Turbulence, critical fluctuations, and intermittency

Mark Nelkin
School of Applied and Engineering Physics, Cornell University, Ithaca, New York 14850
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An explicit analogy is developed between the long-range fluctuations near a critical point and the long-range fluctuations in $\vec{k}$ space in a turbulent fluid in the limit of large Reynolds number. Mathematically the analogy is between the probability functional of the spins on a lattice and the probability functional for the Fourier-transformed velocity fluctuations. The latter is given by the stationary solution of an exact Fokker-Planck equation originally derived by Edwards. This functional, which is not explicitly known, plays the role of a Hamiltonian for stationary turbulence. The limit of zero viscosity is equivalent to the temperature approaching the critical temperature from above. The order parameter is the vorticity-vorticity-correlation function in $\vec{k}$ space, which in three dimensions is the same as the energy spectrum $E(k)$. The mechanism of energy dissipation constrains the critical exponent γ to satisfy $\gamma = 1$. The fluid is stirred at an external length scale k_0^{-1}. If effects due to the nonzero value of k_0 are ignored, the scaling properties of the full probability functional can be explicitly calculated, and lead to the original Kolmogorov theory. This can be characterized by the critical exponents $\nu = 3/4$, $\eta = 2/3$, indicating that the equivalent Hamiltonian is of long range. For $k_0 \neq 0$, the range of fluctuations in $\vec{k}$ space still diverges for zero viscosity. This limit no longer leads to a probability functional with the scale symmetry of a fixed point. We examine the cascade process under the assumption that $\vec{r}$-space correlation functions go as some power of r. The original Kolmogorov theory is recovered by neglecting fluctuations in the cascade process. When these fluctuations are included, correlation functions of different order do not scale in the same way. This is in accord with phenomenological theories of intermittency and with experiment. An effective critical exponent $\eta' = 2/3 + \zeta$ is still defined by the energy spectrum $E(k)$. The parameter ζ is one of a family of intermittency exponents which are intrinsic properties of the Navier-Stokes equations. Their calculation from the dynamical equations remains an objective for future work.

I. INTRODUCTION

The small-scale fluctuations of a strongly turbulent flow are nearly homogeneous and isotropic and nearly independent of the large-scale motions whose instability is the source of the turbulent energy. A quantitative description of these fluctuations is most natural when Fourier transformed to $\vec{k}$ space. As the viscosity is decreased the range of correlations in $\vec{k}$ space becomes increasingly long, diverging in the limit of zero viscosity. Fluctuations of divergent range, whose behavior is independent of the particular anisotropic conditions on the scale of small wave numbers, are reminiscent of $\vec{r}$-space fluctuations near a critical point.

We consider fully developed turbulence as a statistically steady state far from equilibrium. In order to achieve this description mathematically we replace the source of turbulent energy in the large-scale flow by a statistically defined external stirring force. This completely begs the question of the origin of turbulence. Since the small-scale fluctuations are expected to be independent of the details of the large-scale motions, the latter are treated in the most mathematically convenient way without regard to physical realizability. As we see in Sec. II the problem is defined by three parameters plus the underlying Navier–Stokes equations. These parameters are the kinematic viscosity μ, the total rate of energy input per unit mass ϵ, and the characteristic stirring length k_0^{-1} at which energy is put into the fluid.

We concentrate our attention on equal-time correlation functions relating the fluctuating velocity field at varying spatial separation. The simplest of these is the two-point function $\langle u(\vec{x})u(\vec{x}+\vec{r})\rangle$ whose Fourier transform is the energy spectrum $E(k)$, the turbulent energy content per unit mass and per unit wave number. Since the energy is dissipated via viscosity, the Navier–Stokes equations imply[1]

$$\int_0^\infty k^2 E(k)\, dk = \tfrac{1}{2}\epsilon\mu^{-1}. \qquad (1.1)$$

The energy spectrum $E(k)$ is, in fact, a natural choice of order parameter in three dimensions. (For simplicity we stay in three dimensions throughout this paper. Effects of dimensionality are discussed in the Appendix.) If we consider the viscosity μ as analogous to $T-T_c$, and the parameter ϵ to be fixed, then Eq. (1.1) constrains the critical exponent[2] γ to satisfy $\gamma=1$.

Suppose further that for $k \gg k_0$, the spectrum $E(k)$ is independent of k_0. Then on purely dimensional grounds we must have

$$E(k)=\epsilon^{2/3}k^{-5/3}f(k/\xi)\,, \tag{1.2}$$

where

$$\xi=\epsilon^{1/4}\mu^{-3/4}. \tag{1.3}$$

This is just the result first given by Kolmogorov in 1941, and is discussed in many standard references.[1,3] Beyond the dimensional arguments the essential physical idea is that the function $f(x)$ have a finite limit for small x. Thus the energy spectrum in the "inertial subrange" $k_0 \ll k \ll \xi$ goes as the $-\frac{5}{3}$ power of the wave number.

Equations (1.2) and (1.3) should be compared with the corresponding expressions near a critical point in three dimensions,

$$g(r)=r^{-(1+\eta)}f(r/\xi)\,, \tag{1.4}$$

with

$$\xi=\xi_0(T-T_c)^{-\nu}. \tag{1.5}$$

We see that the Kolmogorov theory corresponds to a critical point with the critical exponents $\gamma=1$, $\nu=\frac{3}{4}$, and $\eta=\frac{2}{3}$. The scaling relation $\gamma=(2-\eta)\nu$ is satisfied. The finite value of $f(0)$ in Eq. (1.4) indicates that $g(r)$ has a finite limit as $T-T_c$ goes to zero. The corresponding result in Eq. (1.2) is that $E(k)$ has a finite limit as μ goes to zero. In the case of turbulence this is not proven, but is a basic assumption of our formulation. Up to this point we have argued strictly by analogy. We shall see later that these results arise from an approximate scale symmetry of the dynamical equations.

It was pointed out by Landau (see footnote on page 126 of Ref. 3) that the stochastic nature of the energy cascade made the universal validity of Eq. (1.2) questionable. In 1962 Kolmogorov[4] and Obukhov[5] suggested a modified universal theory taking Landau's objection into account. Their statistical models were given a physical interpretation by Yaglom[6] in 1966, and have since been considerably refined by Novikov.[7,8] The essential feature is that the buildup of fluctuations during the cascade leads to intermittency which increases with increasing wave number. The most striking qualitative feature is a spatial inhomogeneity of the small-scale motions. In more quantitative form the probability distribution of any scale-dependent random variable changes shape as the length scale changes. The smaller the length scale or the larger the value of wave number, the more the probability distribution develops a long tail and a large amplitude near zero, and the less it resembles a Gaussian. The simplest quantitative consequence is that the inertial subrange spectrum takes the revised form

$$E(k)=\epsilon^{2/3}k^{-5/3}(k_0/k)^{\zeta}\,, \tag{1.6}$$

with ζ a small positive number. In the usual phenomenological theory, Eq. (1.6) is obtained as a rather indirect and approximate consequence. We present in Sec. III a modification of Novikov's[8] argument, which gives Eq. (1.6) directly.

There is considerable experimental evidence for the intermittency of turbulent flows.[9] Much of this evidence comes from atmospheric measurements at very large Reynolds number,[10,11] and tends to support the model of Novikov. The measurements are not accurate enough to test Eq. (1.6) directly, but the idealized conditions of isotropic homogeneous turbulence and a well-defined inertial subrange are reasonably well established. The inference of a nonzero value for the parameter ζ thus comes from more complicated statistical properties of the flow than $E(k)$. We take the point of view that the combination of phenomenological theory and experiment gives strong support to Eq. (1.6) with $\zeta\neq 0$, but a more conservative point of view[12] leaves the question completely open.

The key theoretical problem is to understand why the external length scale k_0^{-1} is important. Several authors have shown[13-16] that if the external length scale is taken to infinity from the beginning, then the original Kolmogorov theory follows from the Navier-Stokes equations. In three dimensions the convergence of all needed $\vec{k}$-space integrals has been demonstrated. (Novikov[13] noted that this limiting procedure is inappropriate if intermittency is present.) In the phenomenological theories of Yaglom and others all that matters is that k_0 be different from zero. Its actual numerical value is irrelevant. One might think of ζ in Eq. (1.6) as a modified critical exponent. The parameter k_0 is the small "length" in our problem, and might be considered analogous to a lattice constant in critical phenomena. We will see, however, that intermittency implies that ζ is not a critical exponent in the usual sense.

A theoretical formulation which is useful for the problems described above must have a delicate balance between generality and tractability. It should allow the problem to be formulated in a steady-state form without the need to consider the additional complications of time evolution. Furthermore, we are not interested in details of the mechanism of energy input at small wave numbers. This problem has been nicely solved by Edwards,[15,17] who derived an exact Fokker-Planck equation for the probability functional of the Fourier transform of the velocity field. The external stirring force is taken as a Gaussian white-noise process characterized by a single function $h(k)$ with the property

$$\int_0^{k_0} h(k)\, d^3k = \epsilon L^3 .$$

The fluid is contained in a cube of side L. This choice of stirring force makes ϵ an independent parameter of the problem, unaffected by the response of the fluid to the external force. This arises from the assumed white-noise spectrum of the external force, and is convenient for our study of very large Reynolds-number turbulence. We will further assume that the only important parameters of the function $h(k)$ are ϵ and k_0.

To formalize the analogy with equilibrium critical phenomena we need a "Hamiltonian" for the turbulence problem. This is obtained almost trivially when we recognize that the Hamiltonian in equilibrium statistical mechanics is by definition what appears in the expression

$$P = e^{-H/k_B T} .$$

The Hamiltonian is just a convenient way to express the probability functional for the full set of spins on a lattice, and need not have any dynamical meaning. In particular, we are interested, not in the Hamiltonian itself, but in its scaling properties under a renormalization-group transformation[18] near the critical point. The Hamiltonian for our turbulence problem is just the logarithm of the probability functional given by the solution of the Edwards exact Fokker–Planck equation. We do not know this Hamiltonian explicitly, but can examine its scaling properties.

In Sec. II we write down the Fokker–Planck equation, and examine its behavior under a transformation which scales wave numbers and velocities. In order to constrain the solution to describe a steady state far from equilibrium, this transformation must be carried out at fixed ϵ. The scale symmetry of the equation is broken by the viscosity and by the nonzero value of k_0. If the latter is neglected, the dependence on viscosity is easily calculated. When $\mu = 0$, the probability functional is invariant under the transformation

$$k' = ck, \quad u'_{k'} = c^{-10/3} u_k , \tag{1.7}$$

thus describing a fixed point of the renormalization-group transformation. The energy spectrum at the fixed point goes as $k^{-5/3}$ so that the critical exponent η takes the value $\eta = \frac{2}{3}$. Examining the behavior of the probability functional in the neighborhood of the fixed point yields $\nu = \frac{3}{4}$. Thus the original Kolmogorov theory of Eqs. (1.2) and (1.3) is recovered with the constraint of Eq. (1.1) satisfied. The large value of η indicates that the effective Hamiltonian is of long range[19] in $\vec{k}$ space. This is consistent with the slow convergence of the $\vec{k}$-space integrals which appear in the Kolmogorov theory.[14]

If k_0 is not zero, we can measure k in units of k_0, and the appropriate dimensionless viscosity is the reciprocal of the Reynolds number

$$R = \mu^{-1} k_0^{-4/3} \epsilon^{1/3} .$$

As the viscosity goes to zero the range of correlations in $\vec{k}$ space diverges, independent of the value of k_0. The analogy with critical phenomena suggests that the correlation "length" should be of the form $\xi = k_0 R^{\nu}$, with the exponent ν not restricted to have the value $\frac{3}{4}$.

In Sec. III we argue that the probability functional $P(\cdots u_k \cdots)$ does not have the scaling symmetry appropriate to a fixed point in the limit of infinite Reynolds number. We do this by examining the probability distribution $p(y(r))$, where $y(r)$ is the difference in velocity between spatial points separated by a distance r. We study the moments $\langle y^n(r)\rangle$, and assume that they go as some power of r. Following Novikov[8] these moments can be related to properties of the probability distribution $p_c(q)$, where $q = y(r)/y(c^{-1}r)$. The third moment of $p_c(q)$ is restricted by the energy-conserving property of the inertial terms in the Navier–Stokes equation. The original Kolmogorov theory is obtained if $p_c(q)$ is replaced by a δ function. For any nonsingular $p_c(q)$, the behavior of $p(y(r))$ is not consistent with the scale symmetry of a fixed point.

We are not able to relate the statistical arguments of Sec. III to the dynamical arguments of Sec. II. We cannot therefore be sure that $p_c(q)$ is nonsingular. The essential physical feature of strong turbulence is, however, the tendency of the nonlinear terms to amplify fluctuations. Thus the cascade process driven by these nonlinearities appears as a random process even though the underlying Navier–Stokes equations are deterministic. It is thus extremely plausible that $p_c(q)$ is a nonsingular probability distribution. To derive this from the dynamical equations involves a fundamentally deeper understanding of the theory than has yet been achieved.

The fluctuations in the cascade process from small to large wave number have no obvious analog in critical phenomena. These fluctuations are manifest through a set of intermittency parameters ζ_n which describe the difference between the Kolmogorov values of $\langle y^n(r)\rangle$ and the actual ones. In some sense these can be thought of as fluctuation corrections to a "Kolmogorov mean-field theory," and they do appear as changes in certain exponents in power-law expressions for correlation functions.[20] On the other hand, they represent a breakdown of scale symmetry in a regime of infinite-range correlations. The parameters ζ_n can, in principle, be calculated from the formalism of Sec. II. Until such a connection has been established the relation of these parameters to parameters of other physical problems remains conjectural. In particular, we have relied heavily on our intuitive understanding of turbulence in three dimensions. Whether these fluctuation corrections vanish in some higher dimensionality is not known.

II. KOLMOGOROV CRITICAL POINT

Consider an incompressible fluid which satisfies the Navier–Stokes equations

$$\frac{\partial \vec{U}}{\partial t} = \mu \nabla^2 \vec{U} - (\vec{U}\cdot\vec{\nabla})\vec{U} - \vec{\nabla}p + \vec{f} , \qquad (2.1)$$

where μ is the kinematic viscosity, p the pressure, and $\vec{f}(r,t)$ an externally applied body force. With the aid of the incompressibility condition

$$\vec{\nabla}\cdot\vec{U} = 0 , \qquad (2.2)$$

the pressure can be eliminated in favor of the velocity field $\vec{U}(r,t)$. It is more convenient to work with the Fourier components in a box of side L, and apply periodic boundary conditions

$$\vec{U}(\vec{r},t) = L^{-3}\sum_k e^{-i\vec{k}\cdot\vec{r}}\vec{U}_k(t) , \qquad (2.3)$$

where $\vec{k} = (2\pi/L)(n_1, n_2, n_3)$. Following Ref. 15 we transform Eq. (2.1) to the form

$$\frac{\partial U_k^\alpha}{\partial t} = -\mu k^2 U_k^\alpha + \sum_{j,l,\beta,\gamma} M_{kjl}^{\alpha\beta\gamma} U_j^\gamma U_l^\beta + \sum_\beta f_k^\beta D_k^{\alpha\beta} , \qquad (2.4)$$

where the condition $k^\alpha U_k^\alpha = 0$ has been used to eliminate the pressure. The kinematic factor $D_k^{\alpha\beta}$ restricts the coupling to transverse velocity fluctuations and is given by

$$D_k^{\alpha\beta} = \delta^{\alpha\beta} - k^\alpha k^\beta (k)^{-2} . \qquad (2.5)$$

The mode-coupling term M in Eq. (2.4) is given by

$$M_{kjl}^{\alpha\beta\gamma} = -iL^{-3}(k^\beta D_k^{\alpha\gamma} + k^\gamma D_k^{\alpha\beta})\delta_{kjl} , \qquad (2.6)$$

where $\delta_{kjl} = 1$ if $\vec{k}+\vec{j}+\vec{l} = 0$, and zero otherwise. Equation (2.4) is too general to be useful for our problem. We specialize to the case where the random force $f_k^\alpha(t)$ is a Gaussian white-noise process with correlation function

$$\langle f_k^\alpha(t) f_k^\beta(t')\rangle = 2h_k\,\delta(t-t')\delta^{\alpha\beta} . \qquad (2.7)$$

With this specialization the nonlinear Langevin equation, Eq. (2.4), can be transformed by standard methods to a linear Fokker–Planck equation for the probability $P(\cdots u_k \cdots t)$ of finding the U_k to have the values of u_k at time t:

$$\frac{\partial P}{\partial t} + \sum_{k,\alpha}\frac{\partial}{\partial u_k^\alpha}\left(\mu k^2 u_k^\alpha - \sum_{j,l,\beta,\gamma} M_{kjl}^{\alpha\beta\gamma} u_j^\beta u_l^\gamma\right) P + \sum_{k,\alpha} h_k \frac{\partial^2 P}{2 u_k^\alpha u_{-k}^\alpha} = 0 . \qquad (2.8)$$

Equation (2.8) has a strong formal similarity to the Fokker–Planck equation used in the mode-mode coupling theory of dynamical critical phenomena,[21] or in the theory of the long-time tails of equilibrium time-correlation functions.[22] This similarity is, however, only formal because in the present case we are far from thermal equilibrium. At thermal equilibrium the fluctuation-dissipation theorem $h_k = \mu k^2 \times (k_B T)$ applies. This leads to a stationary solution of Eq. (2.8) which is of the equipartition form

$$P_{eq} = \exp\left(-(k_B T)^{-1}\sum_{k,\alpha} u_k^\alpha u_{-k}^\alpha\right)$$

even in the presence of the mode-coupling terms. At thermal equilibrium the physical interest is in the time dependence of $P(\cdots u_k \cdots t)$. In the turbulence problem, on the other hand, we consider a steady state far from equilibrium. The forcing term h_k differs from zero only in a small region of $\vec{k}$ space, $k < k_0$. The viscous term is by contrast important only for large values of k. Thus the energy enters the system at small k, cascades to large k through the mode-coupling terms, and is dissipated at large k. Our entire problem reduces to studying the time-independent solution of Eq. (2.8) under these conditions. More accurately, it is not the stationary probability functional $P(\cdots u_k \cdots)$ which is of physical interest, but the reduced probability functions and correlation functions obtained from it by functional integration. It should be noted that turbulence measurements[10,11] record realizations of the random process $\vec{U}(\vec{r},t)$, and are thus capable of yielding much more detailed statistical information than is normally obtained for equilibrium fluctuations. This leads eventually to important differences in the kind of theoretical questions that one asks.

Of particular importance is the condition of overall energy balance obtained by introducing

$$D_k^{\alpha\beta} q_k = \int u_k^\alpha u_{-k}^\beta P\,\delta\boldsymbol{u} . \qquad (2.9)$$

The energy-balance expression is

$$\mu\int_0^\infty k^2 q_k\, d^3k = \int_0^{k_0} h_k\, d^3k = \epsilon L^3 . \qquad (2.10)$$

The energy spectrum $E(k)$ is given by

$$E(k) = 2\pi k^2 q_k L^{-3}. \tag{2.11}$$

With this identification, Eq. (2.10) will be recognized as Eq. (1.1).

To study the scaling properties of Eq. (2.8) we replace the sums over wave numbers by integrals. The stationary solution satisfies

$$(V+I+D)P=0, \tag{2.12}$$

where the viscous, inertial, and driving terms are given by

$$VP = \mu \int d^3k\, k^2 \frac{\partial}{\partial u_k^\alpha} (u_k^\alpha P), \tag{2.13}$$

$$IP = \iiint d^3k\, d^3j\, d^3l\, m^{\alpha\beta\gamma}(\vec{k}, \vec{j}, \vec{l}) \frac{\partial}{\partial u_k^\alpha} (u_j^\beta u_l^\gamma P), \tag{2.14}$$

$$DP = \int_0^{k_0} d^3k\, h(k) \frac{\partial^2 P}{\partial u_k^\alpha \partial u_{-k}^\alpha}, \tag{2.15}$$

and

$$m^{\alpha\beta\gamma}(\vec{k}, \vec{j}, \vec{l}) = i(2\pi)^{-3}(k^\beta D_k^{\alpha\gamma} + k^\gamma D_k^{\alpha\beta})\delta(\vec{k}+\vec{j}+\vec{l}). \tag{2.16}$$

A summation convention for Cartesian components has been introduced.

Consider now the renormalization-group transformation[23]

$$k' = ck, \quad u_k = c^x u'_{ck}. \tag{2.17}$$

When expressed in terms of the primed variables the viscous term VP retains the same form but is multiplied by c^{-5}. The inertial term also retains the same form but is multiplied by $c^{(-7+x)}$. The driving term is not form invariant since the integral now extends to ck_0. To deal with a steady-state turbulent problem far from equilibrium we must scale $h(k)$ so that the energy input ϵ per unit volume is constant. Thus

$$h(k) = c^6 h'(k')$$

and

$$\int_0^{k_0} h(k)\, d^3k = c^3 \int_0^{ck_0} h'(k')\, d^3k' = \epsilon L^3 = c^3 \epsilon L'^3. \tag{2.18}$$

With this scaling of $h(k)$, the driving term in the scaled variables becomes

$$c^{(3-2x)} \int_0^{ck_0} h'(k')\, d^3k' \frac{\partial^2 P}{\partial u'^\alpha_{k'} \partial u'^\alpha_{-k'}}. \tag{2.19}$$

If we neglect the violation of scale symmetry due to the upper limit of the integral in Eq. (2.19) then the inertial terms and driving terms scale together as $c^{-11/3}$ when $x = \frac{10}{3}$. The limit of zero viscosity thus defines a fixed point of the renormalization-group transformation when $k_0 = 0$. The scaling property

$$k' = ck, \quad u_k = c^{10/3} u'_{ck} \tag{2.20}$$

defines the statistically self-similar probability functional[11] as it appears in generalizations of the original Kolmogorov theory to include more refined statistical properties. In particular, it leads to $E(k)$ proportional to $k^{-5/3}$ in the inertial subrange and corresponds to the critical exponent $\eta = \frac{2}{3}$.

The departure from the fixed point due to viscosity is easily included. Consider the correlation length $\xi(\mu, k_0)$. When $k_0 = 0$ we have from the scaling properties discussed above that

$$\xi(c^{-4/3}\mu, 0) = c\xi(\mu, 0). \tag{2.21}$$

Thus

$$\xi = \xi_0 \mu^{-3/4} \tag{2.22}$$

and the critical exponent $\nu = \frac{3}{4}$. Since we are working in the neighborhood of a fixed point, Eq. (1.2) is applicable.

Consider now the effects of a finite external length scale. When $k_0 \neq 0$, the only conclusion that we can reach from the scaling properties of P is that

$$\xi(c^{-3/4}\mu, ck_0) = c\xi(\mu, k_0).$$

This has the solution

$$\xi = k_0 F(R),$$

where R is the Reynolds number and $F(R)$ is an arbitrary function. We would normally assume $F(R) = R^\nu$ so that

$$\xi = \mu^{-\nu} \epsilon^{\nu/3} k_0^{(1-4\nu/3)}.$$

Note that only in the case $\nu = \frac{3}{4}$ is ξ independent of k_0. There is no *a priori* reason to assume $\nu = \frac{3}{4}$.

It is physically clear that the range of correlations diverges as R goes to infinity whatever the value of k_0. It is thus tempting to assume that the limit $R \to \infty$ defines a fixed point. Using Eq. (1.1) we would then have

$$E(k) = \epsilon^{2/3} k^{-5/3} (k_0/k)^\zeta f(k/k_0 R^\nu), \tag{2.23}$$

with the exponents ζ and ν related through

$$(\tfrac{4}{3} - \zeta)\nu = 1.$$

Equation (2.23) is attractively simple, and seems to extend the analogy with critical phenomena beyond the original Kolmogorov theory. We shall see, however, that Eq. (2.23) with $\zeta \neq 0$ is unlikely to be correct though Eq. (1.6) with $\zeta \neq 0$ is probably true. This leads us to the question of intermittency.

III. INTERMITTENCY

It is convenient to discuss intermittency in terms of a directly measurable probability distribution. Our discussion will be in $\vec{r}$ space, but can be translated to $\vec{k}$ space where necessary. Introduce the random variable

$$y(r)=u(\vec{x}+\vec{r})-u(\vec{x}), \tag{3.1}$$

where u is the component of the velocity along $\vec{r}$. The moments of $y(r)$,

$$B_n(r)=\langle y^n(r)\rangle=\int y^n(r)p(y(r))\,dy(r), \tag{3.2}$$

are known as the nth-order longitudinal-structure functions, and have been studied experimentally[11] under conditions closely approximating the inertial subrange of isotropic homogeneous turbulence.

In the original Kolmogorov theory the probability distribution $p(y(r))$ has the same form at all values of r if the random variable $y(r)$ is scaled as $r^{1/3}$. This leads to the result

$$B_n(r)=C_n(\epsilon r)^{n/3}\quad(k_0=0,\ \mu=0), \tag{3.3}$$

which follows from simple dimensional arguments if ϵ is the only relevant parameter.

We consider now the effects of finite k_0 in the inertial subrange

$$\xi^{-1}\ll r<l\ll k_0^{-1}.$$

The length l is not a parameter of the problem, but just a reminder that the expected power-law behavior

$$B_n(r)=C_n(\epsilon r)^{n/3}(k_0 r)^{\zeta_n} \tag{3.4}$$

does not extend to distances of the order of the external length scale. In Eq. (3.4), C_n is a dimensionless constant, and the dependence on ϵ and k_0 is required by dimensional arguments. The energy spectrum $E(k)$ is the Fourier transform[1] of of $B_2(r)$ so ζ_2 is the same as the parameter ζ introduced in Eq. (1.6). The parameter ζ_3 must be identically zero. This is shown in Sec. 33 of Ref. 3. The equivalent statement in $\vec{k}$ space is that the third-order cumulant

$$T^{(3)}(\vec{p},\vec{q})=\langle u_p u_q u_{-p-q}\rangle$$

must have the scaling property

$$T^{(3)}(c\vec{p},c\vec{q})=c^{-7}T^{(3)}(\vec{p},\vec{q}).$$

This is required[14] in order that the total energy cascading past wave number K be independent of K.

To show that $\zeta_2,\zeta_4,\zeta_5,\ldots$ are not zero, consider the following argument, adapted from Novikov.[8] Introduce a parameter $c<1$, and an integer N such that $(r/l)=c^N$. Introduce the random variables

$$q_j=y(c^j l)/y(c^{j-1}l),\quad j=1,\ldots,N. \tag{3.5}$$

In particular,

$$q_N=y(r)/y(c^{-1}r).$$

The structure function $B_n(r)$ is given by

$$B_n(r)=\langle y^n(l)q_1^n q_2^n q_3^n\cdots q_N^n\rangle. \tag{3.6}$$

In order for $B_n(r)$ to have a power-law form for all $r<l$, and arbitrary c, the q_j must be independently and identically distributed with probability distribution $p_c(q)$ depending only on c. We further require the dependence on c to be constrained so that

$$\langle q^n\rangle=\int q^n p_c(q)\,dq=c^{\mu_n}. \tag{3.7}$$

Using Eq. (3.7) in an appropriately factored Eq. (3.6) and recalling that $c^N=(r/l)$ we obtain

$$B_n(r)=\langle y^n(l)\rangle(r/l)^{\mu_n}. \tag{3.8}$$

If we define

$$\zeta_n=\mu_n-\tfrac{1}{3}n, \tag{3.9}$$

and recall that the final result cannot depend on the arbitrarily chosen length l, we obtain Eq. (3.4). Spatial homogeneity requires $B_1(r)=0$, which follows from $\langle y(l)\rangle=0$, and does not constrain μ_1.

The contraint that $\zeta_3=0$ corresponds to

$$\int q^3 p_c(q)\,dq=c. \tag{3.10}$$

To get the original Kolmogorov theory we require that

$$\int q^n p_c(q)\,dq=c^{n/3}$$

for all n. This would only occur if

$$p_c(q)=\delta(q-c^{1/3}).$$

Thus the Kolmogorov theory corresponds to neglecting fluctuations during the cascade process. Including such fluctuations leads to nonzero values for the exponents ζ_n for $n\neq 3$.

Our argument has certain differences from that of Novikov.[8] Instead of the variable $y(r)$ he considers an intrinsically positive random variable such as $\epsilon(r)$, the viscous dissipation averaged over a region of length r. For a variable of this type he obtains upper bounds on the exponents corresponding to μ_n. He can also construct the characteristic function for the logarithm of the corresponding "breakdown coefficient" $\epsilon(cr)/\epsilon(r)$. This allows him to explicitly construct $p_c(q)$ for any value of c if it is known for one value of c. In our case the random variables $y(r)$ and q can take on negative values. We can prove less about the probability distributions of interest. On the other hand, we study the structure function $B_n(r)$ directly where Novikov assumes[4,6] that it is proportional to $\langle[\epsilon(r)]^{n/3}\rangle$. This latter assumption is probably wrong. There is no physical reason why the statistical behavior of the inertial subrange should depend on the mechanism of energy dissipation at smaller scales. A random variable which relates simply to the structure function $B_n(r)$ should refer to energy transfer by the nonlinear inertial terms, not to viscous dissipation.

Earlier log-normal models of intermittency[4-6] give $\zeta_n = bn(n-3)$, with b an arbitrary constant. From Novikov's work we now know that this dependence on n is too simple. There is no reason to expect any simple *a priori* relation among the various parameters ζ_n. To get a very rough numerical estimate we guess that $p_{1/2}(q)$ is a constant for $\alpha < q < 1$ and zero elsewhere. Using Eqs. (3.7), (3.9), and (3.10) we find $\alpha = 0.544$, $\zeta_2 = +0.038$, and $\zeta_4 = -0.073$. These are qualitatively reasonable when compared to experiment,[11] but such simple numerical games tell us only that the elementary physical ideas involved are not obviously wrong.

Our principal result is a departure from the statistical self-similarity[11] of $p(y(r))$ in the inertial subrange. As r is decreased the probability distribution $p(y)$ does not maintain the same shape after a scaling of y. When translated back into $\vec{k}$ space this means that the probability functional studied in Sec. II does not satisfy scale symmetry in the limit of infinite Reynolds number even though the range of correlations in $\vec{k}$ space becomes infinite. Correlation functions retain their power law form, and an exponent

$$\eta' = \tfrac{2}{3} + \zeta_2$$

can be defined. An earlier speculation[20] that ζ_2 is related to a fluctuation correction to a "Kolmogorov mean-field theory" has thus been put in more precise form. This correction seems to differ, however, in important ways from corrections to mean-field theory in critical phenomena. The random cascade of energy from small to large wave number in a fluid far from equilibrium has no direct analog in the behavior of equilibrium correlation functions.

Since we no longer have a fixed point at infinite Reynolds number, the usual analyticity assumption of the renormalization-group approach is not applicable. Equation (2.23), which follows naturally[23] from an expansion about a fixed point, has no theoretical support in the turbulence problem. It cannot, however, be categorically excluded. The degree of analogy between turbulence and critical fluctuations remains unclear. We do not know whether the scaling of Eq. (2.23) is valid, whether the fluctuation corrections due to intermittency vanish in high enough dimensionality, or whether the renormalization-group approach can be modified to apply to the turbulence problem.

Finally there remains the problem of calculating the intermittency parameters ζ_n from the Navier-Stokes equations. There now seems little doubt that these parameters are intrinsic features of the dynamics, and not artifacts of the boundary conditions. The formulation given in Sec. II seems sufficiently general to allow such a calculation in principle. The logical structure of the problem seems clear. We know at what formal level to pose the problem. We know the parameters which characterize the solution. It remains to devise techniques which allow these parameters to be calculated from the dynamical equations.

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APPENDIX

Extending the discussion of the text to dimensionality $d > 3$ does not lead to any immediately interesting results. We take as the order parameter the vorticity correlation function

$$\langle \Omega\Omega\rangle = k^2\langle u\,u\rangle = k^{3-d}\,E(k).$$

The dimensional arguments of the original Kolmogorov theory give an inertial subrange spectrum $E(k)$ going as $k^{-5/3}$ in d dimensions. The order parameter in the inertial subrange thus goes as

$$\langle\Omega\Omega\rangle = \Omega_0 k^{-(d-2+\eta)},$$

with $\eta = \frac{2}{3}$. In critical phenomena with a long-range potential

$$V(r) = v_0 r^{-(d+\sigma)},$$

it is known[19] that $\eta = 2-\sigma$. The Kolmogorov theory corresponds to $\sigma = \frac{4}{3}$. The physical significance of this observation is obscure.

The condition of energy balance between driving term and viscous dissipation in d dimensions becomes

$$\int \langle\Omega\Omega\rangle\, k^{d-1}\,dk = C\mu^{-1},$$

so that the Kolmogorov theory continues to give $\gamma = 1$, $\nu = \frac{3}{4}$. Nothing is known about intermittency corrections for $d > 3$.

The inertial subrange in two dimensions has special properties because the inviscid Navier–Stokes equations conserve both energy and vorticity. An energy cascade to large wave numbers is not possible, but it is possible to have an enstrophy (mean-square vorticity) cascade.[25] If the inertial subrange spectrum were to be independent of the stirring length and depend only on the rate at which enstrophy cascades, then the usual dimensional arguments give $E(k)$ proportional to k^{-3}. The problem has been clearly outlined by Kraichnan,[25] and is the subject of considerable current interest. In two dimensions there are known to be divergence difficulties[16,25] at small k. Thus it is not possible to take $k_0 = 0$ from the beginning and get a self-consistent result. We expect that intermittency corrections will modify the k^{-3} energy spectrum in two dimensions more strongly than they modify the $k^{-5/3}$ spectrum in three dimensions. The problem is clearly deserving of further study.

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