A transform to calculate scattering amplitudes

C. M. Rosenthal
University of Bradford, Bradford, Yorkshire, England
Department of Chemistry, Drexel University, Philadelphia, Pennsylvania 19104 *

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A line-integral representation for a transform of the free-particle Green's function is derived. This representation gives rise to a line-integral equation for the Yukawa-potential scattering amplitude. The second term in the Born series is recovered by iterating the equation once. Generalization to superpositions of Yukawa potentials and angular-dependent potentials is discussed.

I. INTRODUCTION

A transform is introduced with which the scattering amplitude for a central Yukawa potential may be expressed as the solution of an inhomogeneous line-integral equation. The only integral occurring in this equation is a line integral over the scattering amplitude in the space of the transform variables. The second term in the Born series, for which an analytic expression is available,[1] is recovered by iterating the solution in the usual Born fashion. Generalizations to central potentials expressible as superpositions of Yukawa potentials are described in Sec. IV. Further generalizations to angular-dependent potentials are also indicated in this section. Expressing the scattering amplitude in this fashion seems ideally suited to numerical study as only a one-dimensional integral must be approximated. Similarly, to calculate higher-order terms in the Born series only a one-dimensional integral has to be iterated. Finally, as a consequence of this study a new representation of a transform of the free-particle Green's function is obtained which may prove useful in other applications.

II. DESCRIPTION OF THE PROBLEM

All of the experimentally verifiable information concerning potential scattering is contained in the asymptotic form of the wave function,

$$\psi_+(\vec{r}) \underset{r\to\infty}{\longrightarrow} e^{i\vec{k}_0\cdot\vec{r}} + f_+(\Omega)\, e^{ik_0 r}/r$$

with $|f_+(\Omega)|^2 = d\sigma/d\Omega$, the differential cross section. $f_+(\Omega)$ satisfies

$$f_+(\Omega) = -\frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}}\, V(\vec{r})\psi_+(\vec{r})\,d\vec{r}\,, \tag{1}$$

where $\psi_+(\vec{r})$ is a solution of the Schrödinger equation $\mathcal{H}\psi_+ = \frac{1}{2}k_0^2\psi_+$, regular at the origin and having the above asymptotic form. The vector $\vec{k}_f$ is in the direction Ω, and $|\vec{k}_f| = |\vec{k}_0| = k_0$. Such a function $\psi_+(\vec{r})$ solves the Lippmann-Schwinger equation,

$$\psi_+(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}} + \lim_{\epsilon\to 0}\frac{1}{E - \mathcal{H}_0 + i\epsilon}\, V(\vec{r})\psi_+(\vec{r})\,, \tag{2}$$

with $V(\vec{r})$ the potential of interaction, $\mathcal{H}_0 = \frac{1}{2}p^2$, the free-particle Hamiltonian, and $E = \frac{1}{2}k_0^2$, the incident energy.[2] Vector $\vec{k}_0$ is the incident wave vector. The coordinate form of Eq. (2) will be particularly useful for our purposes:

$$\psi_+(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}} - \frac{1}{2\pi}\int \frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, V(\vec{r}\,')\psi_+(\vec{r}\,')\,d\vec{r}\,'\,. \tag{2$'$}$$

As Eq. (1) reveals, all we need is an integral of $\psi_+(\vec{r})$, $-(1/2\pi)\int e^{-i\vec{k}_f\cdot\vec{r}}\, V(\vec{r})\psi_+(\vec{r})d\vec{r}$. Multiplying Eq. (2′) by $e^{-i\vec{k}_f\cdot\vec{r}}\, V(\vec{r})$, we get the equation

$$\int e^{-i\vec{k}_f\cdot\vec{r}}\, V(\vec{r})\psi_+(\vec{r}) = \int e^{-i\vec{k}_f\cdot\vec{r}}\, V(\vec{r})e^{i\vec{k}_f\cdot\vec{r}}\, d\vec{r} - \frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}}\; V(\vec{r})\,\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \times V(\vec{r}\,')\psi_+(\vec{r}\,')\,d\vec{r}\,d\vec{r}\,'\,. \tag{3}$$

We now specialize to a central Yukawa potential with screening parameter s_0, i.e., $V(r) = e^{-s_0 r}/r$. Regarding s now as a variable, consider the integral

$$\int e^{-i\vec{k}_f\cdot\vec{r}}\,\frac{e^{-sr}}{r}\psi_+(\vec{r})\,d\vec{r}\,,$$

which may be rewritten

$$F(k, s) = \int e^{-i\hat{k}_f\cdot\vec{r}k}\,\frac{e^{-sr}}{r}\psi_+(\vec{r})\,d\vec{r}\,,$$

with $\hat{k}_f$, a unit vector in the direction $\vec{k}_f$ and $k = |\vec{k}_f|$. [Note that $-(1/2\pi)F(k_0, s_0)$ is the desired scattering amplitude.] Let us also define

$$G(k, s, \vec{r}\,') = -\frac{1}{2\pi}\int e^{-i\hat{k}_f\cdot\vec{r}k}\,\frac{e^{-sr}}{r}\,\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,.$$

Then $F(k, s)$ satisfies the equation

$$F(k, s) = F_0(k, s) + \int G(k, s, \vec{r}\,') \frac{e^{-s_0 r'}}{r'} \psi_+(\vec{r}\,') d\vec{r}\,', \tag{4}$$

with

$$F_0(k, s) = \int e^{-\hat{k}_f \cdot \vec{r} k} \frac{e^{-sr}}{r} e^{i\vec{k}_0 \cdot \vec{r}} d\vec{r}.$$

In the next section we derive a line-integral representation for $G(k, s, \vec{r}\,')$ with which we may carry out the integration over $\vec{r}\,'$ in the last term.

III. EQUATION FOR THE TRANSFORM, $G(k,s,\vec{r})$, AND ITS SOLUTION

The free-particle Green's function

$$\frac{-1}{2\pi} \frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = G_0(\vec{r}, \vec{r}\,')$$

satisfies the well-known identity

$$+\tfrac{1}{2}(\nabla_r^2 + k_0^2) G_0(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,').$$

Multiplying both sides by $e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr}$ and integrating, we have

$$\int e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr} \tfrac{1}{2}(\nabla_r{}^2 + k_0^2) G_0(\vec{r}, \vec{r}\,') dr = e^{-i\hat{k}_f \cdot \vec{r}\,' k} e^{-sr'}.$$

Using Green's theorem, and provided that $\mathrm{Re}(s) > 0$, so that no surface term arises, we have

$$\int G_0(\vec{r}, \vec{r}\,') \tfrac{1}{2}(\nabla^2 + k_0^2) e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr} d\vec{r} = e^{-i\hat{k}_f \cdot \vec{r}\,' k} e^{-sr'}.$$

$$\nabla^2 e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr} = \nabla \cdot (\nabla e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr}) = \nabla \cdot (-i\hat{k}_f k - s\hat{r}) e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr}$$

$$= [(i\hat{k}_f k + s\hat{r})^2 - 2s/r] e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr} = [s^2 - k^2 + 2isk(\hat{k}_f \cdot \vec{r}/r) - 2s/r] e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr},$$

so that

$$\tfrac{1}{2}(\nabla^2 + k_0^2) e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr} = \tfrac{1}{2}(s^2 - k^2 + k_0^2) e^{-sr} e^{-i\hat{k}_f \cdot \vec{r} k} + [isk(\hat{k}_f \cdot \vec{r}/r) - s/r] e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr}$$

$$= \left(-\tfrac{1}{2}(s^2 - k^2 + k_0^2) \frac{\partial}{\partial s} - s - sk \frac{\partial}{\partial k}\right) \frac{e^{-i\hat{k}_f \cdot \vec{r} k} e^{-sr}}{r}.$$

If we let

$$G(s, k, \vec{r}\,') = \int \frac{e^{-sr}}{r} e^{-i\hat{k}_f \cdot \vec{r} k} G_0(\vec{r}, \vec{r}\,') d\vec{r},$$

then $G(s, k, \vec{r}\,')$ satisfies the *first-order* partial differential equation

$$\tfrac{1}{2}(s^2 - k^2 + k_0^2) \frac{\partial G}{\partial s} + sG + sk \frac{\partial G}{\partial k} = -e^{-i\hat{k}_f \cdot \vec{r}\,' k} e^{-sr'}. \tag{5}$$

$G(s, k, \vec{r}\,')$ is a transform of the free-particle Green's function which is very similar in form to a transform suggested for bound-state perturbation problems involving the hydrogen-atom Hamiltonian as H_0. The only difference is that the $\hat{z}$ has been replaced by $\hat{k}_f$, which is not regarded as fixed in space but rather as a variable parameter.[3]

Before discussing the solution of Eq. (5), one point should be made. As with all first-order equations the solution will involve some sort of integral over the inhomogeneity. As far as this integration is concerned however, $\vec{r}\,'$ will not be involved: $\vec{r}\,'$ occurs simply as a *parameter* labeling the inhomogeneity of Eq. (5); i.e., $\vec{r}\,'$ does not occur in the operator

$$\tfrac{1}{2}(s^2 - k^2 + k_0^2) \frac{\partial}{\partial s} + s + ks \frac{\partial}{\partial k}.$$

Since this is the case, we may carry out the integration $\vec{r}\,'$ in Eq. (4) and express the result directly in terms of F; this is possible because the combination

$$\int e^{-s'r'} e^{-i\hat{k}_f \cdot \vec{r}\,' k'} \frac{e^{-s_0 r'}}{r} \psi_+(\vec{r}\,') dr' = F(k', s_0 + s').$$

We now obtain an explicit solution of Eq. (5). The equation is linear and first order, so the method of characteristics is applicable.[4] Dividing through by ks, we must then solve

$$\frac{\partial k}{\partial \tau} = 1, \quad \frac{\partial s}{\partial \tau} = \frac{1}{2} \frac{s^2 + k_0^2 - k^2}{ks}.$$

This pair has the solution $k = \tau$, $s^2 = \sigma\tau - \tau^2 - k_0^2$. That is, $\tau = k$, $\sigma = (s^2 + k^2 + k_0^2)/k$ are the two characteristic functions. (τ = constant and σ = constant define the two sets of characteristic lines in the k, s plane.) In terms of the variables τ and σ, Eq. (5) becomes

$$\frac{\partial G}{\partial \tau} + \frac{G}{\tau} = -\frac{e^{-s(\tau, \sigma) r'} e^{-i\hat{k}_f \cdot \vec{r}\,' \tau}}{\tau s(\tau, \sigma)}, \tag{5'}$$

where $s(\tau, \sigma)$ is some branch of $s^2 = \sigma\tau - \tau^2 - k_0^2$. This gives for G

$$G(\tau,\sigma,\vec{r}\,')=-\frac{1}{\tau}\left(\int_0^\tau \frac{e^{-s(\tau,\sigma)r'}e^{-i\hat{k}_f\cdot\vec{r}\,'\tau}}{s(\tau,\sigma)}+A(\sigma,\vec{r}\,')\right) \tag{6}$$

the desired line-integral representation. $A(\sigma,\vec{r}\,')$ is an as yet undertermined function of σ and $\vec{r}\,'$ which arises as a constant (as far as the variable τ is concerned) of integration in the solution of (5').

There are three major difficulties associated with this representation for G which must be resolved before it may be used in Eq. (4). First there is the ambiguity of $A(\sigma,\vec{r}\,')$. The only value k for which

$$G(s,k,\vec{r}\,')=-\frac{1}{2\pi}\int\frac{e^{-sr}}{r}e^{-i\hat{k}_f\cdot\vec{r}k}\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\vec{r}$$

may be easily evaluated is $k=0$. When $k=0$, however, $\sigma=\infty$, so that setting $k=0(=\tau)$ in (6) gives $A(\sigma,\vec{r}\,')=0$ at the single point $\sigma=\infty$, which is not sufficient; we need $A(\sigma,\vec{r}\,')$ for all σ. This is related to the fact that the line $k=0$ cuts all of the characteristic lines $\sigma=$constant in the single point $s^2=-k_0^2$. Furthermore at this point, and for τ close enough to 0, s must be pure imaginary, and Eq. (5) was derived assuming that s had a positive real part (so that no surface term arose from the application of Green's theorem). Finally, there remains another sort of ambiguity in Eq. (6): for $s^2(\tau,\sigma)<0$, which branch should be chosen for $s(\tau,\sigma)$: positive pure imaginary or negative?

In order to resolve these difficulties, we present here an alternate derivation of Eq. (6). Consider the quantity

$$D(\vec{r}\,')=\frac{-1}{2\pi}\int\frac{e^{-sr}e^{-i\hat{k}_f\cdot\vec{r}k}}{r}\frac{e^{k_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}d\vec{r}+\frac{1}{k}\int_0^k\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}\,'\tau}}{s'}d\tau \tag{7}$$

with s' a definite but as yet unspecified branch of $s'^2=\sigma\tau-\tau^2-k_0^2$, and $\sigma=(s^2+k^2+k_0^2)/k$, as before. Then

$$\tfrac{1}{2}(\nabla'^2+k_0^2)\frac{1}{k}\int_0^k\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}\,'\tau}}{s'}d\tau=\frac{1}{k}\int_0^k\left(\tfrac{1}{2}(s'^2-\tau^2+k_0^2)-\frac{s'}{r'}+i\tau s'\frac{\hat{k}_f\cdot\vec{r}\,'}{r'}\right)\frac{1}{s'}e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}\,'\tau}d\tau$$

$$=-\frac{1}{k}\int_0^k\frac{\partial}{\partial\tau}\frac{\tau e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}\,'\tau}}{r'}d\tau\quad\left(\text{since }\frac{\partial s'}{\partial\tau}=\frac{1}{2}\frac{s'^2-\tau^2+k_0^2}{\tau s'}\right)=-\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}\,'k}}{r'},$$

so that $D(\vec{r}\,')$ satisfies

$$\tfrac{1}{2}(\nabla'^2+k_0^2)D(\vec{r}\,')=0\,.$$

Writing

$$D(\vec{r}\,')=\int e^{i\vec{x}\cdot\vec{r}\,'}d(\vec{x})d\vec{x},$$

we see that

$$d(\vec{x})=\delta(|x|-k_0)g(\hat{x})/k_0^2,$$

where $g(\hat{x})$ is an undetermined function of unit vector $\hat{x}$. Then $D(\vec{r}\,')=\int e^{ik\vec{r}\,'\cdot\hat{x}}g(\hat{x})d\Omega_{\hat{x}}$. We fix $g(\hat{x})$ by looking at the asymptotic form of $D(\vec{r}\,')$ for large $\vec{r}\,'$. For large $\vec{r}\,'$,

$$e^{ik_0\vec{r}\,'\cdot\hat{x}}=\frac{2\pi}{ik_0r'}\left[e^{ik_0r'}\delta(\Omega_{\hat{r}'}-\Omega_{\hat{x}})-e^{-ik_0r'}\delta(\Omega_{\hat{r}'}+\Omega_{\hat{x}})\right]+O(1/r'^2)\,.$$

Thus for large $\vec{r}\,'$,

$$D(\vec{r}\,')=(2\pi/ik_0r')[e^{ik_0r'}g(\hat{r}')-e^{-ik_0r'}g(-\hat{r}')]+O(1/r'^2)\,. \tag{8}$$

Comparing this with the asymptotic form of Eq. (7), we obtain directly an equation for the unknown function $g(\hat{r}')$. We actually get two equations, one for $g(\hat{r}')$ and one for $g(-\hat{r}')$ as $e^{ik_0r'}$ and $e^{-ik_0r'}$ are linearly independent.

We first need the asymptotic form of

$$-\frac{1}{2\pi}\int\frac{e^{-sr}}{r}e^{-i\hat{k}_f\cdot\vec{r}k}\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}d\vec{r}$$

$$=\frac{-2e^{ik_0r'}}{r'}\frac{1}{s^2+k^2+k_0^2+2kk_0\hat{k}_f\cdot\hat{r}'}+O\left(\frac{1}{r'^2}\right)$$

$$=\frac{-2e^{ik_0r'}}{kr'}\frac{1}{\sigma+2k_0\hat{k}_f\cdot\hat{r}'}+O\left(\frac{1}{r'^2}\right).$$

The asymptotic form of the second term of (7) is also required; its value depends on which branch is chosen for

$$s'=i(\tau^2+k_0^2-\sigma\tau)^{1/2}$$
$$=i[(\tau-\tfrac{1}{2}\sigma)^2-\tfrac{1}{4}\sigma^2+k_0^2]^{1/2}$$
$$=i(\tau-\tfrac{1}{2}\sigma-p)^{1/2}(\tau-\tfrac{1}{2}\sigma+p)^{1/2},$$

with $p^2=\frac{1}{4}\sigma^2-k_0^2$. The two branch points of s' are therefore at $\tau=\frac{1}{2}\sigma+p$, $\tau=\frac{1}{2}\sigma-p$. We clearly desire s' positive real for $\tau=k$. As $\frac{1}{2}\sigma-p<k<\frac{1}{2}\sigma+p$, there are thus only two possibilities for choosing the branches (see Fig.1): Choice I makes s'/i positive on $0\leq\tau\leq\frac{1}{2}\sigma-p$; choice II makes s'/i negative on $0\leq\tau\leq\frac{1}{2}\sigma-p$.

Let us first investigate the consequences of choosing the branch as in I. The integral over τ must be broken up into two pieces:

$$+\frac{1}{k}\int_0^k \frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s'}d\tau = +\frac{1}{k}\int_0^{\sigma/2-p}\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s'}d\tau + \frac{1}{k}\int_{\sigma/2-p}^{k}\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s'}d\tau\,. \tag{9}$$

For $0\leqslant\tau\leqslant\frac{1}{2}\sigma-p$ and the branch of s' in I,

$$\begin{aligned} s' &= i\,|\tau-(\sigma/2+p)|^{1/2}\,|\tau-\sigma/2+p|^{1/2}\\ &= i\,(\sigma/2+p-\tau)^{1/2}(\sigma/2-p-\tau)^{1/2}\\ &= i\,(\tau^2-\sigma\tau+k_0^2)^{1/2}\,. \end{aligned}$$

Thus the first term on the right becomes

$$\frac{1}{k}\int_0^{\sigma/2-p}\frac{e^{-i(\tau^2-\sigma\tau+k_0^2)^{1/2}r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{i(\tau^2-\sigma\tau+k_0^2)^{1/2}}d\tau\,.$$

Letting $u=\hat{k}_f\cdot\hat{r}'$, $x=(\tau^2-\sigma\tau+k_0^2)^{1/2}+u\tau$, and integrating by parts, this term goes like

$$\frac{1}{ki}\left(-\frac{1}{ir'}\right)\left(-e^{-ir'u(\sigma/2-p)}\frac{1}{p}-\frac{e^{-ir'k_0}}{-\sigma/2+k_0u}\right)+O\left(\frac{1}{r'^2}\right). \tag{10}$$

As for the second term on the right-hand side of (9),

$$+\frac{1}{k}\int_{\sigma/2-p}^{k}\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s'}d\tau = \frac{1}{k}\int_{\sigma/2-p}^{k}\frac{e^{-(\sigma\tau-\tau^2-k_0^2)^{1/2}}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{(\sigma\tau-\tau^2-k_0^2)^{1/2}}d\tau\,.$$

Setting $u=\hat{k}_f\cdot\hat{r}'$, $x=(\sigma\tau-\tau^2-k_0^2)^{1/2}+iu\tau$, and integrating by parts again, we get only a contribution from the lower end point for large r',

$$\frac{1}{k}\left(+\frac{1}{r'}\right)\left(\frac{e^{-ir'u(\sigma/2-p)}}{p}\right),$$

which just cancels the first term of (10). Thus for large $\vec{r}'$,

$$\frac{1}{k}\int_0^k\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\hat{r}'\hat{r}'\tau}}{s'}d\tau = -\frac{1}{kr'}\;\frac{e^{-ir'k_0}}{-\sigma/2+k_0\hat{k}_f\cdot\hat{r}'}+O\left(\frac{1}{r'^2}\right).$$

We have therefore,

$$D(\vec{r}') = -\frac{2}{kr'}\left(\frac{e^{ir'k_0}}{\sigma+2k_0\hat{k}_f\cdot\hat{r}'} - \frac{e^{-ir'k_0}}{\sigma-2k_0\hat{k}_f\cdot\hat{r}'}\right)+O\left(\frac{1}{r'^2}\right).$$

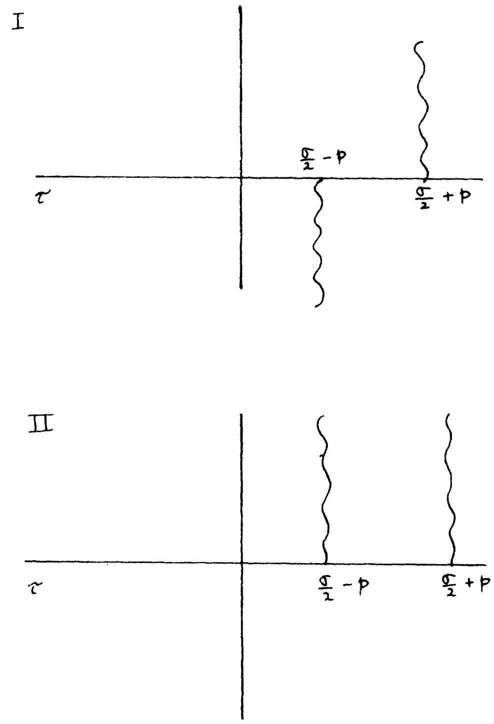

FIG. 1. Branches of $s'(\tau)$.

Direct comparison with Eq. (8) shows that

$$g(\hat{r}') = -\frac{ik_0}{\pi k}\,\frac{1}{\sigma+2k_0\hat{k}_f\cdot\hat{r}'}\,,$$

$$g(-\hat{r}') = -\frac{ik_0}{\pi k}\,\frac{1}{\sigma-2k_0\hat{k}_f\cdot\hat{r}'}\,.$$

Thus a single function $g(\hat{r}')$ exists which satisfies the two independent equations implicit in (8). Furthermore

$$g(\hat{x}) = -\frac{ik_0}{\pi k}\,\frac{1}{\sigma+2k_0\hat{x}\cdot\hat{k}_f}$$

shows that

$$D(\vec{r}') = \int e^{ik_0\vec{r}'\cdot\hat{x}}g(\hat{x})d\Omega_{\hat{x}}$$

is in fact a function of the form $A(\sigma,\vec{r}')/k$ as Eq. (6) implies it must be.

Without going through the same details, it can be shown that the second choice of branch for s', as in II, gives an even simpler result, $g(\hat{x})\equiv 0$, i.e.,

$$-\frac{1}{2\pi}\int\frac{e^{-sr}e^{-i\hat{k}_f\cdot\vec{r}k}}{r}\,\frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dr = -\frac{1}{k}\int_0^k\frac{e^{-s'r'}e^{-i\hat{k}_f\cdot\vec{r}'}}{s'}d\tau\,. \tag{11}$$

Inserting this form for $G(s,k,\vec{r}')$ into (4), we see that

$$F(k,s)=F_0(k,s)-\frac{1}{k}\int_0^k \frac{e^{-s'\tau}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s'}\times\frac{e^{-s_0r'}}{r'}\psi_+(\vec{r}')d\vec{r}'d\tau$$

$$=F_0(k,s)-\frac{1}{k}\int_0^k \frac{F(\tau,s_0+s')}{s'}d\tau\,. \qquad (12)$$

The absence of any additional term $D(\vec{r}')$ thus gives rise to an inhomogeneous integral equation for the Yukawa-potential scattering amplitude with one important distinguishing feature. The amplitude must be integrated over only a line in k-s space; i.e., an ordinary one-dimensional integration must be performed. Thus (12) is ideally suited to either iteration or direct numerical attack.[5]

IV. COMPARISON WITH THE BORN SERIES AND GENERALIZATIONS TO OTHER POTENTIALS

Perhaps the most obvious way to solve Eq. (12) is to iterate it. Writing

$$F_n(k,s)=-\frac{1}{k}\int_0^k \frac{F_{n-1}(\tau,s'+s_0)}{s'}d\tau,\quad n\geq 1\,,$$

generates the series for $F(k,s)$,

$$F(k,s)=\sum_{n=0}^{\infty}F_n(k,s)\,.$$

It is clear that the evaluation of the nth iterate depends on a knowledge of the $(n-1)$th, $F_{n-1}(x,y)$, but only along the line in the x-y plane $x=\tau, y=s(\tau,\sigma)+s_0,\ 0\leq\tau\leq k$.

This is to be contrasted with the ordinary way of calculating the terms in the Born Series in which the integral

$$T_n(\vec{k}_f,\vec{k}_i)=\lim_{\epsilon\to 0}\int T_0(\vec{k}_f,\vec{k})\frac{1}{\frac{1}{2}k_0^2-\frac{1}{2}k^2+i\epsilon}\times T_{n-1}(\vec{k},\vec{k}_i)d\vec{k}$$

is required, where

$$T_0(\vec{k}_f,\vec{k}_i)=\int e^{-i\vec{k}_f\cdot\vec{r}}\frac{e^{-s_0r}}{r}e^{i\vec{k}_i\cdot\vec{r}}d\mathbf{r}=F_0(k,s_0)\,.$$

To calculate T_n, one must have the $(n-1)$th iterate for all values of the vector $\vec{k}$, carry out a difficult volume integration, and then let $\epsilon\to 0$.

Generalization to central potentials which may be expressed as a superposition of Yukawa-type potentials follows the usual pattern. Writing

$$V(r)=\int_0^\infty \frac{e^{-s'r}}{r}v(s')ds'\,,$$

we must solve

$$F(k,s)=F_0(k,s)-\frac{1}{k}\int_0^\infty\int_0^k \frac{F(\tau,s(\tau,\sigma)+s')}{s(\tau,\sigma)}\times v(s')d\tau ds' \qquad (12')$$

with

$$f_+(\Omega)=-\frac{1}{2\pi}\int F(k_0,s')v(s')ds'\,,$$

and

$$F(k,s)=\int\frac{e^{-sr}}{r}e^{-i\hat{k}_f\cdot\vec{r}k}\psi_+(\vec{r})d\vec{r}$$

as before.

Again the most obvious way to solve (12′) is to iterate. At each state, one has the extra integral over $v(s')$ to do, but of course this is also the case with the iterative scheme for the T_n's discussed above. Thus a calculation based on (12′) is simpler for the same reason that the case $v(s')=\delta(s'-s_0)$ is simpler.

Before illustrating this point in Sec. V, where the second term in the Born series for $V(r)=e^{-r}/r$ is recovered, some remarks concerning generalizations to angle-dependent potentials might be included here.

The derivation of Eq. (12) depended upon the representation

$$-\frac{1}{2\pi}\int\frac{e^{-sr}}{r}e^{-i\hat{k}_f\cdot\vec{r}k}\frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d\vec{r}=-\frac{1}{k}\int_0^k\frac{e^{-s(\tau,\sigma)r'}e^{-i\hat{k}_f\cdot\vec{r}'\tau}}{s(\tau,\sigma)}d\tau\,.$$

Replacing $k\hat{k}_f$ by $k\hat{k}_f+\alpha\hat{z}$, we have then

$$-\frac{1}{2\pi}\int\frac{e^{-sr}}{r}e^{-i\hat{k}_f\cdot\vec{r}k}e^{-i\alpha\vec{r}\cdot\hat{z}}\frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d\vec{r}=-\frac{1}{K}\int_0^k\frac{e^{-s(\tau,\Sigma)r'}e^{-i\hat{n}\cdot\vec{r}'\tau}}{s(\tau,\Sigma)}d\tau\,,$$

where

$$K=(k^2+\alpha^2+k\alpha\hat{k}_f\cdot\hat{z})^{1/2}\,,$$

$$\hat{n}=(k\hat{k}_f+\alpha\hat{z})/K\,,$$

$$\Sigma=(s^2+K^2+k_0^2)/K\,.$$

Differentiation on the left with respect to α evaluated at $\alpha=0$ will bring down as many factors of $(r\cos\theta)$ as desired. This makes possible the evaluation of scattering amplitudes due to superpositions of potentials of the type $[e^{-s_0r}/r](r\cos\theta)^n$ as follows: Letting

$$F(k, s; \hat{k}_f) = \int e^{-i\hat{k}_f \cdot \vec{r} k} \frac{e^{-sr}}{r} (r\cos\theta)^n \psi_+(\vec{r}) d\vec{r}$$

$$= \int e^{-i\hat{k}_f \cdot \vec{r} k} \frac{e^{-sr}}{r} (r\cos\theta)^n e^{i\vec{k}_i \cdot \vec{r}} d\vec{r} - \frac{1}{2\pi} \int \frac{e^{-sr}}{r} e^{-i\hat{k}_f \cdot \vec{r} k} (r\cos\theta)^n \frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \frac{e^{-s_0 r'}}{r'} (r'\cos\theta')^n \psi_+(\vec{r}') d\vec{r} d\vec{r}'$$

$$= F_0(k, s; \hat{k}_f) - \frac{1}{2\pi}\left(-\frac{1}{i}\right)^n \frac{\partial^n}{\partial\alpha^n}\Big|_{\alpha=0} \int \frac{e^{-sr}}{r} e^{-i\hat{k}_f \cdot \vec{r} k} e^{-i\alpha\hat{z}\cdot\vec{r}} \frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \frac{e^{-s_0 r'}}{r'} (r'\cos\theta')^n \psi_+(\vec{r}') d\vec{r} d\vec{r}'$$

$$= F_0(k, s; \hat{k}_f) + \left(-\frac{1}{i}\right)^n \frac{\partial^n}{\partial\alpha^n}\Big|_{\alpha=0} \left(-\frac{1}{K} \int \int_0^K \frac{e^{-s(\tau,\Sigma)r'}}{s(\tau,\Sigma)} e^{-i\hat{n}\cdot\vec{r}'\tau} \frac{e^{-s_0 r'}}{r'} (r'\cos\theta')^n \psi_+(\vec{r}') d\vec{r}' d\tau\right.$$

$$= F_0(k, s; \hat{k}_f) - \left(-\frac{1}{i}\right)^n \frac{\partial^n}{\partial\alpha^n}\Big|_{\alpha=0} \left(\frac{1}{K} \int_0^K \frac{F(\tau, s(\tau,\Sigma)+s_0; \hat{n})}{s(\tau,\Sigma)} d\tau\right),$$

where the dependence of F on the vector $\hat{n}$ has been explicitly indicated here. This equation may also be treated by iteration as before.

V. SECOND BORN TERM FOR THE YUKAWA POTENTIAL

As a check on the procedure outlined above, Eq. (12) was iterated once, with $V(\vec{r}) = e^{-r}/r$. This must generate the second Born term for which an analytic expression is available.

The first iterate of Eq. (12) gives

$$F_1(k, s) = -\frac{1}{k} \int \frac{F_0(\tau, s'+1)}{s'} d\tau,$$

with

$$F_0(k, s) = \int e^{-i\vec{k}_f \cdot \vec{r}} \frac{e^{-sr}}{r} e^{i\vec{k}_i \cdot \vec{r}} d\vec{r} = \frac{4\pi}{s^2 + (\vec{k}_i - \vec{k}_f)^2}$$

so on dividing each F_n by $-1/2\pi$, we have

$$F_1(k, s) = \frac{2}{k} \int_0^k \frac{d\tau}{[(s'+1)^2 + (\vec{k}_i - \hat{k}_f \tau)^2]} \frac{1}{s'}$$

with the branch for s' chosen as in Sec. IV. We are of course interested in only $F_1(k_0, 1)$, corresponding to elastic scattering off e^{-r}/r. Dividing the τ integral up into the two regions:

$$0 \leq \tau \leq \sigma/2 - p \text{ and } \sigma/2 - p \leq \tau \leq k,$$

with

$$\sigma = (1 + 2k_0^2)/k_0, \quad p = (\sigma^2/4 - k_0^2)^{1/2},$$

$$F_1(k_0, 1) = \frac{2}{k_0}\left(\int_0^{\sigma/2-p} + \int_{\sigma/2-p}^{k}\right) \times \left(\frac{d\tau}{[(s'+1)^2 + (\vec{k}_i - \hat{k}_f\tau)^2]s'}\right). \tag{13}$$

In the first of these integrals, I_1, the branch of s' has been chosen such that

$$s' = -i|\tau - (\sigma/2 - p)|^{1/2}|\tau - (\sigma/2 + p)|^{1/2} = -i(\sigma/2 - p - \tau)^{1/2}(\sigma/2 + p - \tau)^{1/2}.$$

Setting $\tau = \sigma/2 - p\cosh\theta$, $\theta_0 = \cosh^{-1}(\sigma/2p)$ (and > 0),

$$s' = -i[p(\cosh\theta - 1)]^{1/2}[p(\cosh\theta + 1)]^{1/2} = -ip\sinh\theta,$$

we have

$$I_1 = \frac{2}{ik_0} \int_{\theta_0}^{0} \frac{d\theta}{[(-ip\sinh\theta + 1)^2 + k_0^2 - 2\vec{k}_i \cdot \hat{k}_f(\sigma/2 - p\cosh\theta) + \sigma^2/4 - p\sigma\cosh\theta + p^2\cosh^2\theta]}$$

$$= \frac{2}{ik_0} \int_{\theta_0}^{0} \frac{d\theta}{p^2 - 2ip\sinh\theta + k_0^2 + \sigma^2/4 - 2k\cos\chi(\sigma/2 - p\cosh\theta) - p\sigma\cosh\theta}$$

$$= -\frac{2i}{k_0} \int_{\theta_0}^{0} \frac{d\theta}{C^2 - Ai\sinh\theta - B\cosh\theta},$$

with

$$C^2 = p^2 + 1 + \tfrac{1}{4}\sigma^2 + k_0^2 - k_0\cos\chi,$$

$A = 2p, \quad B = p\sigma - 2pk_0 \cos\chi ,$

$\hat{k}_i \cdot \hat{k}_f = k_0 \cos\chi ,$

$$I_1 = \frac{4i}{k_0(C^2+B)} \int_{x_0}^{0} \frac{dx}{x^2 + 2iA/(B+C^2) + (B-C^2)/(C^2+B)} \; [x_0 = 2k_0/(\sigma+2p)]$$

$$= \frac{4i}{k_0(C^2+B)} \int_{x_0}^{0} \frac{dx}{[x + iA/(C^2+B)]^2 - Q^2/(C^2+B)^2} \quad (Q^2 = C^4 - A^2 - B^2)$$

$$I_1 = \frac{2i}{k_0 Q} \ln\left(\frac{[x_0 + iA/(C^2+B) + Q/(C^2+B)][iA/(C^2+B) - Q/(C^2+B)]}{[x_0 + iA/(C^2+B) - Q/(C^2+B)][iA/(C^2+B) + Q/(C^2+B)]}\right) . \tag{14}$$

The second term of (13), I_2, may be worked out in a similar fashion. Setting $\tau = \frac{1}{2}\sigma - p\cos\theta$, the appropriate branch for s' is $s' = p\sin\theta$. Omitting further algebraic detail, we have

$$I_2 = \frac{2i}{k_0 Q} \ln\left(\frac{[x_1 + A/(C^2+B) + Q/(C^2+B)][A/(C^2+B) - iQ/(C^2+B)]}{[x_1 + A/(C^2+B) - iQ/(C^2+B)][A/(C^2+B) + Q/(C^2+B)]}\right), \tag{15}$$

with $x_1 = 2k_0/(1+2k_0 p)$, and everything else as before. Clearly the imaginary part of the scattering amplitude comes entirely from I_1; both I_1 and I_2 contribute to the real part. Reference 1 shows that $I_1 + I_2$ must reduce to

$$\frac{2}{k_0 S \sin(\sigma/2)} \tan^{-1}\left(\frac{k_0 \sin(\theta/2)}{S}\right) + \frac{i}{2} \ln\left(\frac{S + 2k_0^2 \sin(\theta/2)}{S - 2k_0^2 \sin(\sigma/2)}\right), \tag{16}$$

with

$$S^2 = 1 + 4k_0^2 + 4k_0^4 \sin^2(\theta/2), \quad \hat{k}_i \cdot \hat{k}_f = \cos\theta .$$

This is demonstrated in the Appendix.

VI. CONCLUSIONS

A transform of the free-particle Green's function has been examined which has a particularly simple line-integral representation. Using this representation the Yukawa-potential scattering amplitude emerges as the solution of an inhomogeneous *line*-integral equation. Compared with the usual inhomogeneous integral equation for the scattering amplitude, this equation should be far simpler to handle, either by direct numerical attack or by iteration, as only a one-dimensional integration must be performed. Generalizations to general central potentials and angle-dependent potentials were also considered. Finally, an example, the second Born approximation for the Yukawa potential, was recovered by iterating the equation once.

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APPENDIX

First let us extract the imaginary part of I_1; this comes from the real part of the logarithm. Thus

$$\mathrm{Im}F_1(k_0, 1) = \frac{2}{k_0 Q}\,\frac{1}{2} \ln\left(\frac{[x_0 + Q/(C^2+B)]^2 + A^2/(C^2+B)^2}{[x_0 - Q/(C^2+B)]^2 + A^2/(C^2+B)^2}\right) .$$

In terms of $\sin(\theta/2)$,

$$p^2 = 1 + 1/4k_0^2, \quad C^2 + 2p^2 + (2 + 4k_0^2)\sin^2(\theta/2),$$

$$B = p/k_0 + 4pk_0 \sin^2(\theta/2), \quad A = 2p,$$

so that

$$Q^2 = C^4 - A^2 - B^2 = \sin^2(\theta/2)[4 + 16k_0^2 + 16k_0^4 \sin^2(\theta/2)],$$

$$Q = 2\sin(\theta/2)\, S,$$

where S is given in (16). Then one has

$$\mathrm{Im}F_1(k_0, 1) = \frac{1}{k_0 Q} \ln\left(\frac{x_0^2(C^2+B)^2 + 2x_0(C^2+B)Q + Q^2 + A^2}{x_0^2(C^2+B)^2 - 2x_0(C^2+B)Q + Q^2 + A^2}\right)$$

$$= \frac{1}{k_0 Q} \ln\left(\frac{x_0^2(C^2+B) + 2x_0 Q + C^2 - B}{x_0^2(C^2+B) - 2x_0 Q + C^2 - B}\right)$$

$$= \frac{1}{k_0 Q} \ln\left(\frac{C^2 + 2x_0 Q/(x_0^2+1) + B(x_0^2-1)/(x_0^2+1)}{-C^2 - 2x_0 Q/(x_0^2+1) + B(x_0^2-1)/(x_0^2+1)}\right) .$$

But

$$\frac{2x_0}{1+x_0{}^2}=\frac{2k_0}{\sigma} \text{ and } \frac{x_0{}^2-1}{x_0{}^2+1}=-\frac{2p}{\sigma},$$

so that

$$\mathrm{Im}F_1(k_0,1)=\frac{1}{k_0Q}\ln\left(\frac{C^2+2k_0Q/\sigma-2pB/\sigma}{C^2-2k_0Q/\sigma-2pB/\sigma}\right)$$

$$=\frac{1}{k_0Q}\ln\left(\frac{[2p^2+(2+4k_0^2)\sin^2(\theta/2)](1+2k_0^2)+2k_0^2Q-2p^2[1+4k_0^2\sin^2(\theta/2)]}{[2p^2+(2+4k_0^2)\sin^2(\theta/2)](1+2k_0^2)-2k_0^2Q-2p^2[1+4k_0^2\sin^2(\theta/2)]}\right)$$

$$=\frac{1}{k_0Q}\ln\left(\frac{\sin^2(\theta/2)[(1+2k_0^2)(2+4k_0^2)-2(1+4k_0^2)]+4p^2k_0^2+2k_0^2Q}{\sin^2(\theta/2)[(1+2k_0^2)(2+4k_0^2)-2(1+4k_0^2)]+4p^2k_0^2-2k_0^2Q}\right)$$

$$=\frac{1}{k_0Q}\ln\left(\frac{8\sin^2(\theta/2)k_0^4+1+4k_0^2+2k_0^2Q}{8\sin^2(\theta/2)k_0^4+1+4k_0^2-2k_0^2Q}\right)$$

$$=\frac{1}{k_0Q}\ln\left(\frac{Q^2/[4\sin^2(\theta/2)]+4k_0^4\sin^2(\theta/2)+2k_0^2Q}{Q^2/[4\sin^2(\theta/2)]+4k_0^4\sin^2(\theta/2)-2k_0^2Q}\right)$$

$$=\frac{1}{k_0Q}\ln\left(\frac{Q^2+8k_0^2\sin^2(\theta/2)Q+16k_0^4\sin^4(\theta/2)}{Q^2-8k_0^2\sin^2(\theta/2)Q+16k_0^4\sin^4(\theta/2)}\right)$$

$$=\frac{2}{k_0Q}\ln\left(\frac{Q+4k_0^2\sin^2(\theta/2)}{Q-4k_0^2\sin^2(\theta/2)}\right)=\frac{1}{k_0S\sin(\theta/2)}\ln\left(\frac{S+2k_0^2\sin(\theta/2)}{S-2k_0^2\sin(\theta/2)}\right).$$

This checks with the imaginary part of (16).

Combining I_1 and I_2 [Eqs. (14) and (15)] for the calculation of $\mathrm{Re}F_1(k_0,1)$,

$F_1(k_0,1)$

$$=\frac{2i}{k_0Q}\ln\left(\frac{\left[x_0+\frac{Q}{(C^2+B)}\right]\left[x_1+\frac{A}{(C^2+B)}\right]-\frac{AQ}{(C^2+B)}+i\left\{\left[x_0+\frac{Q}{(C^2+B)}\right]\left[\frac{Q}{(C^2+B)}\right]+\left[x_1+\frac{A}{(C^2+B)}\right]\left[\frac{A}{(C^2+B)}\right]\right\}}{(Q\to-Q)}\right).$$

We are now only interested in $\mathrm{Re}F_1(k_0,1)$:

$$\mathrm{Re}F_1(k_0,1)=-(2/k_0Q)\,\mathrm{Im}\ln z,$$

where z is the argument of ln above, i.e.,

$$z=\frac{x_0x_1(C^2+B)+Qx_1+Ax_0+i(Ax_1+Qx_0+C^2-B)}{(Q\to-Q)}.$$

Noting first that

$$C^2-B=[p/k_0+(2k_0p-1)\sin^2(\theta/2)](2pk_0-1),$$

replacing Q by $2\sin(\theta/2)S$ of the previous section, and multiplying z by

$$\frac{(\sigma+2p)(1+2k_0p)}{(\sigma+2p)(1+2k_0p)},$$

one obtains

$$z = \frac{4k_0^2(C^2+B) + 4k_0 p(1+2k_0 p) + 4k_0 \sin(\theta/2)\,(\sigma+2p)S}{(S \rightarrow -S)}$$

$$+ \frac{i[4k_0 p(\sigma+2p) + 4k_0 \sin(\theta/2)\,S(1+2k_0 p) + (C^2-B)(\sigma+2p)(1+2k_0 p)]}{(S \rightarrow -S)}$$

$$= \frac{4k_0^2[2p^2 + (2+4k_0^2)\sin^2(\theta/2) + p/k_0 + 4pk_0 \sin^2(\theta/2)]}{(S \rightarrow -S)}$$

$$+ \frac{4k_0 p(1+2k_0 p) + 4k_0 \sin(\theta/2)\,S(\sigma+2p) + 4k_0^2}{(S \rightarrow -S)}$$

$$+ \frac{i\{4k_0 p(\sigma+2p) + 4k_0 \sin(\theta/2)\,S(1+2k_0 p) + 4k_0^2(\sigma+2p)[p/k_0 + (2k_0 p - 1)\sin^2(\theta/2)]\}}{(S \rightarrow -S)}$$

$$= \frac{8pk_0[1 + 2k_0^2 \sin^2(\theta/2) + S\sin(\theta/2)] + 4 + 16k_0^2 + \sin^2(\theta/2)\,(8k_0^2 + 16k_0^4) + [4 + 8k_0^2 \sin(\theta/2)]\,S}{(S \rightarrow -S)}$$

$$+ \frac{i\{p[8 + 16k_0^2 + 8k_0^2 \sin(\theta/2)\,S + 16k_0^4 \sin^2(\theta/2)] + (1/k_0)[16k_0^2 + 4 + 4k_0^2 \sin(\theta/2)\,S + 8\sin^2(\theta/2)k_0^4]\}}{(S \rightarrow -S)}$$

$$= \frac{8pk_0\{1 + \sin(\theta/2)\,[2k_0^2 \sin(\theta/2) + S]\} + 4[S + 2k_0^2 \sin(\theta/2)][S + \sin(\theta/2)]}{(S \rightarrow -S)}$$

$$+ \frac{i(4p\{1 + S[S + 2k_0^2 \sin(\theta/2)]\} + (4/k_0)[S + 2k_0^2 \sin(\theta/2)][S - k_0^2 \sin(\theta/2)])}{(S \rightarrow -S)}$$

$$= \frac{[S + 2k_0^2 \sin(\theta/2)][S + \sin(\theta/2) + 2pk_0] + 2pk_0}{(S \rightarrow -S)}$$

$$+ \frac{i\{[S + 2k_0^2 \sin(\theta/2)][S/k_0 - k_0 \sin(\theta/2) + Sp] + p\}}{(S \rightarrow -S)}.$$

Writing $z = z_1/z_2$,

$$\operatorname{Re} z = \frac{1}{|z_2|^2}\,(\{[S + 2k_0^2 \sin(\theta/2)][S + \sin(\theta/2) + 2pk_0 \sin(\theta/2)] + 2pk_0\}$$

$$\times \{[S - 2k_0^2 \sin(\theta/2)][S - \sin(\theta/2) - 2pk_0 \sin(\theta/2)] + 2pk_0\}$$

$$+ \{[S + 2k_0^2 \sin(\theta/2)][Sp + S/k_0 - k_0 \sin(\theta/2)] + p\}$$

$$\times \{[S - 2k_0^2 \sin(\theta/2)][Sp + S/k_0 + k_0 \sin(\theta/2)] + p\})$$

and

$$\operatorname{Im} z = -\frac{1}{|z_2|^2}\,(\{[S + 2k_0^2 \sin(\theta/2)][S + \sin(\theta/2) + 2pk_0 \sin(\theta/2)] + 2pk_0\}$$

$$\times \{[S - 2k_0^2 \sin(\theta/2)][Sp + S/k_0 + k_0 \sin(\theta/2)] + p\}$$

$$- \{[S - 2k_0^2 \sin(\theta/2)][S - \sin(\theta/2) - 2pk_0 \sin(\theta/2)] + 2pk_0\}$$

$$\times \{[S + 2k_0^2 \sin(\theta/2)][Sp + S/k_0 - k_0 \sin(\theta/2)] + p\});$$

omitting further algebraic detail,

$$\mathrm{Re}z = \frac{1}{|z_2|^2}\,[S^2 - k_0^2 \sin^2(\theta/2)]$$
$$\times \left(\frac{2}{k_0^2}(k_0^2+1)(1+4k_0^2) + \frac{p}{k_0}(4+12k_0^2) \right)$$

and

$$\mathrm{Im}z = -\frac{1}{|z_2|^2}\,\frac{2\sin(\theta/2)}{k_0}$$
$$\times\,[2(1+4k_0^2)(1+k_0^2) + pk_0(4+12k_0^2)],$$

so

$$\frac{\mathrm{Im}z}{\mathrm{Re}z} = -\frac{2Sk_0\sin(\theta/2)}{S^2 - k_0^2\sin^2(\theta/2)},$$

so that

$$\mathrm{Re}F_1(k_0, 1) = \frac{2}{k_0 2\sin(\theta/2)\,S}\tan^{-1}\left(\frac{2Sk_0\sin(\theta/2)}{S^2 - k_0^2\sin(\theta/2)}\right)$$
$$= \frac{2}{k_0\sin(\theta/2)\,S}\tan^{-1}\left(\frac{k_0\sin(\theta/2)}{S}\right),$$

which agrees with Eq. (16).

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*Present address.

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[5]It has been suggested that Eq. (12) is related to the method of A. Martin [Nuovo Cimento 14, 403 (1959)], outlined by R. G. Newton [*Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), p. 423]. There are several distinctions between the two approaches however; the most important is that Martin's method is tied to a partial-wave analysis, whereas Eq. (12) is not. The connection between the two may be seen as follows:

$$F(0,s) = \int \frac{e^{-sr}}{r}\,\Psi_+(r)\,dr$$
$$= \sum \int \frac{e^{-sr}}{r}\,P_l(\cos\theta)\,\frac{y_l(r)r^2}{r}\,\sin\theta\,d\theta\,d\phi$$
$$= \int e^{-sr}\,y_0(r)\,dr = \int e^{-sr} f(k,r)\,dr,$$

in Martin's notation. If we assume expansion (14.18) for $f(k,r)$, then $F(0,s) = 1/(s-ik) + \int da\, s(a,k)/(s+a-ik)$. When we let $k \to 0$ in Eq. (12) and insert this form for $F(0,s)$, it is readily seen that $s(a,k)$ satisfies Eq. (14.19) of Newton's summary. One way of expressing this result is to observe that Martin's approach provides an infinite series expansion of the $k = 0$ limit of Eq. (12).