_m t) + \varphi,$$

where ω_p is the rate of precession due to H_{dc} .

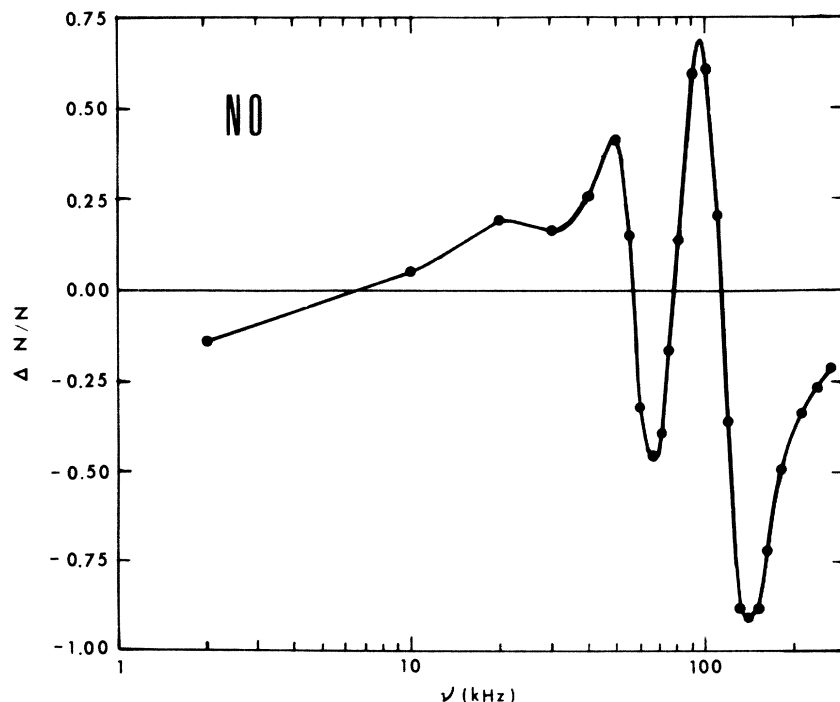


FIG. 3. Typical example of resonant behavior of $\Delta N/N$. Dots represent data of Smith and Scott (Ref. 4) for the gas NO with $\alpha = 7.86$ and $M = 1.52$. [Figure is from G. Smith and G. Scott, Phys. Rev. 188, 433 (1969).]

alone and φ is an arbitrary constant.

The basic idea of the model will be stated in relation to Fig. 4 as follows. The representative molecule (1) emerges from a collision at r'' such that its velocity is in some specific direction in the r, φ plane and such that its rotation axis lies in the same plane and makes an angle ζ'' with its velocity; (2) precesses in the magnetic field $H(t)$ while it travels with speed v toward r' ; (3) has its rotation axis oriented in a definite manner when it undergoes a second collision at r' , and tends to be deflected to one side in a manner which depends systematically on this orientation.

To translate this model into a formula for the deflection, we need to include factors proportional to the probabilities of the following: (1) that the molecule will have an initial orientation of ζ'' ; (2) that it will go from r'' to r' without suffering a collision; and (3) that it will undergo a collision at r' . In addition, we need to (4) include a factor proportional to the average value of the lateral deflection of the molecule when it undergoes the collision at r' ; (5) average over initial orientations ζ'' ; (6) integrate from zero to infinity with respect to the separation of the points r'' and r' ; and (7) average over one period of modulation of the magnetic field (because the period of modulation is always much shorter than the period of the torsion pendulum). We will attend to these seven ingredients in sequence.

(1) We need the distribution for the probability that a molecule emerges from the collision at r''

having the angle between its rotation axis and its velocity equal to ζ'' . What comes out of the model will depend on what is put in at this point. We will show that the deflection predicted by the model is zero if it is assumed that the molecules at which we are looking have their rotation axes oriented randomly in the r, φ plane. Subsequent to that we will indicate what sort of rotation axis polarization needs to be assumed in order for the mean deflection given by the model to be nonzero.

Proceeding on the assumption that the rotation axes are uniformly distributed, we note that the rate at which collisions cause fresh molecules to acquire the particular velocity v and rotation axis orientation ζ'' is nearly equal to the rate at which collisions cause molecules which have had those parameters to acquire different ones. Thus, we take the probability distribution for ζ'' to be proportional to

$$A + B \cos 2\zeta''.$$

It will be convenient to use the expression obtained above for ζ as a function of time. The symbol t will denote the time at which the molecule reaches the point r' . The distance from r'' to r' will be denoted by x . Thus

$$\begin{aligned} \zeta'' &= \omega_p(t - x/v) + (\omega_p M / \omega_m) \\ &\quad \times \sin[\omega_m(t - x/v)] + \varphi'' \end{aligned}$$

and the factor representing the distribution of probability for the initial orientation becomes

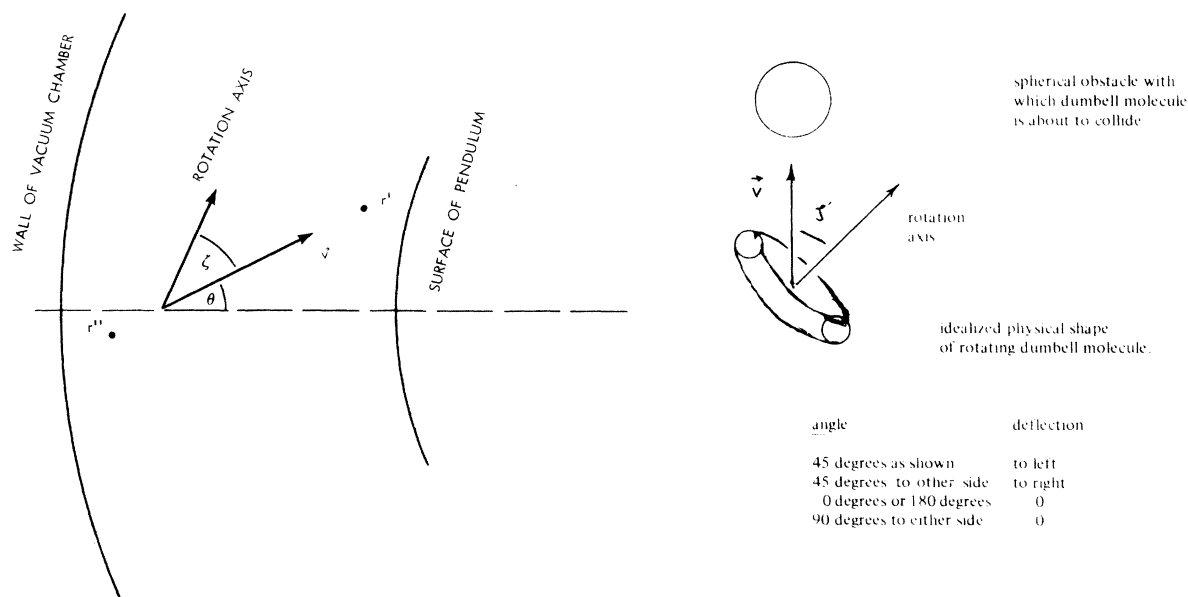


FIG. 4. (a) Molecular velocity and rotation axis vectors in r, φ plane. r'' and r' represent the end points of a single free flight. (b) Illustration of why the form $\sin \zeta' \cos \zeta''$ is used to represent the dependence of the lateral deflection of the dumbbell molecule on its orientation.

$$A + B \cos\{2\omega_p(t - x/v) + (2\omega_p M/\omega_m) \times \sin[\omega_m(t - x/v)] + 2\varphi''\}.$$

(2) The probability $q(x)$ that a molecule will travel a distance x without collision satisfies

$$dq = -[A + B \cos 2\zeta(x)]q dx,$$

where $\zeta(x)$ is the angle between the rotation axis and velocity at x . Thus

$$q(x) = e^{-Ax} \exp\left[-B \int_0^x dy \cos 2\zeta(y)\right].$$

To obtain a useful expression for q we need to replace the second exponential by the first two terms of its power series. To see that this is legitimate, we recall that $B \ll A$ and note accordingly that the exponent of the second exponential will be $\ll 1$ as long as x is small enough for q as a whole to be significantly greater than zero. Thus we have

$$q(x) = e^{-Ax} \left[1 - B \int_0^x dy \cos 2\zeta(y)\right].$$

An explicit expression for $\zeta(y)$ is obtained as follows. The point r'' at which the molecule starts is a distance x away from r' , and when the molecule has gotten to the point y in the above integral it is a distance $x - y$ from r' . Thus

$$\zeta(y) = \omega_p \left(t - \frac{x - y}{v}\right) + \frac{\omega_p M}{\omega_m} \sin\left[\omega_m \left(t - \frac{x - y}{v}\right)\right] + \varphi'',$$

and the probability that the representative molecule travels the distance x from r'' to r' without collision is

$$q(x) = e^{-Ax} \left(1 - B \int_0^x dy \cos\left\{2\omega_p \left(t - \frac{x - y}{v}\right) + \frac{2\omega_p M}{\omega_m} \sin\left[\omega_m \left(t - \frac{x - y}{v}\right)\right] + 2\varphi''\right\}\right).$$

(3) The probability that the representative molecule will undergo collision at r' is proportional to the collision rate, and thus to

$$A + B \cos 2\zeta'$$

and, inserting the expression for the rotation axis coordinate at r' , this is

$$A + B \cos[2\omega_p t + (2\omega_p M/\omega_m) \sin(\omega_m t) + 2\varphi''].$$

(4) We picture the rotating dumbbell as a round pill with flattened sides orthogonal to the rotation axis. See Fig. 4(b). For the average lateral deflection (due to collision with a spherical object) we use the simplest expression which behaves as

required by this picture, namely,

$$2 \sin \zeta' \cos \zeta' = \sin 2\zeta'.$$

The foregoing is one of the main parts of the model.

As this factor will also depend on the nonsphericity of the molecular cross section, we multiply the preceding expression by B/A and use

$$(B/A) \sin[2\omega_p t + (2\omega_p M/\omega_m) \sin(\omega_m t) + 2\varphi''].$$

(5) The integration from 0 to 2π with respect to the additive constant φ'' can be carried out in a straightforward manner. The integrand consists of the eight terms which are obtained when the four basic factors just enumerated are multiplied together. These terms can be grouped according to the number of factors of B which they contain. The first group consists of a single term with only one factor of B . This term vanishes when integrated with respect to φ'' . Of the three terms which contain two factors of B , one vanishes on integration with respect to φ'' . The integrals of the other two are

$$\pi B^2 e^{-Ax} \sin\left(2\omega_p \frac{x}{v} + \frac{2\omega_p M}{\omega_m} [\sin(\omega_m t) - \sin\omega_m(t - x/v)]\right)$$

and

$$-\pi B^2 e^{-Ax} A \int_0^x dy \sin\left\{\frac{2\omega_p}{v}(x - y) + \frac{2\omega_p M}{\omega_p} [\sin(\omega_m t) - \sin\omega_m\left(t - \frac{x - y}{v}\right)]\right\}.$$

The three terms proportional to B^3 all vanish when integrated with respect to φ'' . The term proportional to B^4 is ignored because it is smaller by a factor of $(B/A)^2$ than either of the two terms just written out.

A physical interpretation can be given to the two terms which are being carried forward at this point. Each term deals with a component of the group of molecules scattered into the "stream" at r'' , keeps track of that component as it propagates to r' , and sums up the contribution which that component makes to the mean transverse deflection at r' . Each term takes account of the effect of molecular asymmetry in two correlated collisions, and in each term the second of these two collisions is the collision at r' . The two terms differ as to where the first collision of the pair occurs. The component dealt with in the first term is the component representing the (quadrupole type) polarization of rotation axis directions in the in-scattered group. For this term it is the in-scattering at r'' which is the other collision

in which molecular asymmetry is taken into account. The attenuation of this component by collisions between r'' and r' is evaluated without regard to molecular asymmetry. The second term deals with the unpolarized component of the in-scattered group. Account is taken of the average polarization which this component acquires by virtue of the effect of molecular asymmetry in one collision tending to attenuate this component between r'' and r' .

(6) Coming to the integration with respect to x , one finds, using integration by parts, that the integral of the second of the preceding two terms is equal and opposite to the integral of the first term.

This completes the demonstration that the model gives zero net deflection if the molecules being considered are unpolarized prior to effect of the magnetic field.

$$\pi BA \int_0^\infty dx e^{-Ax} a_2 \cos\left(\frac{2\omega_p x}{v} + \frac{2\omega_p M}{\omega_m} [\sin(\omega_m t) - \sin\omega_m(t-x/v)]\right)$$

and

$$\pi BA \int_0^\infty dx e^{-Ax} b_2 \sin\left(\frac{2\omega_p x}{v} + \frac{2\omega_p M}{\omega_m} [\sin(\omega_m t) - \sin\omega_m(t-x/v)]\right).$$

Each of these two terms takes account of molecular asymmetry in the collision at r' but ignores molecular asymmetry in the collision at r'' and in the attenuation of the stream of molecules between r'' and r' .

Thus, the model yields nonzero mean deflections for molecules corresponding to distribution functions containing $\cos 2\zeta$ or $\sin 2\zeta$. These polarizations are a part of the response of the gas to the nonuniform temperature distribution and to the combination of the temperature distribution and the magnetic field.

(7) Finally, the time averages of the preceding two terms over one period of oscillation of the modulating field may be written in terms of the Bessel function J_0 as

$$\pi BA \int_0^\infty dx e^{-Ax} a_2 \cos\left(\frac{2\omega_p x}{v}\right) \times J_0\left[\frac{4\omega_p M}{\omega_m} \sin\left(\frac{\omega_m x}{2v}\right)\right]$$

and

$$\pi BA \int_0^\infty dx e^{-Ax} b_2 \sin\left(\frac{2\omega_p x}{v}\right) \times J_0\left[\frac{4\omega_p M}{\omega_m} \sin\left(\frac{\omega_m x}{2v}\right)\right].$$

When we consider the more general form

$$1 + a_1 \sin\zeta + b_1 \cos\zeta + a_2 \sin 2\zeta + b_2 \cos 2\zeta$$

for the zero-field probability distribution of the coordinate ζ , the corresponding distribution for ζ'' just after the collision at r'' is

$$(1 + a_1 \sin\zeta'' + b_1 \cos\zeta'' + a_2 \sin 2\zeta'' + b_2 \cos 2\zeta'') \times (A + B \cos 2\zeta'').$$

One can repeat the preceding development and can again do the integration over φ'' in a straightforward manner. One finds that the terms $a_1 \sin\zeta$ and $b_1 \cos\zeta$ do not make any contribution to the mean deflection. Bearing in mind that a_2 and b_2 will both be of order B/A , one finds that among the various terms which $a_2 \sin 2\zeta$ and $b_2 \cos 2\zeta$ contribute to the mean deflection at r' , the only ones which are of order B^2 are

III. RELATIONSHIP BETWEEN DEFLECTION AND SHEAR STRESS

So far we have been considering molecules traveling in some particular direction in the r, φ plane, and we have found that the rotation axes must exhibit quadrupole polarization relative to the direction of travel in order for the mean deflection at the end of a free flight to be nonzero. We want to select from the distribution function for the gas the leading terms which have this polarization and to determine how each one will contribute to the r, φ element $\Sigma_{r, \varphi}$ of the stress tensor. Each term of the distribution function will exhibit the symmetry of the apparatus. This symmetry can be expressed in terms of the angle ζ between the molecular rotation axis and velocity (both still in the r, φ plane) and the angle θ between velocity and the radial direction.

We will look first at the terms which can be present in the distribution function when it has not as yet been affected by the magnetic field. In this case symmetry requires that each term remain unchanged when both ζ and θ change sign. The permissible terms are therefore

$$\sin(2\zeta)[s_1 \sin\theta + s_2 \sin 2\theta + \dots],$$

$$\cos(2\zeta)[1 + c_1 \cos\theta + c_2 \cos 2\theta + \dots].$$

Let us first consider the term $\cos 2\zeta \cos 2\theta$. Let

the mean deflection of molecules with axis polarization $\cos 2\xi$ be denoted by δ_c . Then molecules traveling in directions such that $\cos 2\theta > 0$ will have mean deflection δ_c and molecules traveling in the remaining directions will have mean deflection $-\delta_c$. Thus, following the collision at r' , the distribution of the velocity directions of these molecules may be represented by

$$\delta f = \cos[2(\theta - \delta_c)] h[\cos 2(\theta - \delta_c)] - \cos[2(\theta + \delta_c)] h[-\cos 2(\theta + \delta_c)],$$

where

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The contribution of δf to $\Sigma_{r,\varphi}$ is proportional to

$$\int_0^{2\pi} d\theta \sin\theta \cos\theta \delta f(\theta)$$

which upon substitution is found to be

$$\pi \sin 2\delta_c.$$

Since δ_c will be small, this contribution to $\Sigma_{r,\varphi}$ will be proportional to the mean deflection δ_c . Applying an analogous treatment to the other terms which are not forbidden by symmetry, one finds that their contributions to $\Sigma_{r,\varphi}$ all vanish. Having found that the surviving contribution is due to the term $\cos 2\xi \cos 2\theta$ in the distribution function, we may note that terms of this form arise from terms such as $\vec{J}\vec{J}$: $\nabla\nabla T$ and $\vec{v}\cdot\vec{J}\vec{v}\vec{J}$: $\nabla\nabla T$ in the conventional expansions of the distribution function. (\vec{J} denotes the molecular angular momentum, and T represents temperature.)

We next consider the additional terms which can arise as the distribution function responds to the presence of the magnetic field. The term $\sin 2\xi \times \cos 2\theta$, which is ruled out by symmetry when there is no magnetic field, becomes legitimate if it is multiplied by a factor which is an odd function of the magnetic field. Just what this odd function should be cannot be determined from a free-path approach such as that being given here. However, since this odd function is a response of the molecular distribution function to the presence of the temperature distribution and the magnetic field, it

seems reasonable to assume that it will be something like δ_∞ by which we mean the value which δ_c would have if the magnetic field were unmodulated. Denoting the mean deflection due to the axis polarization $\sin 2\xi$ by δ_s , a distribution function term $\delta_{c0} \sin 2\xi \cos 2\theta$ makes a contribution $\delta_{c0} \pi \times \sin 2\delta_s$ to the r, φ element of the stress tensor. As in the earlier case, this is the only term which contributes.

Thus we will try to include free-path contributions to $\Sigma_{r,\varphi}$ owing to prior effects of the magnetic field on the distribution function by adding to $\Sigma_{r,\varphi}$ a term $P\delta_{c0}\delta_s$ where P is a disposable parameter. We will then expect $\Sigma_{r,\varphi}$ to be proportional to

$$\delta_c + P\delta_\infty\delta_s.$$

IV. MANIFESTATION OF RESONANCES

Recalling the results obtained at the end of Sec. II, introducing the parameters $a \equiv Av/2\omega_p$ and $R \equiv \omega_m/\omega_p$, altering the variable of integration, and dropping various constants, the basic mean deflections may be expressed as

$$\delta_c \propto a \int_0^\infty dy e^{-ay} \sin(y) J_0 \left[\frac{4M}{R} \sin \left(\frac{Ry}{4} \right) \right],$$

$$\delta_s \propto a \int_0^\infty dy e^{-ay} \cos(y) J_0 \left[\frac{4M}{R} \sin \left(\frac{Ry}{4} \right) \right].$$

Since these expressions will be developed in parallel, it will be convenient to write them in a parallel notation, namely,

$$\left\{ \begin{array}{l} \delta_c \\ \delta_s \end{array} \right\} \propto a \int_0^\infty dy e^{-ay} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (y) J_0 \left[\frac{4M}{R} \sin \left(\frac{Ry}{4} \right) \right].$$

The integral for δ_c is found upon evaluation to exhibit resonances as a function of R . The resonances which are implicit in the integral may be made explicit with the help of the following transformation.¹¹

Using the formula¹²

$$J_0(z \sin \alpha) = \sum_{i=-\infty}^{\infty} \cos(2i\alpha) J_i^2 \left(\frac{z}{2} \right),$$

one finds

$$\left\{ \begin{array}{l} \sin y \\ \cos y \end{array} \right\} J_0 \left[\frac{4M}{R} \sin \left(\frac{Ry}{4} \right) \right] = \sum_{i=-\infty}^{\infty} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} \left[\left(1 + \frac{1}{2} lR \right) y \right] J_i^2 \left(\frac{2M}{R} \right),$$

so that

$$\left\{ \begin{array}{l} \delta_c \\ \delta_s \end{array} \right\} \propto a \sum_{i=-\infty}^{\infty} \int_0^\infty dy e^{-ay} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} \left[\left(1 + \frac{1}{2} lR \right) y \right] J_i^2 \left(\frac{2M}{R} \right).$$

Carrying out the integral over y , this becomes

$$\left\{ \begin{array}{l} \delta_c \\ \delta_s \end{array} \right\} \propto a \sum_{i=-\infty}^{\infty} \left(\left\{ \begin{array}{l} 1 + \frac{1}{2} lR \\ a \end{array} \right\} \right) / [a^2 + (1 + \frac{1}{2} lR)^2] J_i^2 \left(\frac{2M}{R} \right).$$

V. INITIAL INTERPRETATION

In order to see the physical meaning of these results, let us assume in this section that the shear stress $\Sigma_{r,\varphi}$ is proportional to δ_c alone. Then, when there is no modulation, $M=0$, and the unperturbed torque is proportional to

$$\frac{a}{1+a^2} = \left(\frac{Av}{2\omega_p} \right) / \left[1 + \left(\frac{Av}{2\omega_p} \right)^2 \right].$$

Since $Av/2\omega_p \propto p/H$, this expression for the static torque depends on p/H in the appropriate manner.

The expression for the relative change in torque due to modulation becomes (with l changed to $-l$)

$$\frac{\Delta N}{N} = (1+a^2) \sum_{l=-\infty}^{\infty} \frac{1-lR/2}{a^2 + (1-\frac{1}{2}lR)^2} J_l^2 \left(\frac{2M}{R} \right) - 1.$$

Considering the l th term of this sum (with $l > 0$), one notes that the factor

$$(1 - \frac{1}{2}lR) / \{a^2 + (1 - \frac{1}{2}lR)^2\}$$

is the common Lorentzian form of resonance with the midpoint of the resonance occurring at $R=2/l$ and that $\Delta N/N$ will therefore contain a sequence of resonances at

$$R \equiv \omega_m / \omega_p = \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \dots$$

Thus, the above expression also reproduces the resonances which are the most conspicuous feature of the experimental results obtained by modulation.

The above expression for $\Delta N/N$ is identical to that obtained by Hess and Waldman.⁸ Fleming and Martin⁹ obtain an identical result for each basic contribution which they consider, but obtain a final result which is somewhat more complicated because of superposition of several contributions. The results obtained by McCourt and co-workers⁸ are equivalent to those of Fleming and Martin.

VI. AVERAGE OVER MOLECULAR SPEEDS

The expressions developed so far assume a single value of $a = Av/2\omega_p$ for all the molecules which contribute to the shear stress. Since the speed v will follow some distribution such as a Maxwellian, it is of interest to determine how the foregoing results are affected when they are averaged with respect to v . The specific form of the integrand in such an average depends on the expression used for A as a function of v . In order for the results to be expressible in closed form, we have had to approximate A as a constant. Fortunately this approximation is not an unreasonable one for the averages in question.

The molecular speed distributions corresponding to the simplest distribution function terms will be

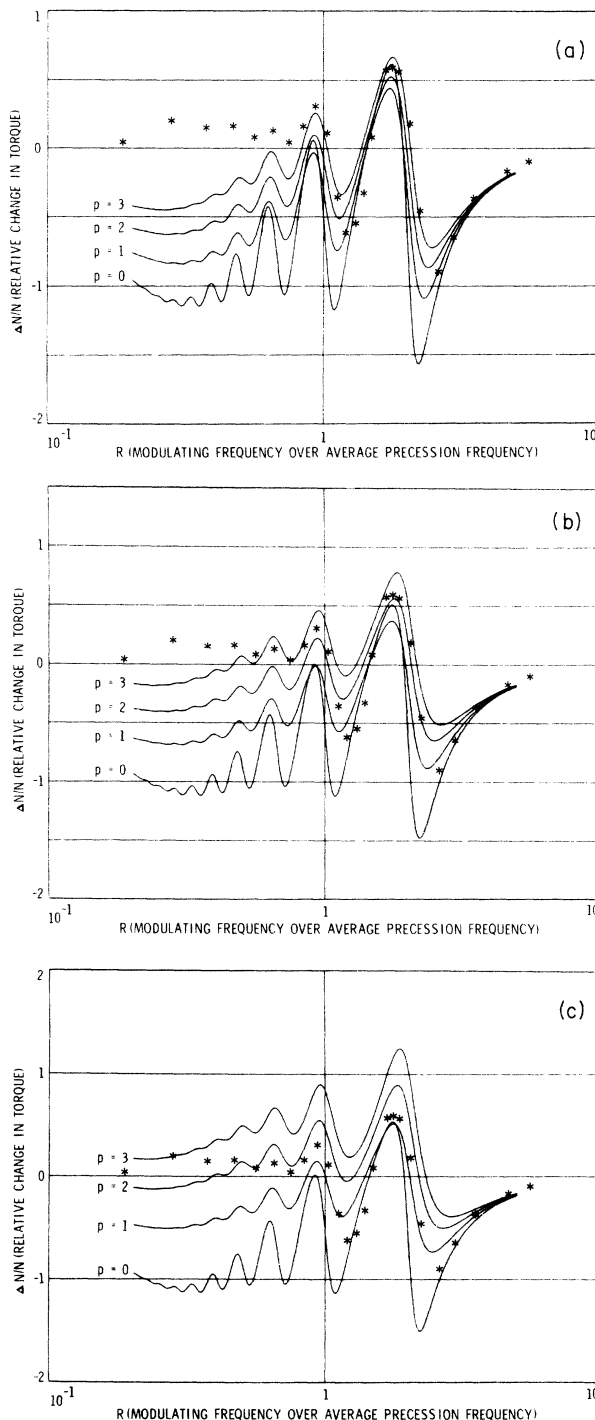


FIG. 5. Asterisks represent data of Smith and Scott for NO with $\alpha = 7.89$ and $M = 1.42$. The curves are from the model and are all for those same values of α and M . The curves in (a) assume that all molecules travel at a single speed. The curves in (b) and (c) assume molecular speed distributions corresponding to $n=5$ and 7, respectively. Each curve is for a particular value of the disposable parameter P as labeled.

as follows:

$$\begin{aligned}\vec{J}\vec{J} &: \nabla\nabla T - v^2 e^{-mv^2/2kT}, \\ \vec{v}\cdot\vec{J}\vec{v}\vec{J} &: \nabla\nabla T - v^4 e^{-mv^2/2kT}.\end{aligned}$$

There will be an additional factor of v^2 because the average represents transport of momentum.

Applying the average over v to the next to last equation of Sec. IV, we need to evaluate

$$\int_0^\infty dv v^n e^{-mv^2/2kT} e^{-Ayv/2\omega_p},$$

where n will be 5 or 7. By Ref. 13, Sec. 19.5.3, this is

$$\left(\frac{kT}{m}\right)^{(n+1)/2} n! \exp\left[\frac{1}{4} \frac{kT}{m} \left(\frac{Ay}{2\omega_p}\right)^2\right] D_{-(n+1)}\left[\left(\frac{kT}{m}\right)^{1/2} \frac{Ay}{2\omega_p}\right],$$

D being a Whittaker function.

Proceeding now to the integral over y , we need

$$\int_0^\infty dy \left\{ \frac{\sin}{\cos} \right\} [(1 + \frac{1}{2}lR)y] \exp\left[\frac{1}{4} \frac{kT}{m} \left(\frac{Ay}{2\omega_p}\right)^2\right] D_{-(n+1)}\left[\left(\frac{kT}{m}\right)^{1/2} \frac{Ay}{2\omega_p}\right],$$

which is given in Ref. 14. Introducing

$$b \equiv (1 + \frac{1}{2}lR) \left(\frac{m}{kT}\right)^{1/2} \frac{2\omega_p}{A},$$

then for $b > 0$ the expression becomes

$$\left(\frac{m}{kT}\right)^{1/2} \frac{2\omega_p}{A} \frac{2^{(n-2)/2}}{\Gamma(n+1)} G_{23}^{22} \left(\frac{b^2}{2} \left| \begin{array}{c} 0, \frac{1}{2} \\ n/2, \left\{ \frac{1}{2} \right\}, \left\{ 0 \right\} \\ \left\{ 0 \right\}, \left\{ \frac{1}{2} \right\} \end{array} \right. \right).$$

In light of the definition of the G function in Ref. 15, p. 207, this reduces to

$$\left(\frac{m}{kT}\right)^{1/2} \frac{2\omega_p}{A} \frac{2^{(n-2)/2}}{\Gamma(n+1)} G_{12}^{21} \left(\frac{b^2}{2} \left| \begin{array}{c} \left\{ \frac{1}{2} \right\} \\ \left\{ 0 \right\} \\ n/2, \left\{ \frac{1}{2} \right\} \\ \left\{ 0 \right\} \end{array} \right. \right).$$

With the help of Ref. 15, p. 216, Eq. (8) and Ref. 13, Sec. 13.1.33, it may next be written as

$$\left(\frac{m}{kT}\right)^{1/2} \frac{\omega_p}{A} \frac{b^n}{n!} \left\{ \Gamma\left(\frac{1}{2}(n+1)\right) U\left(\frac{1}{2}(n+1), \frac{1}{2}(n+1), \frac{1}{2}b^2\right) \right\} \left\{ \Gamma\left(\frac{1}{2}(n+2)\right) U\left(\frac{1}{2}(n+2), \frac{1}{2}(n+2), \frac{1}{2}b^2\right) \right\},$$

U being a confluent hypergeometric function. Using Ref. 13, Sec. 13.6.28, this becomes

$$\left(\frac{m}{kT}\right)^{1/2} \frac{\omega_p}{A} \frac{b^n}{n!} e^{b^2/2} \left\{ \Gamma\left(\frac{1}{2}(n+1)\right) \Gamma\left(-\frac{1}{2}(n-1), \frac{1}{2}b^2\right) \right\} \left\{ \Gamma\left(\frac{1}{2}(n+2)\right) \Gamma\left(-\frac{1}{2}n, \frac{1}{2}b^2\right) \right\}.$$

Finally, recalling that n is an odd integer and using Ref. 13, Sec. 6.5.9, this becomes

$$\left(\frac{m}{kT}\right)^{1/2} \frac{\omega_p}{A} \frac{2^{n/2}}{n!} \left\{ \Gamma\left(\frac{1}{2}(n+1)\right) (b/\sqrt{2}) e^{b^2/2} E_{(n+1)/2}\left(\frac{1}{2}b^2\right) \right\} \left\{ \Gamma\left(\frac{1}{2}(n+2)\right) (\frac{1}{2}b^2)^{n/2} e^{b^2/2} \Gamma\left(-\frac{1}{2}n, \frac{1}{2}b^2\right) \right\},$$

where $E_n(x)$ is one of the exponential integrals (cf. Ref. 13, Chap. 5). Referring back to the integral which leads to the G functions, one sees that these last expressions hold for $b < 0$ as well as for $b > 0$ (i.e., the upper and lower expressions are respectively odd and even in b). Reintroducing various factors which were temporarily omitted (but dropping common factors not depending on H or p), we find

$$\left\{ \begin{array}{c} \delta_c \\ \delta_s \end{array} \right\} \propto \sum_{i=-\infty}^{\infty} \left\{ \frac{\Gamma\left(\frac{1}{2}(n+1)\right) (b/\sqrt{2}) e^{b^2/2} E_{(n+1)/2}\left(\frac{1}{2}b^2\right)}{\Gamma\left(\frac{1}{2}(n+2)\right) (\frac{1}{2}b^2)^{n/2} e^{b^2/2} \Gamma\left(-\frac{1}{2}n, \frac{1}{2}b^2\right)} \right\} J_1^2\left(\frac{2M}{R}\right).$$

The above functions involving b are similar to (and generalizations of) the Lorentzian "resonance" expressions to which the average over molecular speed was applied.

VII. RESULTS

Using the foregoing expressions in the relation

$$N(\text{torque}) \propto \delta_c + P\delta_{c_0}\delta_s$$

and using a computer to evaluate and plot $\Delta N/N$, one obtains curves such as those shown in Fig. 5. The superposed asterisks represent experimental measurements of Smith and Scott.⁵ (The asterisk abscissa values have all been multiplied by a single disposable factor to permit their alignment with the curves.) Each of these figures shows curves for $P=0, 1, 2$, and 3 as labeled. Figure 5(a) shows the results obtained when all molecules are assumed to travel at a single speed. Figures 5(b) and 5(c) show the speed averaged results assuming the speed distributions represented by $n=5$ and 7, respectively.

Examining the curves, one may observe that inclusion of the term $P\delta_{c_0}\delta_s$ appears to permit improved fit to the experimental data. However, none of the fits are particularly good, and there

is no apparent basis for preferring one molecular speed distribution over another.

Several factors are presumed to contribute to the observable discrepancies between the model and experiment. First, the basic free-path approach is quite crude compared to any systematic approximate solution of the Boltzmann equation. Second, while inclusion of a function such as δ_{c_0} seems reasonable, the specific form δ_{c_0} is only a guess. Third, the curves of Fig. 5 assume that all molecules have a single g value (i.e., gyro-magnetic ratio) but are compared there to experimental points for the gas NO which exhibits various g values distributed over a fairly wide range. (For this reason it would be interesting to see experimental results for a gas such as N_2 in which all molecules do have the same g value.) Another feature of the model which may be mentioned is that several vectors which are physically free to point in any direction were constrained to lie in the r, φ plane. However, we do not expect that inclusion of other directions would alter the results.

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