Critical behavior of an Ising model of classical spins in a transverse field

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The critical behavior of an Ising model of classical spins in a transverse field H is derived for dimension $d = 4 - \epsilon$ using an ϵ expansion based on the Wilson renormalization group. To first order in ϵ the model exhibits a line of critical points $T_c(H)$ with the same fixed point and critical exponents as in previous studies of the Wilson effective Hamiltonian.

INTRODUCTION

Recently, the critical exponents for the classical Ising and n-vector models were derived for dimension $d=4-\epsilon$ using an ϵ expansion based on the Wilson renormalization group.[1-4] Here we present a similar derivation, valid to order ϵ, for an Ising-like model of classical spins $\vec{\sigma}$ in a transverse field.

The model we consider has the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}J \sum_{\vec{R},\vec{\delta}} \sigma_1(\vec{R})\sigma_1(\vec{R}+\vec{\delta}) - H\sum_{\vec{R}}\sigma_n(\vec{R}) . \qquad (1)$$

$\vec{R}$ denotes the position of a point in the lattice and $\vec{R}+\vec{\delta}$ is the position of a neighboring lattice point. Each spin is coupled to its neighbors through the component σ_1 and to the transverse field H through the component σ_n. This Hamiltonian is of interest as a model of a ferromagnet with strong uniaxial anisotropy in a transverse magnetic field. Also, the case $d=n=3$ with Pauli spin operators instead of classical spins occurs in pseudospin formalisms for a variety of physical systems,[5-8] as, for example, in theories of rare-earth magnetism,[5,6] where H represents a crystal field, and in theories of displacive-ferroelectric transitions,[7] where H is related to the tunneling frequency. An extensive list of these and other systems is given by Stinchcombe.[8] Various properties of the model have been examined using mean-field or random-phase approximations,[5-8] expansions in the reciprocal of the lattice-coordination number,[8] expansions[9] in powers of $1/n$, and expansions[10] in H/J or J/H.

To investigate the critical behavior of the above model, we obtain a Hamiltonian of the form discussed by Wilson by integrating over the variables $\sigma_2 \cdots \sigma_n$ in the partition function, keeping contributions of order ϵ. We are then in a position to apply the results of previous studies of the Wilson Hamiltonian based on the ϵ expansion.[1-4] The calculation yields a line of critical points $T_c(H)$ with the same fixed point and critical exponents as in the previous studies. Similar results have been obtained by Suzuki[9] with the $1/n$ expansion and by Elliott, Pfeuty, and Wood with a series expansion method.[10]

CRITICAL BEHAVIOR OF THE MODEL

Choosing a phase-space factor[1-4] of the form $\exp(-\frac{1}{2}\Gamma\vec{\sigma}^2 - \frac{1}{4}\Delta\vec{\sigma}^4)$ where Δ is of order ϵ, we write the partition function in the form

$$Z = \int_{-\infty}^{\infty} \prod_{\vec{R}} [d\sigma_1(\vec{R}) \cdots d\sigma_n(\vec{R}) \times e^{-\Gamma\vec{\sigma}(\vec{R})^2/2 - \Delta\vec{\sigma}(\vec{R})^4/4}] e^{-\beta\mathcal{H}} . \qquad (2)$$

Expanding Z to first order in Δ and then integrating over $\sigma_2 \cdots \sigma_n$, one may readily compute $\mathcal{H}(\sigma_1)$ defined by

$$Z = \int_{-\infty}^{\infty} \prod_{\vec{R}} [d\sigma_1(\vec{R})] e^{-\beta\mathcal{H}(\sigma_1)} . \qquad (3)$$

A Hamiltonian of the type discussed by Wilson is obtained from $\mathcal{H}(\sigma_1)$ by introducing[4,11] Fourier transforms $\sigma(\vec{q})$ of the spin variables $\sigma_1(\vec{R})$,

$$-\beta\mathcal{H}_0 = -\frac{1}{2}\int_{\vec{q}} (q^2 + r_0)\sigma(\vec{q})\sigma(-\vec{q}) - u_0 \int_{\vec{q}}\int_{\vec{q}\,'}\int_{\vec{q}\,''} \sigma(\vec{q})\sigma(\vec{q}\,')\sigma(\vec{q}\,'') \times \sigma(-\vec{q}-\vec{q}\,'-\vec{q}\,'') + F(\Gamma, \Delta, \beta H) . \qquad (4)$$

The spin variable $\sigma(\vec{q})$ has been scaled to yield the coefficient of q^2 shown in Eq. (4) with the $\vec{q}$ integration[4,11] over the interval $0 < |\vec{q}| < 1$ instead of $-\pi < q_\alpha < \pi$. The quantities r_0 and u_0 are given by

$$r_0 = \pi^{-2}\frac{2d}{Q\beta J}\left[\Gamma - Q\beta J + \Delta\left(\frac{n-1}{\Gamma} + \frac{(\beta H)^2}{\Gamma^2}\right)\right] ,$$
$$u_0 = \pi^{d-4}(d/Q\beta J)^2 \Delta , \qquad (5)$$

where Q is the coordination number of the lattice. The explicit form of F, which has no spin dependence, will not be needed.

In the renormalization group procedure[1-4] a sequence of Hamiltonians is generated from the initial Hamiltonian. The parameters r_l and u_l, which characterize the sequence after enough iterations are performed so that irrelevant parameters are no longer involved, approach fixed-point values if the starting parameters are chosen so that r_0 is at its appropriate critical value r_{0c}. As discussed by Wilson and Fisher,[1,4] the Hamiltonian of Eq. (4) has a stable Gaussian fixed point for $d>4$ and a stable non-Gaussian fixed point for $d<4$. Taking over these results, one finds that to first order in ϵ and Δ the system we are considering has the same fixed point and critical exponents as discussed by Wilson and Fisher for a line of critical points.

If one neglects irrelevant parameters representing corrections to Eq. (4), the line of critical points is determined by the criticality condition $r_0 = r_{0c}(u_0)$. To obtain an expression for $\beta_c(H)$ in terms of interaction parameters and phase-space factors, one needs to know $r_{0c}(u_0)$ explicitly. However, even without this information an expression for the change in β_c due to the transverse field can be derived to first order in ϵ and Δ. Assuming that r_{0c} and Δ are of order ϵ and that $r_{0c}(u_0)$ can be expanded about its value for zero field, one finds

$$\beta_c(H) - \beta_c(0) = \Delta H^2/(QJ)^3 \,. \qquad (6)$$

Unfortunately, it is not clear how to relate Δ to the physical parameters of a system with discrete rather than continuous spins.

On the basis of a random-phase calculation with $d=n=3$ and with quantum-mechanical spins, Wang and Cooper[5,6] suggest the existence of a first-order transition in the system we are considering. Within the limits of our calculation we see no evidence for a first-order transition,[12] a result which is consistent with Refs. 9 and 10.

In concluding, we point out that these results may readily be extended to a more general class of Hamiltonians with the form

$$\mathcal{H} = -\frac{1}{2}\sum_{\alpha=1}^{n-1}\sum_{\vec{R},\vec{\delta}} J_\alpha \sigma_\alpha(\vec{R})\sigma_\alpha(\vec{R}+\vec{\delta}) - H\sum_{\vec{R}} \sigma_n(\vec{R}) \,. \qquad (7)$$

Since the coupling between neighboring spins is independent of σ_n, the σ_n integration in the partition function of Eq. (2) may be carried out to first order in Δ to yield an $(n-1)$-component effective Hamiltonian of the form discussed by Fisher and Pfeuty.[2] One obtains a line of critical points with the same fixed point and critical exponents as in the $n-1$ vector model.

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[12] The critical properties of the Hamiltonian of Eq. (1) can readily be calculated in the spherical approximation [G. S. Joyce, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), p. 375] for all values of n and d. One evaluates the partition function of Eq. (2) exactly in the case $\Delta = 0$ and chooses Γ to be a function of β and H so that $\langle\vec{\sigma}^2\rangle$ is normalized to a constant value. The calculation yields the usual spherical exponents (Joyce) and a critical line $\beta_c(0)/\beta_c(H) = 1 - H^2/\langle\vec{\sigma}^2\rangle(QJ)^2$. There is no first-order transition.