Coherent-optical-pulse propagation as an inverse problem*

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Various coherent-optical-pulse profiles are known to exhibit lossless propagation during transmission through a resonant atomic medium. It is shown that the requirement of lossless propagation provides asymptotic information that is sufficient to determine these profiles by the inverse method. Pulse profiles are obtained as solutions to the Marchenko equation. The analysis is applied explicitly to the propagation of 2π, 4π, and 0π pulses in an inhomogeneously broadened medium.

I. INTRODUCTION

Analytical expressions for the various pulse profiles that exhibit lossless propagation in an attenuating medium consisting of inhomogeneously broadened, nondegenerate, two-level atoms (i.e., $2n\pi$ pulses, $n=0, 1, 2, \ldots$) have been found[1] to be closely related to Bargmann potentials.[2] Since the Bargmann potentials provide a class of particularly simple solutions to the integral equations of inverse-scattering theory,[3-7] one expects that the pulse profiles of coherent-optical-pulse propagation are obtainable by the inverse method.

This leads to yet another parallel between the analysis of coherent-optical-pulse propagation and that of the Korteweg-deVries and related equations.[8-17] Previous research[18] has shown that propagation of the isolated hyperbolic secant pulse of self-induced transparency[19] is quite similar to the soliton solution of the Korteweg-deVries equation.[8] Sequences of such isolated coherent-optical pulses have been found to evolve from a single larger initial pulse,[19-22] and the equations that govern coherent pulse propagation have been found to possess higher conservation laws.[18] As in the case of the Korteweg-deVries equation,[14,15] these higher conservation laws have been used to determine the amplitude of each of the isolated pulses that ultimately emerge from an initial pulse of arbitrary size.[18,23,24]

The equations governing the temporal response of an atomic medium to a coherent-optical pulse (i.e., the Bloch equations) have been shown[18] to be equivalent to a Sturm-Liouville or Schrödinger equation in which the potential function is related to the pulse profile. The equation governing the spatial evolution of the pulse profile follows from the Maxwell equations. The situation is thus similar to that in which the potential of a Schrödinger equation is governed by the Korteweg-deVries equation.[11] In this latter case, the relationship between the equation of evolution and the Sturm-Liouville equation has been used to obtain the potential and thus the solution of the Korteweg-deVries equation from information concerning the asymptotic form of the solution of the Sturm-Liouville equation. The analytical techniques available for carrying out this procedure are known as the inverse method and have been developed primarily for application to quantum scattering theory.

In the present application of the inverse method, the function playing the role of a potential is found to be complex. The non-Hermitian nature of the associated Hamiltonian leads to new types of solutions that are not obtained for the Korteweg-deVries equation. One of these solutions is the 0π pulse.

The equations governing coherent-optical-pulse propagation are reviewed in Sec. II. A summary of the inverse method in terms of the Marchenko equation is then given in Sec. III. Section IV contains examples of the use of the inverse method to obtain some of the simpler analytical solutions of the equations of coherent-pulse propagation.

II. BASIC EQUATIONS FOR COHERENT-OPTICAL-PULSE PROPAGATION

Propagation of a plane electromagnetic wave in a medium consisting of nondegenerate two-level atoms may be described by a simultaneous solution of the Maxwell equations and the Schrödinger-Bloch equations. When the usual slowly varying envelope approximation is made, the electric field may be written

$$E(x, t) = \bar{\mathcal{E}}(x, t)\cos(k_0 x - \omega_0 t), \tag{2.1}$$

where k_0 and ω_0 are the wave number and angular frequency, respectively, of the optical wave. Propagation is assumed to take place in a positive x direction. The envelope function $\bar{\mathcal{E}}(x, t)$ is assumed to vary slowly on the length and time scales of the carrier wave, i.e., $\partial\bar{\mathcal{E}}/\partial x \ll k_0\bar{\mathcal{E}}$, $\partial\bar{\mathcal{E}}/\partial t \ll \omega_0\bar{\mathcal{E}}$. For a medium consisting of an assemblage of two-level systems distributed with a uniform density n_0, the macroscopic polarization is $n_0\int d\Delta\omega\, g(\Delta\omega)p(\Delta\omega, x, t)$ where $p(\Delta\omega, x, t)$ is the polarization due to an individual atom that is detuned from line center by an amount $\Delta\omega = \omega_0 - \omega$ and $g(\Delta\omega)$ is the normalized spectrum of the detuning. (The present definition of $\Delta\omega$, which differs by a minus sign from that used previously,[18] has been chosen so that the inverse problem to be developed in Sec. III will be in a standard form.) Since depletion of the group of atoms detuned by some particular $\Delta\omega$ would only affect the magnitude of the spectrum at $\Delta\omega$, this type of broadening is referred to as inhomogeneous. It is assumed that the broadening is symmetric about ω_0. The limiting case in which all atoms are on resonance (i.e., all $\Delta\omega = 0$) will be referred to as the unbroadened or sharp-line case. The polarization may be written

$$p(\Delta\omega, x, t) = \wp[\mathcal{P}(\Delta\omega, x, t)\sin(k_0 x - \omega_0 t) + \mathcal{Q}(\Delta\omega, x, t)\cos(k_0 x - \omega_0 t)] . \quad (2.2)$$

For pulses much shorter than both the population relaxation time T_1 and the atomic dephasing time T_2 these relaxation times may be taken to be infinite. The in-phase and in-quadrature envelope functions $\mathcal{Q}(\Delta\omega, x, t)$ and $\mathcal{P}(\Delta\omega, x, t)$, respectively, then obey the undamped Bloch equations[19, 22, 18]

$$\frac{\partial \mathcal{P}}{\partial t} = \kappa\bar{\mathcal{E}}\mathcal{N} - \Delta\omega\mathcal{Q} , \quad (2.3a)$$

$$\frac{\partial \mathcal{Q}}{\partial t} = \Delta\omega\mathcal{P} , \quad (2.3b)$$

$$\frac{\partial \mathcal{N}}{\partial t} = -\kappa\bar{\mathcal{E}}\mathcal{P} , \quad (2.3c)$$

where $\kappa = \wp/\hbar$ and $\mathcal{N}(\Delta\omega, x, t)$ is the population inversion for a single atom. The initial condition $\mathcal{N}(\Delta\omega, x, -\infty) = -1$ corresponds to the case of an attenuator in which all atoms are initially in the lower level. This is the situation that will be considered here.

The wave equation for the electric field in slowly varying envelope approximation reduces to[19, 22, 18]

$$\kappa\left(\frac{\partial \bar{\mathcal{E}}}{\partial t} + c\frac{\partial \bar{\mathcal{E}}}{\partial x}\right) = \alpha' c\langle\mathcal{P}\rangle , \quad (2.4)$$

where $\alpha' \equiv 2\pi n_0\omega_0\wp^2/\hbar c$ and c is the light velocity in the host medium. The angular brackets signify an average over the inhomogeneously broadened atomic spectrum $g(\Delta\omega)$ as given by

$$\langle\cdots\rangle = \int_{-\infty}^{\infty} d\Delta\omega\, g(\Delta\omega)(\cdots) . \quad (2.5)$$

As mentioned above, this broadening will be assumed to be symmetric about ω_0.

Equations (2.3) have exactly the same structure as the Frenet-Serret equations of differential geometry.[25] It is known that the solution to such a set of equations is equivalent to the solution of a Riccati equation.[25] This is seen by first observing that an integral of Eqs. (2.3) is

$$\mathcal{N}^2 + \mathcal{P}^2 + \mathcal{Q}^2 = 1 . \quad (2.6)$$

Two new functions are then introduced by writing

$$\frac{\mathcal{N} + i\mathcal{P}}{1 - \mathcal{Q}} = \frac{1 + \mathcal{Q}}{\mathcal{N} - i\mathcal{P}} = \varphi , \quad (2.7a)$$

$$\frac{\mathcal{N} - i\mathcal{P}}{1 - \mathcal{Q}} = \frac{1 + \mathcal{Q}}{\mathcal{N} + i\mathcal{P}} = -\frac{1}{\psi} = \varphi^* . \quad (2.7b)$$

Equations (2.7) may be inverted to yield

$$\mathcal{N} = \frac{1 - \varphi\psi}{\varphi - \psi} = \frac{2\mathrm{Re}\varphi}{|\varphi|^2 + 1} , \quad (2.8a)$$

$$\mathcal{P} = i\frac{1 + \varphi\psi}{\varphi - \psi} = \frac{2\,\mathrm{Im}\varphi}{|\varphi|^2 + 1} , \quad (2.8b)$$

$$\mathcal{Q} = \frac{\varphi + \psi}{\varphi - \psi} = \frac{|\varphi|^2 - 1}{|\varphi|^2 + 1} . \quad (2.8c)$$

Equations describing the time dependence of φ and ψ are readily deduced by inserting Eqs. (2.8) into Eqs. (2.3). It is found that the equations for φ and ψ are uncoupled; φ satisfies the Riccati equation

$$\frac{\partial\varphi}{\partial t} = i\kappa\bar{\mathcal{E}}\varphi - \tfrac{1}{2}i\,\Delta\omega(\varphi^2 - 1) \quad (2.9)$$

and ψ satisfies the same equation. One may now employ the usual transformation to convert this Riccati equation to a second-order linear equation. One obtains

$$\frac{\partial^2 w}{\partial t^2} + \frac{1}{4}\left((\Delta\omega)^2 + (\kappa\bar{\mathcal{E}})^2 + 2i\kappa\frac{\partial\bar{\mathcal{E}}}{\partial t}\right)w = 0 . \quad (2.10)$$

The new dependent variable w is related to φ through the transformations

$$w(t) = u(t)\exp\left(-\frac{i\kappa}{2}\int_{-\infty}^{t} dt'\,\bar{\mathcal{E}}(t')\right) , \quad (2.11)$$

$$\varphi(t) = \frac{-2i}{\Delta\omega}\frac{d(\ln u)}{dt} . \quad (2.12)$$

A parametric dependence upon the spatial location x has been suppressed.

The alternate factorization of Eq. (2.6) and redefinition of φ according to

$$\frac{\mathcal{P} - i\mathcal{Q}}{1 - \mathcal{N}} = \frac{1 + \mathcal{N}}{\mathcal{P} + i\mathcal{Q}} = \varphi ,$$

etc., would ultimately lead to a Schrödinger equation with a much more complicated potential than that obtained here in Eq. (2.10). However, this more complicated second-order differential equation is equivalent to the pair of first-order equations to which the inverse method has been applied by Zakharov and Shabat.[25a] The two-component formulation has been found to be particularly convenient for the coherent-optical-pulse problem when phase variation is included[25b] since the approach used in the present paper becomes rather unwieldy in that case.

In terms of the dimensionless variables

$$\mathcal{E} = \kappa\overline{\mathcal{E}}/\Omega \tag{2.13}$$

and

$$\tau = \Omega(t - x/c), \quad \xi = \Omega x/c, \tag{2.14}$$

where

$$\Omega^2 = \alpha' c, \tag{2.15}$$

Eqs. (2.4) and (2.10) become

$$\frac{\partial \mathcal{E}}{\partial \xi} = \langle \mathcal{P} \rangle, \tag{2.16a}$$

$$\frac{\partial^2 w}{\partial t^2} + (\nu^2 - \mathcal{V})w = 0. \tag{2.16b}$$

Here, ν is the dimensionless detuning variable

$$\nu = \Delta\omega/2\Omega \tag{2.17}$$

while $\mathcal{V}$ is the dimensionless potential function

$$\mathcal{V} = -\frac{1}{4}\left(\mathcal{E}^2 + 2i\frac{\partial \mathcal{E}}{\partial \tau}\right). \tag{2.18}$$

From Eqs. (2.3c) and (2.16a) it follows that $\mathcal{V}$ satisfies

$$2\frac{\partial \mathcal{V}}{\partial \xi} = \frac{\partial}{\partial \tau}\langle \mathfrak{N} - i\mathcal{P} \rangle. \tag{2.19}$$

One can readily verify that Eq. (2.16b) is equivalent to the pair of linear equations

$$\frac{\partial w(\nu, \tau)}{\partial \tau} + \frac{i}{2}\mathcal{E}w(\nu, \tau) = \nu\overline{w}(\nu, \tau), \tag{2.20a}$$

$$-\frac{\partial \overline{w}(\nu, \tau)}{\partial \tau} + \frac{i}{2}\mathcal{E}\overline{w}(\nu, \tau) = \nu w(\nu, \tau), \tag{2.20b}$$

where $w(\nu, \tau)$ is the function introduced above and $\overline{w}(\nu, \tau) = w^*(-\nu, \tau)$. Also, φ may be expressed in the form

$$\varphi(\nu, \tau) = -i\frac{\overline{w}(\nu, \tau)}{w(\nu, \tau)}. \tag{2.21}$$

For future reference, we now digress to consider a simple model of coherent-optical-pulse propagation. It arises in the limit in which all atoms are exactly on resonance so that $\Delta\omega = 0$. The solution of Eq. (2.9) that corresponds to all atoms also being initially in the ground state is then

$$\varphi = -e^{i\sigma}, \tag{2.22}$$

where

$$\sigma(\xi, \tau) \equiv \int_{-\infty}^{\tau} d\tau' \, \mathcal{E}(\xi, \tau'). \tag{2.23}$$

From Eqs. (2.8b) and (2.16a), one immediately obtains

$$\sigma_{\xi\tau} = -\sin\sigma. \tag{2.24}$$

As described elsewhere,[26,27,18] a Baecklund transformation may be exployed to obtain analytical solutions of Eq. (2.24). Electric field profiles that exhibit the pulse decomposition process are obtained from σ by using the differential form of Eq. (2.23). The main advantage of the inverse method is that it yields the corresponding solution in the presence of inhomogeneous broadening as well.

Returning to the more general situation in which nonresonant atoms are included, one sees from Eqs. (2.8a), (2.11) and (2.12) that $w(\nu, \xi, -\infty) \sim e^{-i\nu\tau}$. In this case $\varphi(\nu, \xi, -\infty) \to -1$ and $\mathfrak{N}(\nu, \xi, -\infty) = -1$ as is required for an attenuator. After passage of an arbitrary pulse the solution for w will, in general, be a linear combination of the form

$$w(\nu, \xi, +\infty) \to a(\nu, \xi)e^{i\nu\tau} + b(\nu, \xi)e^{-i\nu\tau}. \tag{2.25}$$

The atomic-polarization amplitudes $\mathcal{P}$ and $\mathcal{Q}$ for an individual two-level system detuned from the carrier frequency of the pulse by an amount $\Delta\omega$ are then left ringing at the dimensionless frequency ν and the population is not returned completely to the ground state. To have all two-level atoms returned exactly to the ground state, one must require that $w(\nu, \xi, +\infty)$ again be of the form $e^{-i\nu\tau}$, i.e.,

$$a(\nu, \xi) = 0. \tag{2.26}$$

Application of the inverse method is particularly simple when Eq. (2.26) is imposed. It should be emphasized that the only pulse profiles being considered here are the special solutions that result from imposition of Eq. (2.26).

Equation (2.26) is the same requirement as that imposed to obtain reflectionless potentials in a one-dimensional scattering problem[28,29] (with τ replaced by the space variable x, and ν by the propagation constant k) when the particle flux is incident from $+\infty$. In the present problem there is the additional consideration that the arbitrary constants arising in the integration of the Sturm-Liouville equation (2.16b), contain parametric dependence upon ξ, the space variable. The proper spatial dependence is obtained by requiring that the solution also satisfy the equation governing the spatial propagation, i.e., Eq. (2.16a).

III. SUMMARY OF THE INVERSE METHOD

As shown in the previous section, the determination of coherent-optical-pulse profiles is equivalent to finding potentials for the Schrödinger equation when certain information concerning the asymptotic form of the solution is specified. Except for the additional requirement that the arbitrary constants arising in the integration process contain a spatial dependence that must also be determined, one is confronted with a typical example of an inverse problem (at fixed angular momentum). Of the various formulations of the inverse problem that are available,[3] the one that has been found to be particularly suitable for the present problem is that employing the Marchenko equation on the infinite interval as given by Faddeev.[4] This approach is now summarized and applied to the pulse-propagation problem.

The two fundamental solutions of the Sturm-Liouville equation, Eq. (2.16b), that approach $e^{\pm i\nu\tau}$ as $\tau\to\pm\infty$, respectively, may be written[3,4,7]

$$f_1(\nu,\tau)=e^{i\nu\tau}+\int_\tau^\infty dt\,A_1(\tau,t)e^{i\nu t}\,, \tag{3.1a}$$

$$f_2(\nu,\tau)=e^{-i\nu\tau}+\int_{-\infty}^\tau dt\,A_2(\tau,t)e^{-i\nu t}\,. \tag{3.1b}$$

The functions A_1 and A_2 are related to the potential function of Eq. (2.16b) according to[3,7]

$$\mathcal{V}=-2\frac{\partial A_1(\tau,\tau)}{\partial\tau}=2\frac{\partial A_2(\tau,\tau)}{\partial\tau}\,. \tag{3.2}$$

From the definition of $\mathcal{V}$ in Eq. (2.18),

$$\frac{\partial \mathrm{Re}A_2}{\partial\tau}=-\tfrac{1}{8}\mathcal{E}^2 \tag{3.3a}$$

$$\mathrm{Im}A_2=-\tfrac{1}{4}\mathcal{E}\,. \tag{3.3b}$$

The general solution of the Sturm-Liouville equation may be expressed in terms of any two independent solutions such as $f_1(\nu,\tau)$ and $f_1(-\nu,\tau)$, or $f_2(\nu,\tau)$ and $f_2(-\nu,\tau)$. However, since the potential in Eq. (2.16b) is complex, $f_1(\nu,\tau)$ and $f_2(\nu,\tau)$ are no longer equal to $f_1^*(-\nu,\tau)$, and $f_2^*(-\nu,\tau)$, respectively. Since any third solution may be expressed in terms of a pair of independent solutions, one may write

$$f_2(\nu,\tau)=c_{11}(\nu)f_1(\nu,\tau)+c_{12}(\nu)f_1(-\nu,\tau)\,, \tag{3.4a}$$

$$f_1(\nu,\tau)=c_{21}(\nu)f_2(-\nu,\tau)+c_{22}(\nu)f_2(\nu,\tau)\,. \tag{3.4b}$$

The solution $f_2(\nu,\tau)$ is seen to be more directly related to the present problem as discussed above Eq. (2.25); i.e., it approaches $e^{-i\nu\tau}$ as $\tau\to-\infty$ and approaches a linear combination of $e^{\pm i\nu\tau}$ as $\tau\to+\infty$.

Wronskians, among the various solutions, lead to relations between the $c_{ij}(\nu)$. In particular,

$$c_{12}(\nu)=c_{21}(\nu)\,,\quad c_{11}(\nu)=-c_{22}(-\nu)\,. \tag{3.5}$$

Exponentially decaying or "bound-state" solutions of Eq. (2.16b) are again related to the zeroes of $c_{12}(\nu)$ which are located at points $\nu=\nu_n$ in the upper half of the complex ν plane. At these points

$$f_2(\nu_n,\tau)=c_{11}(\nu_n)f_1(\nu_n,\tau)\,; \tag{3.6}$$

i.e., the two solutions are linearly dependent. Since the potential is complex, these zeroes need no longer be confined to the imaginary axis. It will be shown that pairs of zeroes that are symmetrically placed with respect to the imaginary axis lead to the 0π pulse.

Equations (3.4) may be put in conformity with standard notation of one-dimensional scattering theory by dividing them by $c_{12}(\nu)$. One then obtains

$$t(\nu)f_2(\nu,\tau)=r_1(\nu)f_1(\nu,\tau)+f_1(-\nu,\tau)\,, \tag{3.7a}$$

$$t(\nu)f_1(\nu,\tau)=r_2(\nu)f_2(\nu,\tau)+f_2(-\nu,\tau)\,, \tag{3.7b}$$

where

$$t(\nu)=\frac{1}{c_{12}(\nu)}\,,\quad r_i(\nu)=\frac{c_{ii}(\nu)}{c_{12}(\nu)}\,,\quad i=1,2\,. \tag{3.8}$$

The residue at a pole of the transmission coefficient $t(\nu)$ is given by[4]

$$\mathrm{Res}\,t(\nu)|_{\nu=\nu_n}=i\left[\int_{-\infty}^\infty d\tau'\,f_1(\nu_n,\tau')f_2(\nu_n,\tau')\right]^{-1}\equiv i\gamma_n\,. \tag{3.9}$$

Introducing the Fourier transforms

$$R_1(t)=\int_{-\infty}^\infty\frac{d\nu}{2\pi}r_1(\nu)e^{i\nu t}\,,\qquad R_2(t)=\int_{-\infty}^\infty\frac{d\nu}{2\pi}r_2(\nu)e^{-i\nu t}\,, \tag{3.10}$$

and

$$T(t)=\int_{-\infty}^\infty\frac{d\nu}{2\pi}\left[t(\nu)-1\right]e^{-i\nu t}=-\sum\gamma_n e^{-i\nu_n t}\,,\quad t<0 \tag{3.11}$$

one finds that with the help of Eqs. (3.1) and (3.6) the Fourier transform of Eq. (3.7b) may be reduced to[4]

$$A_2(\tau,t)+\Omega_2(\tau+t)+\int_{-\infty}^\tau dy\,A_2(\tau,y)\Omega_2(y+t)=0\,,\quad \tau>t \tag{3.12}$$

where

$$\Omega_2(t)\equiv R_2(t)+\sum_{n=1}^N m_n e^{-i\nu_n t}\,, \tag{3.13}$$

N is the number of poles of $t(\nu)$ in the upper half of the ν plane and

$$m_n=\left\{\int_{-\infty}^\infty d\tau\left[f_2(\nu_n,\tau)\right]^2\right\}^{-1}=\gamma_n\,c_{22}(\nu_n)\,. \tag{3.14}$$

The integral equation obtained here is known in the literature of inverse-scattering theory as the Marchenko equation. Similar results, with an integral extending from τ to ∞, may be obtained for Eq. (3.7a).

As discussed in Sec. II, the pulse profiles associated with lossless propagation follow from the requirement that the solution of the Sturm-Liouville equation approach $e^{-i\nu\tau}$ as $\tau\to+\infty$, i.e., $r_1(\nu)=0$. According to Eqs. (3.5) and (3.8), this also implies

$$r_2(\nu)=0\,. \tag{3.15}$$

The Marchenko equation is readily solved in this case. For a given number of poles N, one may introduce the vectors

$$\vec{\Psi}(t)=(m_1e^{-i\nu_1 t},\, m_2e^{-i\nu_2 t},\, \ldots,\, m_N e^{-i\nu_N t}) \tag{3.16a}$$

and

$$\vec{\Phi}(t)=(e^{-i\nu_1 t},\, e^{-i\nu_2 t},\, \ldots,\, e^{-i\nu_N t})\,. \tag{3.16b}$$

The solution of Eq. (3.12) for $t=\tau$ may then be written[3]

$$A_2(\tau,\tau)=-\frac{\partial}{\partial\tau}\ln|\vec{\mathrm{V}}|\,, \tag{3.17}$$

where $|\vec{\mathrm{V}}|$ is the $N\times N$ determinant of the dyadic

$$\vec{\mathrm{V}}=\vec{\mathrm{I}}+\int_{-\infty}^{\tau}dt\,\vec{\Psi}(t)\vec{\Phi}(t) \tag{3.18}$$

in which $\vec{\mathrm{I}}$ represents the unit dyadic.

From Eqs. (2.18), (3.2), and (3.17), one finally obtains the electric field profile in the form

$$\mathcal{E}=4\frac{\partial}{\partial\tau}\,\mathrm{Im}\ln|\vec{\mathrm{V}}|\,. \tag{3.19}$$

To make contact with previous results[18,1] a function $\sigma(\xi,t)$ analogous to that introduced in Eq. (2.23) will be employed. Comparison of the differential form of Eq. (2.23) with Eq. (3.19) shows that

$$\sigma=4\tan^{-1}\left[\frac{\mathrm{Im}|\vec{\mathrm{V}}|}{\mathrm{Re}|\vec{\mathrm{V}}|}\right]. \tag{3.20}$$

Electric field envelopes then follow from the differential form of Eq. (2.23). In the present case the function $\sigma(\xi,\tau)$ will contain appropriate averages over the inhomogeneously broadened spectrum.

The response of the medium follows from Eqs. (2.8) and (2.21). One finds

$$\mathfrak{N}-i\mathcal{P}=-f_2(\nu,\tau)f_2(-\nu,\tau)\,, \tag{3.21}$$

where

$$f_2(\nu,\tau)\equiv w(\nu,\tau)/w(\nu,-\infty)\,.$$

Then, from Eqs. (2.19), (3.2), and (3.21)

$$4\frac{\partial A_2}{\partial\xi}=1-\langle f_2(\nu,\tau)f_2(-\nu,\tau)\rangle\,. \tag{3.22}$$

The imaginary part yields the propagation equation (2.16a) in the form

$$\frac{\partial\mathcal{E}}{\partial\xi}=\mathrm{Im}\langle f_2(\nu,\tau)f_2(-\nu,\tau)\rangle\,. \tag{3.23}$$

The spatial dependence is readily inferred by considering Eq. (3.22) in the limit $\tau\to-\infty$. From the Marchenko equation one may write

$$A_2(\tau,t)\xrightarrow[\tau\to-\infty]{}-\Omega_2(\tau+t)=-\sum_{n=1}^{N}m_n e^{-i\nu_n(\tau+t)}\,. \tag{3.24}$$

The definition of $f_2(\nu,\tau)$ then yields

$$f_2(\nu,\tau)f_2(-\nu,\tau)\xrightarrow[\tau\to-\infty]{}1-2i\sum_{n=1}^{N}\frac{m_n\nu_n}{\nu_n^2-\nu^2}\,e^{-2i\nu_n\tau}\,. \tag{3.25}$$

Using the corresponding limiting form of Eq. (3.22) and equating terms of the same time dependence, as they are linearly independent, one finds that the spatial dependence of the $m_n(\xi)$ are

$$m_n(\xi)=m_n(0)\exp\left(\frac{i}{2}\,\nu_n\xi\left\langle\frac{1}{\nu_n^2-\nu^2}\right\rangle\right). \tag{3.26}$$

From the form of $\mathcal{V}$ as given in Eqs. (2.18) and (3.2), one notes that $\mathrm{Re}A_2\ll\mathrm{Im}A_2$ as $\tau\to-\infty$ for fixed ξ. Hence A_2 becomes purely imaginary in this limit. Equation (3.24) then shows that m_n must be purely imaginary if ν_n is purely imaginary. Also, for pairs of ν_n of the form $\nu_{n2}=-\nu_{n1}^*$, one finds that $m_{n2}=-m_{n1}^*$.

One is now in a position to determine the pulse profile associated with a given set of ν_n. Some of the simpler cases have been observed experimentally. They will be considered in the next section.

IV. SPECIFIC PULSE PROFILES

In Secs. II and III it was shown that the profile of a coherent-optical pulse $\mathcal{E}(\xi,\tau)$ satisfies

$$\frac{\partial\mathcal{E}}{\partial\xi}=-\mathrm{Im}\langle f_2(\nu,\tau)f_2(-\nu,\tau)\rangle \tag{4.1}$$

where $f_2(\nu,\tau)$ is the solution of

$$\frac{\partial^2 f_2}{\partial\tau^2}+(\nu^2-\mathcal{V})f_2=0 \tag{4.2}$$

that approaches $e^{-i\nu\tau}$ as $\tau\to-\infty$ and where $\mathcal{V}(\xi,\tau)$ is related to $\mathcal{E}(\xi,\tau)$ by

$$\mathcal{V}=-\frac{1}{4}\left(\mathcal{E}^2+2i\frac{\partial\mathcal{E}}{\partial\tau}\right). \tag{4.3}$$

From the general theory of the inverse method, it is known that both $f_2(\nu, \tau)$ and the function $\mathfrak{V}(\xi, \tau)$ may be expressed in terms of a function $A_2(\tau, t)$ as

$$f_2(\nu, \tau) = e^{-i\nu\tau} + \int_{-\infty}^{\tau} dt\, A_2(\tau, t) e^{-i\nu t} \tag{4.4}$$

and

$$\mathfrak{V} = 2\frac{\partial A_2(\tau, \tau)}{\partial \tau}, \tag{4.5}$$

respectively. In addition, the function $A_2(\tau, t)$ satisfies the Marchenko equation

$$A_2(\tau, t) + \Omega_2(\tau + t) + \int_{-\infty}^{\tau} dy\, A_2(\tau, y)\Omega_2(y + t) = 0, \quad \tau > t \tag{4.6}$$

in which, for the case of lossless propagation, the function Ω_2 is of the form

$$\Omega_2(t) = \sum_{n=1}^{N} m_n e^{-i\nu_n t}. \tag{4.7}$$

The m_n contain parametric dependence upon the space variable ξ as given by Eq. (3.26). The various pulse profiles depend upon the number of constants ν_n and the values assigned to them. Three of the simplest cases, all of which are of interest experimentally, will now be considered.

A. One pole on positive imaginary axis—self-induced transparency

This first example is the steady-state solution mentioned in the introduction. It is analogous to the single-soliton solution of the Korteweg-deVries equation.

Setting $\nu = ia/2$, $a > 0$ and $m = ic(\xi)$ where $c(\xi)$ is real, one finds from Eq. (3.18) that

$$V = 1 + i[c(\xi)/a]e^{a\tau}. \tag{4.8}$$

According to Eq. (3.26)

$$c(\xi) = c(0)\exp\left(-a\xi\left\langle\frac{1}{a^2 + 4\nu^2}\right\rangle\right). \tag{4.9}$$

Then, from Eq. (3.20),

$$\sigma = 4\tan^{-1}\left[\frac{c(0)}{a}\exp a\left(\tau - \left\langle\frac{1}{a^2 + 4\nu^2}\right\rangle\xi\right)\right]. \tag{4.10}$$

The term $c(0)/a$, which may be written as the phase term $\ln[c(0)/a]$, can be ignored since it is equivalent to a translation of axes.

The associated electric field profile is then

$$\kappa\bar{\mathcal{E}} = \Omega\frac{\partial\sigma}{\partial\tau} = \frac{2}{\tau_p}\operatorname{sech}\left[\frac{1}{\tau_p}\left(t - \frac{x}{V}\right)\right], \tag{4.11}$$

where $a = (\Omega\tau_p)^{-1}$ and the pulse velocity V is given by

$$\frac{c}{V} = 1 + \left\langle\frac{1}{a^2 + 4\nu^2}\right\rangle = 1 + (\Omega\tau_p)^2\left\langle\frac{1}{1 + (\tau_p\Delta\omega)^2}\right\rangle. \tag{4.12}$$

The pulse velocity may be two or three orders of magnitude lower than the phase velocity of the light wave. Numerical values indicative of recent experimental results[21] are $\omega_0 \sim 10^{15}$ sec^{-1}, $n_0 \sim 10^{12}$ cm^{-3}, $\wp \sim 6\times 10^{-18}$ (cgs), $\tau_p \sim 7\times 10^{-9}$ sec. Then, $\alpha' c \sim 2\times 10^{20}$ sec^{-2} and from the definition of Ω given in Eq. (2.15)

$$(\Omega\tau_p)^2 \sim 3000. \tag{4.13}$$

The function to be averaged over the frequency distribution leads to a factor of order unity. Hence the pulse velocity is reduced by three orders of magnitude in this instance. Lossless propagation and pulse breakup were observed for pulse areas up to 6π in these experiments.

B. Two poles on imaginary axis–breakup of 4π pulse

As noted previously, pulses with area much larger than 2π have been found to decompose into a number of isolated 2π pulses. The simplest example is the 4π pulse which decomposes into a pair of 2π pulses. A pulse with *any* initial area between 3π and 5π will also evolve into a pair of 2π pulses[19] and the final amplitude of each pulse may be determined by using higher conservation laws.[18,23,24] Without resorting to numerical integration the detailed evolution of the pulse breakup can only be followed if the initial area is also 4π. The analytical form of this 4π pulse is that to be obtained presently.

Setting $\nu_1 = ia_1/2$, $\nu_2 = ia_2/2$ with $a_1, a_2 > 0$ and also setting $m_1 = ic_1(\xi)$, $m_2 = ic_2(\xi)$ where $c_1(\xi)$ and $c_2(\xi)$ are real, the determinant of the dyadic $\bar{\bar{V}}$ defined in Eq. (3.18) is of the form

$$|\bar{\bar{V}}| = \begin{vmatrix} 1 + i\gamma_{11} & i\gamma_{12} \\ i\gamma_{21} & 1 + i\gamma_{22} \end{vmatrix}. \tag{4.14}$$

The functions that appear in the elements of this determinant are

$$\gamma_{11} = \frac{c_1}{a_1}e^{a_1\tau}, \quad \gamma_{12} = \frac{2c_1}{a_1 + a_2}e^{(a_1 + a_2)\tau/2}, \quad \gamma_{21} = \frac{2c_2}{a_1 + a_2}e^{(a_1 + a_2)\tau/2}, \quad \gamma_{22} = \frac{c_2}{a_2}e^{a_2\tau}. \tag{4.15}$$

From Eq. (3.26)

$$c_i(\xi) = c_i(0)\exp\left(-a_i\xi\left\langle\frac{1}{a_i^2 + 4\nu^2}\right\rangle\right), \quad i = 1, 2. \tag{4.16}$$

Defining

$$g_{ij} = \frac{a_1 - a_2}{a_1 + a_2} \gamma_{ij} \tag{4.17}$$

and noting that

$$\gamma_{12}\gamma_{21} = 4 \frac{a_1 a_2}{(a_1 + a_2)^2} \gamma_{11}\gamma_{22}, \tag{4.18}$$

one finds that

$$\frac{\mathrm{Im}|\overleftrightarrow{V}|}{\mathrm{Re}|\overleftrightarrow{V}|} = \frac{a_1 + a_2}{a_1 - a_2} \frac{g_{11} + g_{22}}{1 - g_{11} g_{22}}. \tag{4.19}$$

If one now sets $g_{11} = \tan(\sigma_1/4)$ and $g_{22} = -\tan(\sigma_2/4)$, then

$$\sigma = 4 \tan^{-1}\left[\frac{a_1 + a_2}{a_1 - a_2} \tan\left(\frac{\sigma_1 - \sigma_2}{4}\right)\right]. \tag{4.20}$$

Again ignoring phase terms by setting

$$\frac{a_1 - a_2}{a_1 + a_2} \frac{c_i(0)}{a_i} = 1, \quad i = 1, 2 \tag{4.21}$$

one finds that the σ_i in Eq. (4.20) are of the form

$$\sigma_i = 4 \tan^{-1}\left[\exp a_i \left(\tau - \left\langle \frac{1}{a_i^2 + 4\nu^2} \right\rangle \xi \right)\right], \quad i = 1, 2. \tag{4.22}$$

The function σ given in Eq. (4.20) has the same form as that obtained previously[18] for the 4π pulse in the sharp-line limit (all $\Delta\omega = 0$). The result given here, however, applies to the inhomogeneously broadened situation as well.

The electric field profile follows from the differential form of Eq. (2.23). The result can be put in the form[18]

$$\mathcal{E} = A \frac{(2/\tau_1)\,\mathrm{sech}X + (2/\tau_2)\,\mathrm{sech}Y}{1 - B(\tanh X \tanh Y - \mathrm{sech}X\,\mathrm{sech}Y)} \tag{4.23}$$

where

$$A = \frac{\tau_2^2 - \tau_1^2}{\tau_2^2 + \tau_1^2}, \quad B = \frac{2\tau_1\tau_2}{\tau_2^2 + \tau_1^2},$$
$$X = \frac{t - x/V_1}{\tau_1}, \quad Y = \frac{t - x/V_2}{\tau_2}. \tag{4.24}$$

The velocities are

$$\frac{c}{V_i} = 1 + (\Omega\tau_i)^2 \left\langle \frac{1}{1 + (\tau_i \Delta\omega)^2} \right\rangle, \quad i = 1, 2. \tag{4.25}$$

By using the Baecklund transformation, the analytical form that describes the decomposition of a 6π pulse in the sharp-line limit has also been obtained.[18] Presumably, the corresponding solution in the presence of a symmetrically broadened line, i.e., the solution to the inverse problem when three poles are located on the positive imaginary axis, could be written down immediately by merely replacing the terms ξ/a_i^2 in the unbroadened solution by $\xi\langle 1/(a_i^2 + 4\nu^2)\rangle$. The close correspondence between broadened and sharp-line cases is not unexpected since it has been observed in numerical solutions.[19, 22, 30] More recently[31] the Baecklund transformation method has been thoroughly systematized and an example of a 12π pulse presented.

C. Two poles symmetric about imaginary axis—0π pulse

As has been noted previously,[18] there are two distinct types of 0π pulses. One is obtained from the solution for the 4π pulse by merely changing the sign of $c_2(0)$ in Eq. (4.21). The pulse profile then contains regions of negative area. After complete separation, each pulse is in the form of a hyperbolic secant, one with area $+2\pi$, the other with area -2π. The difference in sign merely indicates a shift in phase of 180° of one entire pulse with respect to the other. Such pulses would be attenuated by level degeneracy in the same manner as a single 2π pulse, of course, and hence this type of 0π pulse is not of any great experimental interest.

The second type of 0π pulse remains intact as a single undulating profile as shown in Fig. 1. It provides an alternate and somewhat more complicated version of self-induced transparency. From numerical solutions of the equations of coherent-pulse propagation, it has been found that this solution, which has been calculated for a nondegenerate two-level system, also retains its form and propagates with little loss in level-degenerate systems as well.

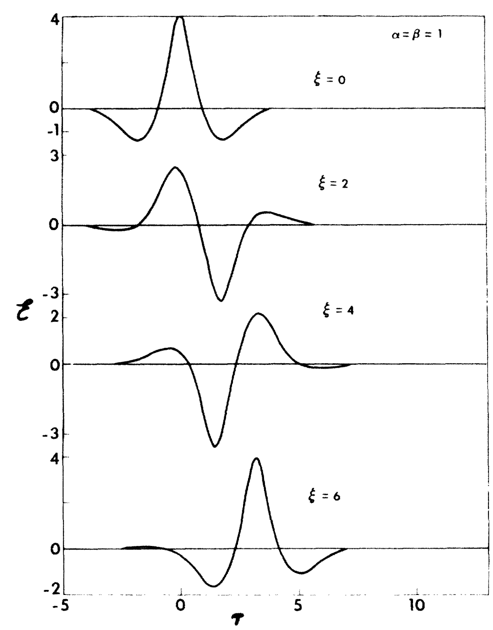

FIG. 1. Envelope of 0π pulse ($t_e = t_s = \Omega^{-1}$).

Experimental confirmation of 0π pulse propagation has thus far been confined to small-amplitude pulses where linear analysis is applicable. Even here the results have been quite remarkable. It has been reported[32] that 65% of the initial pulse energy can be transmitted through 25 e-folding lengths of a level-degenerate absorption cell by using a 0π pulse.

The 0π pulse to be considered here is shown in Fig. 1. It is obtained when the two poles located in the upper half plane are symmetrically located in the first and second quadrants, i.e., $\nu_2 = -\nu_1^*$.

Setting $\nu_1 = \frac{1}{2}(-\beta + i\alpha)$, $\nu_2 = \frac{1}{2}(\beta + i\alpha)$ with $\alpha > 0$, and also requiring $m_2 = -m_1^*$, the real and imaginary parts of $|\tilde{V}|$ are found to be

$$\mathrm{Im}|\tilde{V}| = \frac{1}{2(\alpha^2+\beta^2)}(m_1^*\nu_1 e^{2i\nu_1\tau} + \text{c.c.}), \tag{4.26a}$$

$$\mathrm{Re}|\tilde{V}| = 1 + \frac{\beta^2|m_1|^2 e^{2\alpha\tau}}{\alpha^2(\alpha^2+\beta^2)}. \tag{4.26b}$$

In addition, the spatial dependence of m_1 takes the form

$$m_1(\xi) = m_1(0)e^{(\alpha\Gamma + i\beta\Delta)\xi}, \tag{4.27}$$

where

$$\Gamma \equiv \left\langle \frac{\alpha^2+\beta^2+4\nu^2}{D} \right\rangle, \tag{4.28a}$$

$$\Delta \equiv \left\langle \frac{\alpha^2+\beta^2-4\nu^2}{D} \right\rangle, \tag{4.28b}$$

$$D \equiv [4\nu^2 - (\alpha^2+\beta^2)]^2 + 16\alpha^2\nu^2. \tag{4.29}$$

Introduction of these results into Eq. (3.20) leads to

$$\sigma = 4\tan^{-1}\left(\frac{t_s \sin[(t - x/V_s)/t_s]}{t_e \cosh[(t - x/V_e)/t_e]}\right), \tag{4.30}$$

where $t_e = (\alpha\Omega)^{-1}$, $t_s = (\beta\Omega)^{-1}$. The velocities are

$$c/V_e = 1 + A + (\Omega t_T)^2 B, \tag{4.31a}$$

$$c/V_s = 1 + A - (\Omega t_T)^2 B, \tag{4.31b}$$

where

$$A \equiv \langle 4\nu^2/D \rangle, \tag{4.32a}$$

$$B \equiv \langle (\alpha^2+\beta^2)^2/D \rangle, \tag{4.32b}$$

$$\frac{1}{t_T^2} = \frac{1}{t_e^2} + \frac{1}{t_s^2}. \tag{4.33}$$

A phase term has again been neglected.

The electric field profile follows from the time derivative of σ. The result is[18,1]

$$\mathcal{E} = 4(\Omega t_e)^{-1}\,\mathrm{sech}p\left(\frac{\cos q - (t_s/t_e)\sin q \tanh p}{1 + (t_s/t_e)^2 \sin^2 q\, \mathrm{sech}^2 p}\right), \tag{4.34}$$

where

$$p = \frac{t - x/V_e}{t_e}, \tag{4.35a}$$

$$q = \frac{t - x/V_s}{t_s}. \tag{4.35b}$$

The function $\mathcal{E}(\xi, \tau)$ is shown in Fig. 1. From Eq. (4.34) it is seen that the pulse has a half-width t_e determined by the term sech$p$. The remaining terms contribute the undulating structure shown in the figure. Oscillatory solutions similar to the 0π pulse have also been obtained for the modified Korteweg-deVries equation[17] and the nonlinear Schrödinger equation.[25a]

V. SUMMARY

The inverse method, which has been developed primarily in quantum theory to determine the scattering potential when asymptotic information on the scattering (phase shifts) is specified, has been employed here to determine the coherent-optical-pulse profiles that pass without attenuation through a resonant absorber. The use of the inverse method for this purpose parallels certain recent work on the Korteweg-deVries equation. The method provides a concise cataloging of the possible solutions in terms of singularities in a complex plane. In the present problem the function that plays the role of the potential in a Schrödinger equation is complex. This leads to new types of solutions that do not occur for the Korteweg-deVries equation. The method is applied to pulse shapes that are of recent experimental interest, i.e., 2π, 4π, and 0π pulses. As improvements in experimental techniques are forthcoming, an interest in the more complicated solutions may be expected.

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