Very short optical pulses: A new approximation method*

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(Received 13 December 1972; revised manuscript received 31 December 1973)

Very short optical pulses, traveling in sharp-line resonant absorbers without change of shape, are investigated with a new approximation technique. Instead of employing the long-pulse or slowly varying envelope assumption, we have developed a method of approximation based on a series expansion in powers of a small parameter which is related to the electric field strength. The principal advantages of the method are its ability to treat very short pulses, and the simplicity with which it can be extended, both from lower to higher levels of approximation, and from one type of steady-state pulse to another. In this paper we use the approximation method to study very short pulses which require retention of the second-derivative terms in the Maxwell envelope and phase equations. Corrections to the well-known McCall-Hahn results are easily found to first order in the small parameter. As expected, the corrections begin to be important only for picosecond and shorter pulses.

I. INTRODUCTION

In the seven years since McCall and Hahn[1] first advanced their theory of self-induced transparency (i.e., of lossless optical pulse propagation in a resonant absorbing medium), a number of experimental confirmations of the theory have been reported.[2-4] It is clear from this experimental work that the theory gives an excellent description of a highly nonlinear phenomenon, at least when the optical pulse length τ is five to six orders of magnitude longer than an optical period $2\pi/\omega$, and also 10–100 times longer than T_2^*, the resonant absorber's inhomogeneous lifetime.[1]

However, pulses which are much shorter than those available in 1965 are commonly used in experimental work now, and Gibbs and Slusher have reported[5] results obtained in a "sharp-line" experiment, in which $\tau \sim T_2^*$. Thus it seems desirable to investigate the modifications to the McCall-Hahn results which very short pulses may require. We report such an investigation in this paper.[6]

Since the McCall-Hahn theory of optical pulses is not exact, but is based on the "slowly varying envelope" assumptions,[1,7] it will almost certainly fail for pulses as short as a few optical cycles. What is more interesting is that there is some reason to suspect that modifications may be appropriate even for pulses as long as many thousands of optical cycles, if the pulses are shorter than a certain coherence or "cooperation" time[8,9] which we denote by τ_c.

The issue of pulses shorter than τ_c but longer than $2\pi/\omega$ is not entirely academic; τ_c is actually fairly "long" by present standards, being in the neighborhood of 0.1 nsec for a wide variety of resonant absorbers, and within the limits of present experimental capabilities in several cases.

Thus, in this paper we will have in mind "sharp-line"[5] pulses, those which satisfy $2\pi/\omega < \tau < T_2^* \ll T_2'$, and we will pay particular attention to those pulses which are shorter than τ_c. Just how much shorter than τ_c and longer than ω^{-1} will depend on the individual circumstance. We will occasionally refer to the time domains sketched in Fig. 1. With regard to the six regions or time domains shown there, pulses with lengths falling into regions I-V may propagate coherently with small energy loss. Pulses with pulse length greater than T_2' (pulses in region VI) will be disrupted by homogeneous damping effects. Note that pulses in region I can be too short to be well described within a slowly varying envelope approximation.

This paper is organized as follows. In Sec. II we describe the approximations with which we will work. In Sec. III we begin to analyze the basic quantum optical-pulse equations. Our analysis differs from previous studies because we retain all of the second-derivative terms in the Maxwell wave equation. For this reason we have adopted the term "second order" to distinguish our work from the "first-order" investigations of earlier workers,[7] who commonly used a slowly varying envelope approximation to justify the neglect of

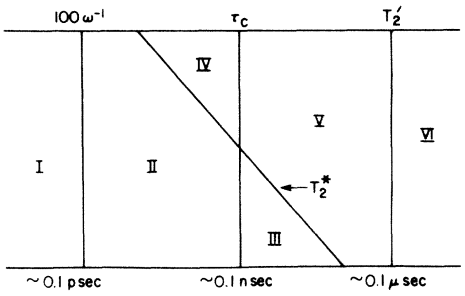

FIG. 1. Six possible steady-state pulse regimes are shown. They are defined by the various time scales which govern pulse propagation, namely, T_2', T_2^*, τ_c, and $2\pi/\omega$. For example, regions I, II, III embrace pulses whose length τ is shorter than T_2^*, "sharp-line" pulses. The T_2^* line is not vertical because in certain absorbers it can be adjusted more or less at will within a range of values (see Gibbs and Slusher, Ref. 5, for an example of such an adjustment). Regions IV, V, VI are those of "broad-line" pulses. Most experimental work to date on self-induced transparency has been done in region V. Pulses in region VI are longer than T_2' and experience rapid incoherent decay. In this paper we treat pulses in regions I, II, III. Typical values for T_2', τ_c, and $100\omega^{-1}$ for resonant optical absorbers are given along the bottom axis.

all second derivatives in the equations of motion.

A first integral of the second order in-phase Maxwell equation derived in Sec. III suggests our new approximation scheme. It is based on the fact that the Rabi frequency of the pulse is typically much smaller than the atomic transition frequency. We denote the ratio of these two frequencies by ρ, and expand every variable in a series of powers of this small parameter. In Sec. IV we derive an equation for ρ itself, and in Sec. V we derive relations between the pulse length and the velocity, the chirp, and the dispersion of pulses in regions I–III of Fig. 1. Graphs are given of these relations for a medium like Rb vapor. The relations between our expressions and those of other treatments of steady-state pulses are pointed out. As anticipated, the corrections to the familiar McCall-Hahn results are small. We find that the "coherence time" τ_c of the resonant medium serves as a kind of threshold for the onset of these small corrections.

Section VI is devoted to a brief summary of our results. Finally, in Appendices A and B, we show qualitatively the physical origin of τ_c in a sharp-line medium; and explain some details of the approximation method introduced here.

II. APPROXIMATIONS

It will be helpful to distinguish two kinds of approximation that are embodied in current optical-pulse theory.[7] The first kind of approximation might be called experiment oriented; the simplest and most ideal experimental environment is invoked. For example, it is frequently assumed[7] that the atoms have only one possible transition, that incoherent scattering from impurities, back scattering, and nonresonant losses are absent, that multimode and diffractive effects can be ignored, etc. The total effect of all of the approximations of this first kind is a theory reduced to its resonant nonlinear core, the Maxwell wave equation for the electric field coupled to the Schrödinger equation for the individual two-level atoms.

The second kind of approximation is made whenever the resonant nonlinear coupled equations themselves are reduced in complexity. The most common approximations of this second kind are made after the electric field and the polarization density have both been written as the product of an envelope function and a sinusoidal carrier wave. It is then usually assumed[7] that: (i) the in-phase and out-of-phase components of the polarization, the field envelope, and the field phase all vary so slowly that all of their second derivatives, with respect to both space and time, may be ignored; and (ii) frequency modulation, i.e., "chirping," does not occur.[10,11]

Corrections to several of the experiment-oriented approximations of the first kind have been reported already. For example, McCall and Hahn[12(a)] have considered the effects due to the presence of a transverse mode of the radiation field within the absorber volume; Slusher and Gibbs[4] have been able to allow for the presence of a nonresonant transition in their computer studies. Some workers have explored the effects of atomic-level degeneracy,[1,12(b)] and others have considered the presence of a second species of resonant atom with a different value of dipole-moment matrix element.[12(c)]

It has also been suggested[12(d)] that the Lorentz field $\frac{4}{3}\pi P$ can make significant changes in the predictions of the theory; recent work on a neoclassical theory of pulse propagation[12(e)] considers the effect of new dispersive terms added to the usual atomic equations; and an approximate theory of Stark effects in sharp-line self-induced transparency has been given.[12(f)] While such corrections may be very important, and even necessary for agreement with careful experimental work (cf. Ref. 4), they probably do not modify one's qualitative understanding of loss-free nonlinear pulse propagation.

On the other hand, to the best of our knowledge no systematic study exists of corrections to the second kind of approximation in absorbers [see Armstrong and Courtens, Ref. 11, for an exact amplifier solution]. There is no system of ap-

proximations which has been exploited to provide successively finer corrections to the basic coupled Maxwell-Bloch theory developed by McCall and Hahn,[1] and Lamb.[7] The main purpose of this paper is to develop the framework for such a hierarchy, and apply it to find the lowest-order corrections.

III. SECOND-ORDER STEADY-STATE PULSE EQUATIONS AND CONSERVATION LAWS

Without assuming them to be slowly varying, we first introduce steady-state envelope functions for the electric field E and the polarization density P:

$$\vec{\mathrm{E}}(t,z)=\mathcal{E}(t-z/V)\{\hat{1}\}, \tag{1}$$

$$\vec{\mathrm{P}}(t,z)=\mathfrak{N}d[u(t-z/V)\{\hat{1}\}+v(t-z/V)\{\hat{2}\}], \tag{2}$$

where the rotating orthogonal unit vectors,

$$\{\hat{1}\}=\hat{x}\cos[\omega t-Kz+\phi(t-z/V)]+\hat{y}\sin[\omega t-Kz+\phi(t-z/V)],$$

$$\{\hat{2}\}=-\hat{x}\sin[\omega t-Kz+\phi(t-z/V)]+\hat{y}\cos[\omega t-Kz+\phi(t-z/V)],$$

contain the field phase ϕ explicitly. Here the magnitude of the carrier wave vector K is generally not equal to the magnitude of the vacuum wave vector $k=\omega/c$, V is the magnitude of the steady-state pulse velocity, $\mathfrak{N}$ is the volume density of resonant atoms, and d is their common dipole matrix element.

Equations of motion for the field envelope $\mathcal{E}$ and phase ϕ follow from the substitution of (1) and (2) into the second-order vector Maxwell wave equation:

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\mathrm{E}}(t,z)=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{\mathrm{P}}(t,z).$$

By taking its components along $\{\hat{1}\}$ and $\{\hat{2}\}$ separately the vector wave equation becomes two second-order coupled scalar equations:

$$\left(\frac{1}{V^2}-\frac{1}{c^2}\right)\ddot{\mathcal{E}}-\left[(K^2-k^2)+2\left(\frac{K}{V}-\frac{k}{c}\right)\dot{\phi}+\left(\frac{1}{V^2}-\frac{1}{c^2}\right)\dot{\phi}^2\right]\mathcal{E}=\frac{2\pi\mathfrak{N}\hbar\kappa}{c^2}[\ddot{u}-(\omega+\dot{\phi})^2u-\ddot{\phi}v-2(\omega+\dot{\phi})\dot{v}], \tag{3}$$

$$\left(\frac{1}{V^2}-\frac{1}{c^2}\right)\ddot{\phi}\mathcal{E}+2\left[\frac{K}{V}-\frac{k}{c}+\left(\frac{1}{V^2}-\frac{1}{c^2}\right)\dot{\phi}\right]\dot{\mathcal{E}}=\frac{2\pi\mathfrak{N}\hbar\kappa}{c^2}[\ddot{v}-(\omega+\dot{\phi})^2v+\ddot{\phi}u+2(\omega+\dot{\phi})\dot{u}]. \tag{4}$$

Here $(\dot{})$ means $d(\,)/d\zeta$, where $\zeta=t-z/V$ is the local time coordinate, and $\kappa\equiv 2d/\hbar$. If the phase variation $\dot{\phi}$ vanishes identically, and if $\ddot{u}$ and $\ddot{v}$ are discarded by comparison with ω^2u and ω^2v, then these two equations reduce to a set studied before.[9] If, in addition to setting $\dot{\phi}\equiv 0$, one also eliminates all but the lowest-derivative term on each side of (3) and (4), one obtains the usual SIT (self-induced transparency) equations for a sharp-line absorber.

The atomic Schrödinger equation, allowing for possible field phase variations, and assuming no effective broadening of the on-resonance line, takes the form[10]:

$$\dot{u}=\dot{\phi}v, \tag{5}$$

$$\dot{v}=-\dot{\phi}u+\kappa\mathcal{E}w, \tag{6}$$

$$\dot{w}=-\kappa\mathcal{E}v. \tag{7}$$

The product $\kappa\mathcal{E}$, which recurs frequently, has the dimensions of inverse time, and may be called the instantaneous Rabi frequency for the pulse. The conservation of probability

$$u^2+v^2+w^2=1 \tag{8}$$

is a consequence of Eqs. (5)–(7) for arbitrary $\dot{\phi}$. The interpretation of u and v, in the rotating frame, as the dispersive and absorptive components of a typical atom's dimensionless dipole-moment expectation value is standard, as is the interpretation of w as the single-atom inversion or the dimensionless energy expectation value.[1-7]

The two second-order Maxwell equations, given in (3) and (4), are both nonlinear and probably not explicitly solvable by known methods. Their complexity can be reduced somewhat by using the optical Bloch equations (5)–(7) to eliminate all of the derivatives of atomic variables. This step is merely algebraic. The reduced equations may be written somewhat more simply than (3) and (4) if we introduce our definition of the sharp-line-medium coherence time τ_c,

$$\frac{1}{\tau_c^2}=\tfrac{1}{2}\pi\mathfrak{N}\hbar\omega\kappa^2, \tag{9}$$

and the following abbreviations for the dimensionless ratio R,

$$R=\tfrac{1}{2}\omega\left(\frac{K}{V}-\frac{k}{c}\right)c^2\tau_c^2, \tag{10}$$

and two auxiliary frequencies P and Q:

$$P^2=(K^2-k^2)/(V^{-2}-c^{-2}) \tag{11}$$

and

$$Q^2=(8/\tau_c^2)[(c/V)^2-1]^{-1}. \tag{12}$$

We find, then, for the two Maxwell equations:

$$\ddot{\mathcal{E}} - \left[P^2 - Q^2 w + \frac{Q^2}{2\omega}(R-w)\dot{\phi} + \dot{\phi}^2\right]\mathcal{E} = -\frac{Q^2\omega}{2\kappa}u, \tag{13}$$

$$\ddot{\phi}\mathcal{E} + \left[2\dot{\phi} + \frac{Q^2}{2\omega}(R-w)\right]\dot{\mathcal{E}} = -\frac{Q^2\omega}{2\kappa}\left[1+\left(\frac{\kappa\mathcal{E}}{\omega}\right)^2\right]v. \tag{14}$$

The phase equation (14) is a first-order linear differential equation for $\dot{\phi}$. Its integrating factor is $\mathcal{E}$, and it has the simple first integral

$$(\kappa\mathcal{E})^2\dot{\phi} = \frac{Q^2}{2\omega}\left[-\tfrac{1}{2}R(\kappa\mathcal{E})^2 + \omega^2(1+w) + (\kappa\mathcal{E})^2 w - \int_0^{\mathcal{E}} w(\mathcal{E}')\,\kappa\mathcal{E}'\,d(\kappa\mathcal{E}')\right]. \tag{15}$$

This relation may be regarded as a conservation law of the pulse. Its importance is that it shows that if the functional relation between the atomic dimensionless energy and the pulse amplitude is known, i.e., if the energy-conservation law $w = w(\mathcal{E})$ can be found, then $\dot{\phi}$ is also a known functional of pulse amplitude, $\dot{\phi}(\mathcal{E})$.

Furthermore, if both $w(\mathcal{E})$ and $\dot{\phi}(\mathcal{E})$ are known, then one may determine, at least formally by using (5) and (7), the functional relation $u = u(\mathcal{E})$. But this last relation is all that is needed, then, to turn the amplitude equation (13) into a nonlinear second-order equation for $\mathcal{E}$ alone. Thus the system may be reduced to quadratures, at least in principle, if only $w = w(\mathcal{E})$ is known.

Finally, one notices that once all of the variables have been expressed as functionals of the basic functional $w(\mathcal{E})$, then the conservation-of-probability relation (8) can be used, again at least in principle, to determine *self consistently* that same basic functional $w(\mathcal{E})$.

IV. POWER-SERIES APPROXIMATION METHOD

In order to carry out a program of self-consistent integration of Eqs. (5)–(7), (13), and (15), one merely needs to adopt a sufficiently general form for the relation $w(\mathcal{E})$. We assume only that w has a power-series expansion about the origin:

$$w(\mathcal{E}) = \sum_{l=0}^{\infty} w_l (\kappa\mathcal{E}/\omega)^{2l}, \tag{16}$$

where the choice of the dimensionless ratio $\kappa\mathcal{E}/\omega$ as the expansion parameter, and the restriction to even powers of the parameter, are both suggested by the first-order McCall-Hahn theory.[1] All of the coefficients are to be determined self consistently except w_0 which must equal -1 in order that the atoms of the absorber have their ground-state energy when $\mathcal{E}=0$.

The convergence of the series (16) is a fairly subtle mathematical question (because the coefficients w_l depend in complicated ways on the maximum value taken by $\mathcal{E}$) that cannot be answered at this time. However, there are good physical grounds for expecting rapid convergence. In the first place, the relation between inversion w and field energy $\mathcal{E}^2$ should be nonsingular. Moreover, there is a simple argument that shows that the expansion parameter $\kappa\mathcal{E}/\omega$ must be smaller than unity: the eigenenergies of the atom are of the order of $\hbar\omega$, and the perturbing atom-field interaction energy is $\vec{d}\cdot\vec{E} \sim d\mathcal{E} \sim \hbar\kappa\mathcal{E}$. Thus one is forced to assume $\kappa\mathcal{E} \ll \omega$ if the interaction energy is to be significantly weaker than the unperturbed energy. If it were not so, if $\vec{d}\cdot\vec{E} \gtrsim \hbar\omega$, it would indicate that the atom's energy levels were poorly defined and transitions between them relatively meaningless. This is a nonmathematical argument, but it offers strong support for the rapid convergence of series of powers of $\kappa\mathcal{E}/\omega$ such as (16).

Obviously, given $w(\mathcal{E})$ in (16), the desired expression for $\dot{\phi}(\mathcal{E})$ is found immediately from (15) by evaluating the definite integral. We choose to write $\dot{\phi}(\mathcal{E})$ without a constant term so that there is no frequency shift where there is no pulse. [Both Courtens and Lee[10(b)] obtain a constant frequency shift for times and distances asymptotically far from the field-atom interaction region. This comes from a different definition of the phase in their case.] The evaluation of the integral in (15) leads to the condition:

$$2w_1 = 1 + R, \tag{17}$$

as well as to a power-series expression for $\dot{\phi}$,

$$\dot{\phi}(\mathcal{E}) = \frac{Q^2}{2\omega}\sum_{l=0}^{\infty} a_l (\kappa\mathcal{E}/\omega)^{2l+2}, \tag{18}$$

where the coefficients a_l are related to the coefficients w_l by

$$a_l = \left(\frac{2l+3}{2l+4}\right)w_{l+1} + w_{l+2}. \tag{19}$$

It is then straightforward to check that $u(\mathcal{E})$, determined by the first and last of the Bloch equations, is given by

$$u(\mathcal{E}) = -\frac{Q^2}{2\omega^2}\sum_{n=0}^{\infty} b_n (\kappa\mathcal{E}/\omega)^{2n+1}, \tag{20}$$

where

$$b_n = \sum_{l=0}^{n} \frac{2l}{2n+1} w_l a_{n-l}. \tag{21}$$

The Maxwell equation (13) for the field amplitude can now be written solely in terms of $\ddot{\mathcal{E}}$, powers of $\mathcal{E}^2$, and the to-be-determined coefficients a_l and w_l, and then integrated.

Rather than continue in the most general way, let us restrict our analysis here to the level of approximation in which the basic atomic-energy expansion is carried through the fourth power of the small parameter $\kappa\mathcal{E}/\omega$.[6(a)] We denote the dimensionless expansion parameter by ρ:

$$\rho \equiv \kappa\mathcal{E}/\omega,$$

and truncate the series in (16):

$$w = -1 + \tfrac{1}{2}(1+R)\rho^2 + w_2\rho^4. \tag{22}$$

Here we have used the requirement that $w \to -1$ as $\rho \to 0$, and Eq. (17), to specify w_0 and w_1.

From the form of (15) it can be deduced that, to the same level of approximation, only the first term in (18) suffices for the instantaneous frequency shift:

$$\dot{\phi} = \left(\frac{Q^2}{2\omega}\right)\left[\tfrac{3}{8}(1+R) + w_2\right]\rho^2; \tag{23}$$

and u is adequately given by

$$u = \frac{-1}{3\omega}\left(\frac{Q^2}{2\omega}\right)\left[\tfrac{3}{8}(1+R) + w_2\right]\rho^3. \tag{24}$$

Note that, at this level of approximation, u vanishes and there is no chirp if $w_2 = -\frac{3}{8}(1+R)$.

Now we integrate (13), the first Maxwell equation, by multiplying by $\dot{\rho}$ after using (22)–(24) for w, $\dot{\phi}$, and u. All of the terms in (13) except $\dot{\phi}^2$ make some contribution at this level. The result of the integration is

$$\dot{\rho}^2 = (P^2+Q^2)\rho^2 - \tfrac{1}{4}Q^2(1+R) \times\left(1 - \frac{2Q^2}{3\omega^2}\left[\tfrac{3}{8}(1+R) + w_2\right]\right)\rho^4, \tag{25}$$

which already implies that the envelope function will be a hyperbolic secant at this level of approximation,[6(b)] but leaves open the question of area.

Another independent equation for ρ can be obtained from (14) merely by solving it for v in terms of ρ, $\dot{\rho}$, and the only still-undetermined coefficient, w_2. Then relation (8) allows v to be eliminated, leaving

$$\dot{\rho}^2 = \frac{\omega^2}{1+R}\rho^2 - \left[\tfrac{1}{4}\omega^2 + 6\omega^2\left(\frac{1}{1+R}\right)^2 w_2\right]\rho^4. \tag{26}$$

We identify the coefficient of ρ^2 in (25) and (26) as the square of the small-signal growth rate of ρ, and denote it by $1/\tau^2$:

$$P^2 + Q^2 = \tau^{-2} = \omega^2(1+R)^{-1}. \tag{27}$$

Note that this physical definition of τ is independent of the question of the shape or duration or even the existence of a well-defined pulse. Also, it is clear that the coefficient of $-\rho^4$ is just $(\rho_0\tau)^{-2}$, where ρ_0 is the peak value of ρ if ρ has a definite peak value:

$$\tfrac{1}{4}Q^2(1+R)\left\{1 - \frac{2Q^2}{3\omega^2}\left[\tfrac{3}{8}(1+R) + w_2\right]\right\} = (\rho_0\tau)^{-2} = \tfrac{1}{4}\omega^2 + 6\omega^2\left(\frac{1}{1+R}\right)^2 w_2. \tag{28}$$

These complicated algebraic relations can be unravelled in a straightforward way. Note that Eqs. (10)–(12) provide an implicit solution for R in terms of P^2 and Q^2:

$$1 + R\left(\frac{1}{(\omega\tau_c)^2} - \frac{16P^2}{Q^4R^2}\right) - \frac{2}{Q^2R}(\omega^2 + P^2) = 0. \tag{29}$$

However, Eq. (27) provides for the elimination of both P^2 and R (say) from (29), leaving a single solution for Q^2 in terms of ω, τ, and τ_c:

$$(Q\tau)^2 = \frac{1 + (\omega\tau)^2 - 8(\tau/\tau_c)^2 + \left\{\left[1 + (\omega\tau)^2 - 8(\tau/\tau_c)^2\right]^2 + 16(\tau/\tau_c)^2\left[2 + R + R^2(\omega\tau_c)^{-2}\right]\right\}^{1/2}}{2 + R + R^2(\omega\tau_c)^{-2}}, \tag{30}$$

where $R = (\omega\tau)^2 - 1$ is understood from (27).

If we restrict our interest to pulses *longer* than about 0.1 psec, we have both

$$(\omega\tau)^2 \gg (\tau/\tau_c)^2,$$

since $\tau_c \sim 0.1$ nsec for many optically resonant media, and

$$(\omega\tau)^2 \gg 1.$$

These inequalities allow (30) to be simplified drastically[13]:

$$(Q\tau)^2 = \frac{2}{1 + (\tau/\tau_c)^2} + O((\omega\tau)^{-2}) + O((\omega\tau_c)^{-2}). \tag{31}$$

Equation (28) can be solved easily within these same limits:

$$w_2 = -\tfrac{3}{8}(\omega\tau)^2(\tau/\tau_c)^4, \tag{32}$$

$$\rho_0^{-2} = (\tfrac{1}{2}\omega\tau)^2. \tag{33}$$

It is the last of these which establishes the link between the basic approximation method of this paper and the neglect of higher powers of $(\omega\tau)^{-2}$ in (31)–(33). It would have been inconsistent to have kept higher powers of $(\omega\tau)^{-2}$ than the first, because we have kept only the lowest power of ρ_0^2 in writing the two forms of Maxwell's equations, (25) and (26) (cf. Appendix B).

Basically the chirped-pulse problem is now solved to the lowest level of approximation. By substitution of (31)–(33) into (11), (12), (23), and (27) we can easily determine the most important pulse characteristics. We do that in Sec. V.

V. CONNECTION WITH EXPERIMENTAL QUANTITIES

A nonlinear transmission threshold, anomalously slow velocities, quantized pulse "area," and nonlinear dispersion have come to be associated with modern work on optical resonant pulse propagation in absorbers. Intrinsic chirping[11] might also be included in this list of novel pulse features.

The question of a threshold can be dispensed with trivially. Equations (25) and (26) both show that if the field strength is low enough so that only the first term on the right need be retained, then exponential growth or decay follows as a matter of course. As the field strength gets higher so that the second term must also be retained, the exponential behavior saturates and gives way to true pulses. The threshold value is obviously determined by ρ_0 in (28). Furthermore, (33) fixes ρ_0 to be $2/\omega\tau$, which is equivalent to fixing the envelope area at 2π, the same "quantized" value of unchirped theories.

In order to display the existence of the other nonlinear characteristics listed above we have graphed our solutions for pulse velocity, maximum chirp, and nonlinear dispersion as a function of pulse length. Experimental points are included which come from the Slusher-Gibbs data on Rb vapor.[4]

We should note that strict comparisons with present experimental data are not possible. This is because the data was obtained in a situation where T_2^* effects may have been critical. In any event $\tau \ll T_2^*$ was not satisfied for the experimental pulses, but our theory has assumed $\tau \ll T_2^*$ from the beginning.[14]

Several features of the curves in Figs. 2–4 should be pointed out. First of all, we see in Fig. 2 that very short pulses are associated with high velocities. [This is the basic assumption of the Courtens and Lee treatments.[10(b)]] Pulses shorter than τ_c satisfy $V = c$ to within a factor of 2 or so. While our sharp-line theory does predict very low velocities when $\tau > \tau_c$, these low velocities must be compared with the broad-line measurements of Slusher and Gibbs only with great caution. Only within the sharp-line regions of Fig. 1 do we expect our predictions to be completely reliable.

The high velocity of second-order pulses is the first feature qualitatively different from the usual picture of self-induced transparency pulses. It is not an unexpected feature however. As a pulse with a given envelope area is made shorter its energy density grows larger. Eventually, a short enough 2π pulse has too great an energy density to share a very large fraction of that energy with the atoms it overlaps. (As we show in Appendix A, in sharp-line situations this inability to share all the pulse energy sets in at $\tau \sim \tau_c$.) But it is understood that just this process of energy sharing is the cause of the slow SIT velocities.[15] Thus $V \sim c$ is a natural expectation for second order pulses.

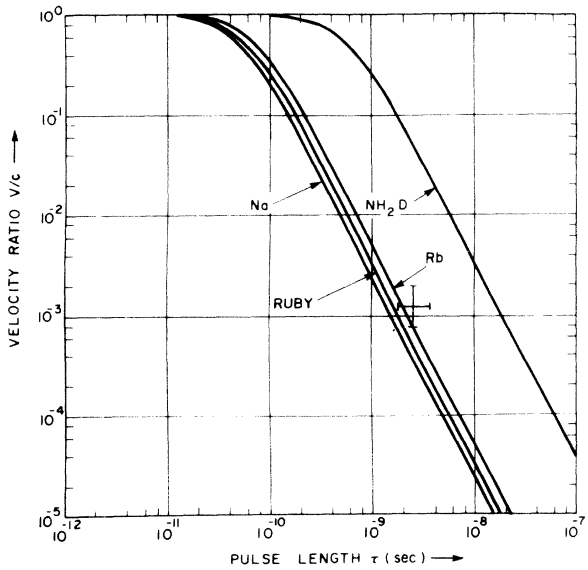

FIG. 2. Velocity ratio V/c vs the pulse length τ for different media. The crossing bars depict experimental data obtained by Slusher and Gibbs (Ref. 4) on Rb vapor.

We may also note that our assumption of zero backscattering is a good assumption which gets better as the pulse velocity approaches c. The approximation which neglects backscattering is thus better for our second-order pulses than for the usual first-order pulses.[16]

The new nonlinear feature of our steady-state pulses is their chirp. In Sec. IV we showed how an expression for pulse chirp can be obtained from the theory. In Fig. 3 we illustrate the nature of the chirp.[17] We have plotted the maximum value taken by $\dot{\phi}$, its value at pulse center,

$$\dot{\phi}_{\max} = \dot{\phi}(\zeta = 0) \equiv \dot{\phi}_0,$$

as a function of pulse length τ. For simplicity we plot only the lowest-level approximate results, obtained from Eqs. (23), (31), and (32) of Sec. IV.

Figure 3 suggests that, in the case of Rb vapor at least, $\dot{\phi}$ is very small in the sense that $\dot{\phi}/\omega \ll 1$ (since $\omega \sim 10^{15}$ sec^{-1}). The sign of the intrinsic chirp changes from positive to negative in going

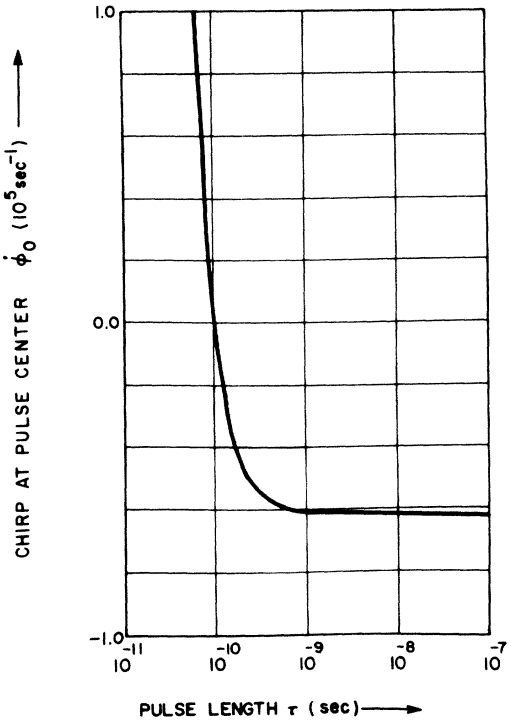

FIG. 3. Maximum frequency shift due to chirping vs the pulse length, for Rb vapor at a pressure typical of the Slusher-Gibbs experiments (Ref. 4). The pulse length for which the chirp is identically zero is $\tau = \tau_c \approx 0.1$ nsec. However, within the range of τ shown, all predicted chirps are too small to measure (see Fig. 4).

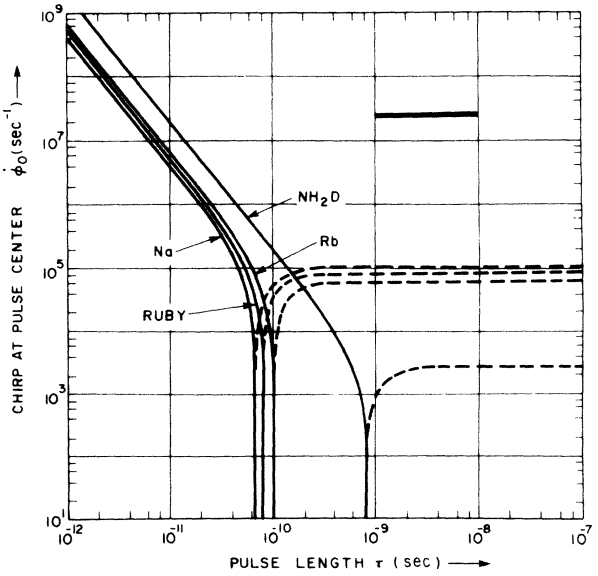

FIG. 4. Maximum frequency shift due to chirping vs the pulse length, for different media. The solid portions of the curves indicate positive values of $\dot{\phi}_0$ while the dashed portion correspond to negative values. The bar shows the experimental bound placed on chirping by the Slusher-Gibbs work (Ref. 4).

from ultrashort pulses to those longer than τ_c, the point of exactly zero chirp occurring at $\tau = \tau_c$.[18] We show in Fig. 4 the same result for Rb, as well as chirping curves for three other optical absorbers, plotted over a wider range of pulse lengths. All show the same features, differing as τ_c differs among the four absorbers. The most interesting of the chirping features is the very rapid increase of $\dot{\phi}_0$ as $\tau \to 100\omega^{-1}$. For Rb we predict $\dot{\phi}_0 \sim 1$ GHz for a picosecond pulse, so that during a propagation time of approximately 1 nsec a phase shift of approximately $\frac{1}{2}\pi$ would accumulate, leading to an effect which might be detectable using interferometric techniques.

Finally, in Fig. 5, we show the effect on resonant spatial dispersion of the nonlinear interaction. In effect, we are showing the dispersion relation $K = K(\omega, \tau)$ by plotting the normalized relative dispersion $K/k - 1$ as a function of pulse length τ, for three possible values of τ_c. For second-order pulses in sharp-line absorbers we show below that measurable phase shifts cannot arise from the difference between K and k.

Having presented the results of our theory of

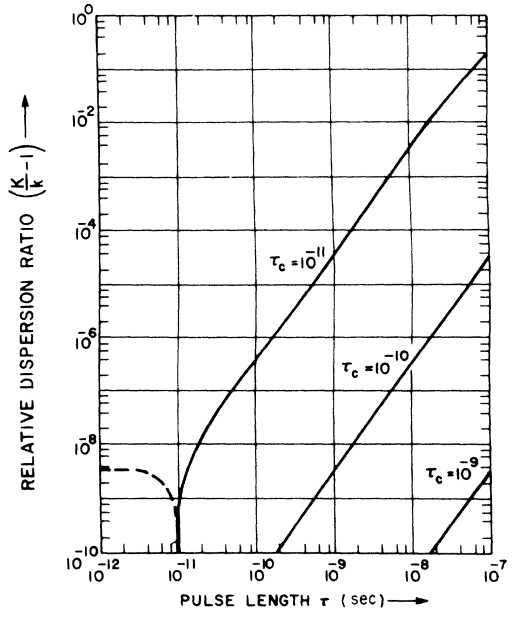

FIG. 5. Plot of the quantity $K/k - 1$, effectively the contribution of "nonlinear dispersion" (Ref. 13) to the index of refraction, vs pulse length, for three values of the cooperation time τ_c. In most optically resonant media studied to date, a typical value is $\tau_c \approx 0.1$ nsec. Dashed portions of the curves indicate negative values, while solid lines correspond to positive values.

second-order pulses graphically, we can also give simple analytic expressions in the lowest level of approximation for the graphed curves. These expressions should be reliable for steady-state pulses whose pulse lengths lie in the "sharp-line" domains of Fig. 1, regions II and III, and possibly also region I if the two-level atom model remains valid.

For the velocity, one has

$$\frac{c}{V} - 1 = 2\left(\frac{\tau}{\tau_c}\right)^2 = \frac{4\pi \mathfrak{N}\omega d^2}{\hbar}\tau^2; \tag{34}$$

the envelope area A,

$$A = 2\pi; \tag{35}$$

maximum frequency shift due to intrinsic chirp,[13]

$$\dot{\phi}_0 = \frac{3}{2\omega\tau^2}\left[1 - \left(\frac{\tau}{\tau_c}\right)^2\right]; \tag{36}$$

and maximum wave-vector shift due to nonlinear dispersion,[13]

$$c(K-k) = \frac{-1}{\omega\tau_c^2}\left[1 - \left(\frac{\tau}{\tau_c}\right)^2\right]. \tag{37}$$

The restriction to the lowest level of approximation, which gave us (33), guarantees that $\omega\tau \gg 1$, which is sufficient, using (36), to ensure that $\dot{\phi}_0 \ll 1/\tau$. Thus the spectral bandwidth of the pulse, very roughly given by

$$\Delta\nu_{\text{pulse}} \simeq \Delta\nu_{\text{envelope}} + \Delta\nu_{\text{phase}} \sim \frac{1}{\tau} + \dot{\phi}_0$$

is due mostly to $1/\tau$ and very little to $\dot{\phi}_0$. That is, merely to measure the pulse spectral width will not be adequate to reveal any intrinsic chirp present.

As we pointed out above, the chirp can in some cases, nevertheless, be large enough to lead to phase shifts which might be detected interferometrically. We can now see that the other possible contributor to a phase shift, the nonlinear wave-vector shift, is much less important since appreciable phase shifts develop only if $\tau \ll \tau_c$, in which case (36) and (37) show that $\dot{\phi}_0 \gg c(K-k)$.

VI. SUMMARY AND CONCLUSIONS

To summarize briefly, we have applied our power-series approximation method to the problem of chirped steady-state sharp-line optical pulses. We have given the explicit solution to this problem analytically and graphically in the lowest level of approximation.

We have found that the unusual and special features of the "preferred" pulse solution[9] have a natural explanation. That is, the present paper shows that as a general rule all steady-state pulses are chirped and that the sign of the chirp is positive for very short pulses and negative for long pulses. The value of the pulse length for which the sign of the chirp changes from positive to negative is exactly $\tau = \tau_c$, corresponding to a pulse velocity $V = \frac{1}{3}c$, exactly the characteristics of the "preferred" pulse of Davidovich and Eberly.[9] This finding, that the only sharp-line unchirped pulse is the pulse that travels with velocity $V = \frac{1}{3}c$, was also obtained by Courtens and by Lee.[10(b)]

What is most interesting, of course, is the collection of findings concerning those pulses which are chirped. We have established, for example, that intrinsic chirping is a very small effect, even for subpicosecond pulses. Not only is it true that $\dot{\phi} \ll \omega$, as expected, but one also has $\dot{\phi} \ll 1/\tau$. Thus intrinsic chirping makes no contribution to pulse bandwidth and must be detected interferometrically if at all. The McCall-Hahn assumption that chirping is entirely absent is, to this degree, now justified theoretically as well as experimentally.

Our conclusions regarding the significance of τ_c for pulses in the sharp-line regions II and III of Fig. 1 follow from Eqs. (34)–(37). As they show, all of the principal pulse characteristics except envelope area change in some way near $\tau = \tau_c$. This is due to the importance of the second-derivative terms in the Maxwell equations for $\tau \lesssim \tau_c$. That the second-derivative terms should make any contribution at all to pulses as long as τ_c ($\omega\tau_c \sim 10^4$ or more) is unexpected.

ACKNOWLEDGMENTS

We are happy to acknowledge conversations at various times with H. M. Gibbs, E. L. Hahn, A. Kujawski, S. L. McCall, P. L. Milonni, and R. E. Slusher. In addition, the first author would like to thank J. R. Klauder for his support and encouragement during the early part of this research, and the third author thanks S. D. Drell and A. L. Schawlow for their hospitality at Stanford University.

APPENDIX A: ROLE OF τ_c, THE COOPERATION TIME

Self-induced transparency is well known to be characterized by an "anomalously" low pulse velocity.[1,4] Such a slow pulse is actually a natural result of the very strong atom-field interaction, and of the ability of the atoms to share large fractions of the pulse energy.[15] In order for maximal sharing to occur, the pulse-energy density must be of the order of, or smaller than, the maximum energy density which can be stored in the atoms: $\mathcal{E}^2 \lesssim \mathfrak{N}\hbar\omega$. For a steady-state single pulse, for

which $\mathcal{S} \sim 1/\kappa\tau$, and which is propagating in a "sharp-line" medium, the implication is that

$$\frac{1}{\tau^2} \lesssim \mathfrak{N}\hbar\omega\kappa^2. \tag{A1}$$

In other words, recalling the definition of τ_c in (9), slow SIT-type pulses necessarily are longer than τ_c. Since in this paper we study pulses which satisfy the opposite of the inequality of (A1), we cannot expect our steady-state pulses to share every SIT pulse characteristic.

APPENDIX B: SYSTEMATIC APPROXIMATION OF THE ENVELOPE EQUATIONS

The field-envelope equations (25) and (26) are of the form

$$\dot{\rho}^2 = (\rho/\tau)^2[1 - A\rho^2 - B\rho^4 - C\rho^6 - \cdots], \tag{B1}$$

The solution depends, of course, on the number of terms kept on the right-hand side. Recall the definition of ρ—the ratio of the pulse Rabi frequency to the optical carrier frequency. Thus ρ is usually very small in practice, and we expect the retention of very many terms will not be required. An iterative approximation scheme suggests itself, in which more terms in the series (B1), are kept to obtain finer approximations.

Let us refer to Fig. 6 in which $\dot{\rho}^2$ is plotted as a function of ρ^2. Each of the several curves drawn corresponds to a different number of terms kept on the right-hand side of Eq. (B1). That is, the curve labeled 1 is obtained from (B1) by keeping only one term on the right-hand side; curve 2 is obtained by keeping two terms on the right-hand side of (B1); and so on. Note that the curves in Fig. 6 are qualitatively dissimilar until curve 3 is reached. In other words, keeping the next higher power of ρ^2 in Eq. (B1) makes a big difference, not a small one, until the ρ^6 term has been reached. For this reason, the term $B\rho^6$ is the first one which can properly be expected to give only a small correction to the terms preceding it. Therefore, we will regard the ρ^2 and ρ^4 terms as both belonging to the lowest level of approximation.

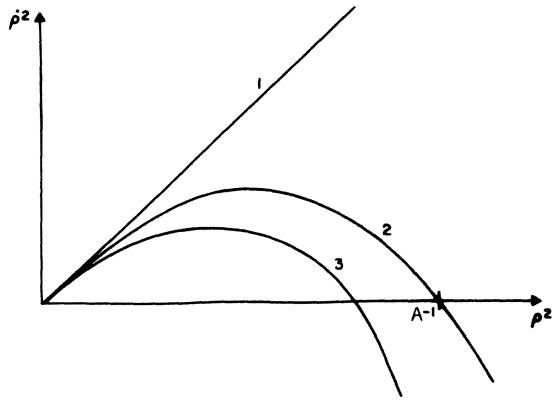

FIG. 6. Several levels of approximation to Eq. (B1) are sketched. The curves correspond to Eq. (B1) when one, two, and three terms are retained on the right-hand side.

We now show that a satisfactory approximation scheme can be developed by emphasizing the existence of a pulse. In order for there to be a pulse, ρ must have a maximum value at some finite value of ζ, and ρ must go to zero at infinite values of ζ. Consequently $\dot{\rho} = 0$ both in the pulse wings and at the pulse peak. These conditions are all met by rewriting (B1) as

$$\dot{\rho}^2 = (\rho/\rho_0\tau_0)^2(\rho_0^2 - \rho^2)[1 - a(\rho_0^2 - \rho^2) - b(\rho_0^2 - \rho^2)^2 - \cdots], \tag{B2}$$

where ρ_0 denotes the maximum value of ρ. In order that (B2) be consistent with (B1) we must have

$$A\rho_0^2 + B\rho_0^4 + C\rho_0^6 + \cdots = 1, \tag{B3}$$

$$\left(\frac{1}{\rho_0\tau_0}\right)^2 = \frac{1}{\tau^2}(A + 2B\rho_0^2 + 3C\rho_0^4 + \cdots), \tag{B4}$$

$$a = \frac{B + 3C\rho_0^2 + \cdots}{[A + 2B\rho_0^2 + \cdots]}, \tag{B5}$$

and so on.

In effect the change from (B1) and (B2) shifts the emphasis of the power series from the wings of the pulse, where $\rho^2 \approx 0$, to the body of the pulse itself, where $\rho^2 \approx \rho_0^2$. Clearly, ρ_0 sets the time scale on which the pulse rises and falls about its maximum, whereas τ does the same for the pulse wings. The interpretation of (B3) is straightforward also; it is the relation that determines the value of ρ_0, the pulse maximum, in any given level of approximation.

Note that the error arising from truncating (B2) at some finite level is easily estimated. More importantly, at the pulse peak the error is identically zero in (B2), in contrast to (B1) where the maximum error due to truncation occurs at the pulse peak.

If we apply the procedure sketched here to the case provided by Eqs. (25) and (26) of the text, we easily determine that

$$Q^2 = \frac{1}{\tau^2}, \quad P^2 = 0 \tag{B6}$$

and

$$A = \rho_0^{-2} = \tfrac{1}{4}(\omega\tau)^2. \tag{B7}$$

Note that (B7) connects pulse amplitude and pulse length in just the way that requires the pulse, in this lowest level of approximation, to be a 2π pulse. Also, at this level $\tau_0 = \tau$, so the pulse amplitude changes at the same rate in the wings as at its peak. It is these expressions which we used in Sec. IV in the text in deriving the lowest-level approximation to the steady-state pulse equations.

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*Based in part on a dissertation submitted by the first author to the Faculty of the Polytechnic Institute of Brooklyn in partial fulfillment of the requirements for the degree Doctor of Philosophy (Electrical Engineering), 1972.

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§Research partially supported by the U.S. Army Research Office-Durham.

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[13]Inspection shows that the transition from (30) to (31) also requires $(\omega\tau)^2 >> (\tau/\tau_c)^4$, which is violated for pulses sufficiently *long* so that $\tau > \omega\tau_c^2$. However such pulses necessarily fall into region VI of Fig. 1 (since $\omega\tau_c^2$ is just the dipole-dipole contribution to T_2') and therefore need not be considered.

[14]Preliminary studies indicate that the assumption $\tau << T_2^*$ can be eliminated from our chirped-pulse theory. See J. H. Eberly, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973) and J. H. Eberly and R. A. Marth, Bull. Am. Phys. Soc. 18, 63 (1973).

[15]See, for example, E. Courtens, Phys. Rev. Lett. 21, 3 (1968).

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[17]Small departures from this rule appear in the next

level of approximation. For details, see R. A. Marth, Ref. 6a.

[18]This is consistent with the results of Courtens and Lee [Ref. 10(b)] and with a result found in earlier work which did *not* ignore inhomogeneous broadening; see L. Davidovich and J. H. Eberly, Ref. 9. See also L. Davidovich, in *Electromagnetic Interactions of Two-Level Atoms*, edited by J. H. Eberly (University of Rochester, Rochester, N.Y., 1970), p. 65.