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Phenomenological ^{3}He quasiparticle scattering amplitude in dilute solutions of ^{3}He in superfluid ^{4}He†

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From measurements of the spin diffusion, thermal conductivity, and viscosity coefficients in dilute solutions of ^{3}He in superfluid ^{4}He, we have phenomenologically determined the amplitude for ^{3}He-^{3}He quasiparticle scattering. Two different parametrizations of the amplitude have been used; one form depends only on the momentum transferred in the scattering event while the other depends on the initial momenta of the quasiparticles as well. The amplitudes are fitted to measurements at temperatures in both the degenerate Fermi-liquid regime ($T < T_F$) and the classical regime ($T \gtrsim 2T_F$), whereas most previously determined amplitudes are valid only in the low-temperature regime. Finally, the possibility that the scattering amplitude is ^{3}He-concentration dependent is investigated by fitting measurements at each concentration independently. This procedure indicates that there is an enhancement of the amplitude at small momentum transfer as the concentration increases.

I. INTRODUCTION

One method of determining the interaction between ^{3}He quasiparticles in dilute solutions of ^{3}He in ^{4}He is by analyzing measurements of the transport properties. The usual procedure[1,2] is to use a parametrized model for the quasiparticle scattering amplitude and to adjust the parameters so as to obtain the best fit of the calculated transport coefficients to the measured ones. For computational reasons this method is usually applied in two distinct temperature regions: at temperatures $T \ll T_F$, the ^{3}He Fermi temperature, where the ^{3}He behaves as a degenerate Fermi liquid; and at T sufficiently large ($T \gtrsim 2T_F$) that the ^{3}He behaves as a Boltzmann (classical) gas. In these two temperature regions (but not at intermediate T), simple analytic solutions of the Boltzmann equation exist so that the interaction between ^{3}He quasiparticles may be easily related to the transport properties.

The transport coefficients that are of interest to us here are the spin diffusion D, the thermal conductivity K, and the viscosity η. Until recently, the measurements available for analysis were of D at both low and high temperature[3] and of K at low temperature.[4] Now that the ^{3}He viscosity in the classical regime has also been deduced from measurements,[5,6] an effort to determine a scattering amplitude consistent with all transport data in both the high- and low-temperature regimes seems worthwhile. The purpose of the present work is to find such an amplitude.

The remainder of this paper is organized as follows: Sec. I is an introduction containing a brief review of other attempts to determine the interaction between ^{3}He quasiparticles with emphasis on phenomenological studies based on measured transport properties. In Sec. II we give a summary of the relevant transport theory with a description of how the scattering amplitude is extracted from the experimental transport coefficients; Sec. III contains a discussion of the results.

In general, the amplitude for ^{3}He-^{3}He quasiparticle scattering in dilute solutions of ^{3}He in ^{4}He may be a function of the atomic fraction of ^{3}He, X, of the spin state of the quasiparticles, and of the temperature, as well as of the initial momenta $\vec{p}$ and $\vec{p}\,'$ of the particles and the momentum transfer $\vec{q}$. For now, we shall assume that the concentration and temperature dependence may be ignored (the possibility of concentration dependence is introduced in Sec. III) and that the only spin dependence is what is required by the Fermi statistics of the ^{3}He. Then the amplitude for the scattering of parallel-spin quasiparticles, $A_{\uparrow\uparrow}(\vec{p}, \vec{p}\,'; \vec{q})$, may be written in terms of the scattering amplitude for antiparallel spin quasiparticles, $A_{\uparrow\downarrow}(\vec{p}, \vec{p}\,'; \vec{q})$:

$$A_{\uparrow\uparrow}(\vec{p}, \vec{p}\,'; \vec{q}) = A_{\uparrow\downarrow}(\vec{p}, \vec{p}\,'; \vec{q}) - A_{\uparrow\downarrow}(\vec{p}, \vec{p}\,'; \vec{p}\,' - \vec{p} - \vec{q}) . \tag{1}$$

Given these assumptions, the problem of finding quasiparticle interaction is considerably simplified, since only $A_{\uparrow\downarrow}$ need be determined. The most direct approach for doing this is to expand the amplitude in powers of inner products of the three vector arguments, with coefficients in the expansion being found by comparing calculated values of the transport coefficients with measurements. Unfortunately, this method is not practical for two reasons. First, in the classical regime, typical momenta are $\sim(mk_BT)^{1/2}$, where m is the ^{3}He effective mass (assuming a parabolic quasiparticle energy-momentum dispersion relation) and k_B is Boltzmann's constant. Such momenta are sufficiently large that many terms must be kept in the expansion to give an adequate representation of $A_{\uparrow\downarrow}$ for all $\vec{p}, \vec{p}'$, and $\vec{q}$ of importance. Second, the amount of information available from experiment is not great enough to permit determination of these coefficients. In particular, in the low-temperature (degenerate) regime, all quasiparticle scattering is on the Fermi surface, $|\vec{p}| = |\vec{p}'| = |\vec{p}+\vec{q}| = |\vec{p}'-\vec{q}| = p_F$, where $\vec{p}_F$ is the Fermi momentum. Thus only a restricted domain of $\vec{p}, \vec{p}'$, and $\vec{q}$ is sampled, strongly limiting the amount of information about $A_{\uparrow\downarrow}$ that may be obtained from these measurements. This point is brought out by the fact that $\vec{p}\cdot\vec{q} = \vec{p}'\cdot\vec{q} = -q^2/2$ for scattering on the Fermi surface. Hence one cannot even distinguish between the inner products $\vec{p}\cdot\vec{q}, \vec{p}'\cdot\vec{q}$, and q^2 in the low-temperature regime. In view of the foregoing, it is more sensible to assume some relatively simple form for $A_{\uparrow\downarrow}(\vec{p}, \vec{p}'; \vec{q})$ than to employ a general expansion. This form should be both physically reasonable and computationally convenient.

Before giving a description of some forms that have been widely used for $A_{\uparrow\downarrow}$ in the past, we wish to discuss briefly the relationship between the experimental data and the scattering amplitude and the conditions under which information about the latter may be extracted from the measurements. The coefficients of thermal conductivity, spin diffusion, and viscosity involve, respectively, currents of heat, spin, and momentum. In any given experiment these currents may in part be determined by excitations other than ^{3}He quasiparticles, such as phonons and rotons. In order to obtain information about ^{3}He-^{3}He scattering, it must be possible to extract from the data that part of the current which is carried by the ^{3}He, and it must also be true that the dominant mechanism limiting this current is the ^{3}He-^{3}He scattering. Depending on the transport coefficient being measured, these conditions can only be met in certain regimes of temperature and ^{3}He concentration. For example, the ^{3}He contribution to the heat current in a 1.3% solution is found[4] to dominate the phonon contribution only for $T \lesssim 0.015$ K. Similarly, the phonon contribution to the momentum current makes it difficult to extract the ^{3}He contribution in the classical regime. (This and other problems related to determining the component of the viscosity limited by ^{3}He-^{3}He scattering are discussed in another paper.[6]) The case of spin diffusion is relatively simple because there is no phonon contribution to the spin current. Furthermore, scattering of parallel-spin quasiparticles cannot alter the spin current, so only $A_{\uparrow\downarrow}(\vec{p}, \vec{p}'; \vec{q})$ enters the computations independent of any assumptions concerning the relation between $A_{\uparrow\uparrow}$ and $A_{\uparrow\downarrow}$. Therefore measurements of D are particularly useful in the present context.

Another important point deals with the question of which measurements provide independent information about the scattering amplitude. Basically, any given measurement involves a certain average over $A_{\uparrow\downarrow}$ in momentum space, so the question is how to obtain different averages. In the degenerate regime, because of restrictions imposed by the Pauli exclusion principle, it turns out that each measurement at a different ^{3}He concentration X gives a distinct average, but that, at a given X, changing T does not lead to anything new so long as $T \ll T_F$. Conversely, in the classical regime, measurements at different X are redundant, but a change of temperature redistributes the quasiparticles in momentum space and so leads to a different average over the scattering amplitude. Thus we are interested in measured transport coefficients at different concentrations in the degenerate regime and at different temperatures in the classical regime. Those experiments which are available for our analysis include, at $T \ll T_F$, the spin diffusion[3] in 1.3% and 5.0% solutions and the thermal conductivity[4] at the same concentrations and, at $T \gtrsim 2T_F$, the spin diffusion[3] and viscosity.[5,6]

A number of attempts[1,2,7-11] have been made at phenomenologically determining the interaction of the ^{3}He quasiparticles in dilute ^{3}He-^{4}He solutions. For a summary of these investigations, the reader may consult Ref. 2; here we shall mainly discuss the approach of Bardeen, Baym, and Pines, from which the present treatment has evolved.

The phenomenological BBP theory assumes a weak effective interaction between two ^{3}He quasiparticles; this interaction depends only on the interparticle distance and has a Fourier transform $V_{\mathrm{BBP}}(q)$. It may be used to calculate static properties as well as transport properties of the solutions, and was, in fact, determined from measurements of the phase separation curve[12] and of the spin-diffusion coefficient at low temperature.[3] The BBP effective interaction is equivalent to the scattering amplitude

$$A_{\uparrow\downarrow}(\vec{p}, \vec{p}'; \vec{q}) \rightarrow A(q) = V_{\mathrm{BBP}}(q)/V, \tag{2}$$

where V is the volume of the system.

It is not possible to determine the over-all sign of the amplitude from transport measurements because they depend only on its absolute value squared. To overcome this difficulty, BBP made use of the measured solubility of ^{3}He in ^{4}He at low T. This consideration plus the two low-temperature measurements of D at $X = 0.013$ and 0.05 led them to the simple two-parameter function

$$V_{\text{BBP}}(q) = -V_{\text{BBP}}(0)\cos(\beta q/\hbar) \tag{3}$$

with $V_{\text{BBP}}(0) = 0.075 m_4 s^2/v_{40}$ and $\beta = 3.16$ Å^{-1}; m_4 is the ^{4}He atomic mass while s and v_{40} are the first sound velocity and volume per atom in pure ^{4}He at $T = P = 0$ (P is the pressure). It is convenient to further define V_c as

$$V_c \equiv m_4 s^2 v_{40} = 1.73 \times 10^{-37}\ \text{erg cm}^3 .$$

In addition to empirically determining the quasiparticle scattering, BBP discuss the physical origins of this scattering. By considering both (i) the static shift in the energy of a ^{3}He quasiparticle produced by the introduction of a second one and (ii) the contribution of single-phonon exchange to the quasiparticle scattering, they find the theoretical amplitude

$$A_{\uparrow\downarrow}(\vec{\mathrm{p}}, \vec{\mathrm{p}}\,'; \vec{\mathrm{q}}) = [-A_0 V_c - A_d(\vec{\mathrm{p}} \cdot \vec{\mathrm{q}})(\vec{\mathrm{p}}\,' \cdot \vec{\mathrm{q}}) v_{40}/m_4 q^2]/V, \tag{4}$$

where $A_0 = \alpha^2$ and $A_d = (m_4/m)^2[1 + \alpha + (m - m_3)/m_4]^2 \approx 1.72$, m_3 is the ^{3}He atomic mass, and α is a dimensionless number such that the volume per atom in a solution of concentration X at $T = 0$ is $v_{40}(1 + \alpha x)$ in the limit of small X. From molar volume measurements one finds $\alpha^2 = 0.081$ at $P = 0$.

In the degenerate regime, all scattering events are such that $\vec{\mathrm{p}} \cdot \vec{\mathrm{q}} = -\vec{\mathrm{p}}\,' \cdot \vec{\mathrm{q}} = -q^2/2$, so $A_{\uparrow\downarrow}$ becomes $(-\alpha^2 V_c + A_d q^2 v_{40}/4m_4)/V$. For small q, $V_{\text{BBP}}(q)$ is also a parabola, so the forms of the phenomenological and theoretical amplitudes are consistent in this limit. At large q, the amplitude is not expected[1,13] to be given correctly by Eq. (4).

Following the work of BBP, Abel *et al.*[4] measured the low-temperature thermal conductivity of solutions at $X = 0.013$ and 0.05; their results turned out to be inconsistent with predictions based on the BBP amplitude. Subsequent improvement of the theoretical expressions relating $A_{\sigma\sigma'}$ and the Fermi liquid transport coefficients[9,14] removes part but not all of the discrepancy; furthur attempts[2,10] have been made to modify $V_{\text{BBP}}(q)$ in such a way as to fit the measurements of both D and K, but it does not appear to be possible to obtain less than a 10% discrepancy between all of the calculated and measured coefficients. The best fit has been achieved using a five-term power series,

$$A_{\uparrow\downarrow}(q) = -(V_c/V)\sum_{n=0}^{4} a_n(q/q_0)^{2n} ,$$

where the a_n are variable parameters. This series is a convenient representation of $A_{\uparrow\downarrow}$ because it is a simple function and, if enough terms are kept, it can accurately represent an arbitrary function on a finite positive interval of q, $0 < q < q_{\max}$ so long as $q_{\max}$ is fairly small.

A quite different modification that has been tried[2] is the "dipolar" amplitude of Eq. (4) with A_0 and A_d as variable parameters. Since this is just equivalent to a parabola in the degenerate regime, i.e., to the first two terms of the power series, one should expect to be even less successful in obtaining a good fit with the dipolar amplitude. This turns out to be the case, the discrepancy between calculated and measured transport coefficients being as large as 25%. The result is disappointing, since one would like to believe the dipolar amplitude has some theoretical justification.

Neither of the scattering amplitudes described above is well-suited for analysis of the high-T transport data; the reasons are, in the case of the series, that a power series is difficult to control at large values of its argument and, in the second case, that there is not enough flexibility in the amplitude to make it work over the wide range of momenta that are important. Previously, an amplitude in the form of a sum of Gaussians was used to analyze high-T data[7]; this turns out to be quite adequate, but it has the drawback of being different in functional form from the amplitudes used to fit the low-T data.

In the present work we want to use a single amplitude to analyze all of the data and prefer a function that more nearly reduced to one of the common choices (power series or dipolar) at small q. To this end we have chosen to work with the following two functions:

(i) Modified power-series amplitude. The power series is easier to control if it is multiplied by some envelope which dominates the series at large q. A simple choice is the Gaussian $\exp[-B(q/q_0)^2]$; thus we have the modified power-series amplitude,

$$A_{\uparrow\downarrow}(\vec{\mathrm{p}}, \vec{\mathrm{p}}\,'; \vec{\mathrm{q}}) \rightarrow A(q) = -(V_c/V)e^{-B(q/q_0)^2}\sum_{n=0}^{N} a_n(q/q_0)^{2n} , \tag{5}$$

where the a_n and B are to be found from the fit. Note that such a function is not more general than a simple power series, at least if N is large; it is, however, easier to control at large q and is therefore more convenient for computations.

(ii) Modified dipolar amplitude. Comparison of

the dipolar amplitude [Eq. (4)] with the experiments shows that it is too strong at large q; therefore one may obtain a better fit if it is multiplied by a factor which approaches 1 at small q and which decreases as q increases. A Gaussian also has this property; we therefore have investigated the modified dipolar amplitude,

$$A_{\uparrow\downarrow}(\vec{p},\vec{p}\,';\vec{q}) = -(1/V)[A_0V_c + A_d(\vec{p}\cdot\vec{q}) \times(\vec{p}\,'\cdot\vec{q})v_{40}/m_4q^2]e^{-B(q/q_0)^2}, \tag{6}$$

where A_0, A_d, and B are the free parameters.

We turn next to the procedure by which the parameters in these two amplitudes are determined.

II. DETERMINATION OF THE SCATTERING AMPLITUDE

A. Transport theory

In order to extract information about the quasiparticle scattering amplitude from the measured thermal conductivity, viscosity, and spin diffusion, it is necessary to have solutions of the appropriate transport equation, which in this case is the Boltzmann equation. We are interested in solutions in both the classical ($T \gtrsim 2T_F$) regime, where the ^{3}He quasiparticles obey Boltzmann statistics, and the degenerate ($T \ll T_F$) regime, where Fermi statistics must be used. In fact, there are also useful measurements at temperatures in the intermediate regime, but no simple solutions of the Boltzmann equation are available for this case.

Given a weakly interacting Fermi gas, which the ^{3}He quasiparticles are assumed to be in dilute mixtures, there is no intrinsic difference between the classical and degenerate regimes. Strictly as a consequence of the change from classical to Fermi statistics, however, the solutions of the Boltzmann equation for the transport coefficients have quite different forms in the two regimes, and so we shall discuss them separately.

For temperatures large compared to the Fermi temperature, we shall use the lowest-order term in the Chapman-Enskog series, a solution that is appropriate for a dilute classical gas.[15] It must be generalized slightly to take into account the fact that the interaction between two ^{3}He quasiparticles need not be independent of the motion of their center of mass. This independence is generally assumed in a classical gas, but in the present system the interaction is partly determined by the background superfluid ^{4}He so that there is no invariance under translation of the ^{3}He. This generalization leads to the following expressions for the transverse viscosity η and for the spin difusion D:

$$\frac{1}{\eta} = \frac{m^{3/2}}{2\hbar^3\pi^6(k_BT)^{1/2}}\int d^3r\,d^3s\,d^3t\,e^{-(r^2+s^2+t^2)}\,\delta(\vec{r}\cdot\vec{s}) \times(r_xs_z+r_zs_x)^2W(\vec{t}+\vec{r}+\vec{s},\vec{t}+\vec{r}-\vec{s};-2\vec{s}) \tag{7}$$

and

$$\frac{1}{D} = \frac{m^{3/2}n_3}{2\pi^6\hbar^3(k_BT)^{1/2}}\int d^3r\,d^3s\,d^3t\,e^{-(r^2+s^2+t^2)}\,\delta(\vec{r}\cdot\vec{s}) \times r_z^2W_{\uparrow\downarrow}(\vec{t}+\vec{r}+\vec{s},\vec{t}+\vec{r}-\vec{s};-2\vec{s})\ , \tag{8}$$

where n_3 is the ^{3}He quasiparticle number density in the mixture and subscripts z and x denote Cartesian components of a vector; also $\vec{r}$, $\vec{s}$, and $\vec{t}$ are reduced momenta, the real momenta being given by $(mk_BT)^{1/2}\vec{r}$, etc. Finally, the quantity $W_{\sigma\sigma'}(\vec{p},\vec{p}\,';\vec{q})$ is proportional to the probability that quasiparticles initially in states $(\vec{p}\sigma)$ and $(\vec{p}\,'\sigma')$ scatter into states $(\vec{p}+\vec{q},\sigma)$ and $(\vec{p}\,'-\vec{q},\sigma)$; σ and σ' are spin indices with $W=\frac{1}{2}W_{\uparrow\downarrow}+\frac{1}{4}W_{\uparrow\uparrow}$. In terms of the scattering amplitude $A_{\sigma\sigma'}(\vec{p},\vec{p}\,';\vec{q})$,

$$W_{\sigma\sigma'}(\vec{p},\vec{p}\,';\vec{q}) = (2\pi/\hbar)|\,VA_{\sigma\sigma'}(\vec{p},\vec{p}\,';\vec{q})|^2\ . \tag{9}$$

In the absence of any explicit form for the scattering amplitude, it is not possible to carry the integration indicated in Eqs. (7) and (8) very far; if one assumes that $A_{\sigma\sigma'}$ depends only on the magnitude of the momentum transfer $q \equiv |-2\vec{s}|$, then the equations reduce to the usual expressions for the viscosity and diffusion coefficient in a classical gas except that the scattering probability is spin-dependent and should be symmetrized in accordance with the requirements of the Pauli principle.

In the degenerate regime, $T \ll T_F$, the linearized transport equation has been solved exactly[14] for D, K, and η in a normal Fermi liquid. The results are

$$\frac{1}{DT^2} = \frac{3m^5k_B^2}{16\pi^2C(\lambda_D)\hbar^6p_F^2(1+F_0^a)} \times\left\langle\frac{W_{\uparrow\downarrow}(\theta,\varphi)\sin^2\frac{1}{2}\theta(1-\cos\varphi)}{\cos\frac{1}{2}\theta}\right\rangle\ , \tag{10}$$

$$\frac{1}{KT} = \frac{3m^4}{4\pi^2H(\lambda_K)\hbar^3p_F^3}\left\langle\frac{W(\theta,\varphi)\sin^2\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right\rangle\ , \tag{11}$$

$$\frac{1}{\eta T^2} = \frac{45m^4k_B^2}{16C(\lambda_\eta)\hbar^3p_F^s}\left\langle\frac{W(\theta,\varphi)\sin^4\frac{1}{2}\theta\sin^2\varphi}{\cos\frac{1}{2}\theta}\right\rangle\ . \tag{12}$$

Here the probabilities $W_{\sigma\sigma'}$ are written as functions of θ, the angle between $\vec{p}$ and $\vec{p}\,'$, and φ, the angle by which the plane containing the quasiparticle momenta is rotated in the scattering process. Such a representation is possible because for $T \ll T_F$, all quasiparticles which can participate in scattering events lie close to the Fermi surface, so that to a very good approximation $|\vec{p}| = |\vec{p}\,'| = |\vec{p} + \vec{q}| = |\vec{p}\,' - \vec{q}| = p_F$. Thus the only way in which $W_{\sigma\sigma'}$ can enter expressions for the transport coefficients is through an average $\langle \cdots \rangle$ over the Fermi surface; this average is defined by

$$\langle f(\theta, \varphi) \rangle \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\pi} \frac{\sin\theta \, d\theta}{2} f(\theta, \varphi) \ . \tag{13}$$

The functions $C(\lambda)$ and $H(\lambda)$ are given by

$$C(\lambda) = \frac{1-\lambda}{4} \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)[(n+1)(2n+1)-\lambda]} \tag{14}$$

and

$$H(\lambda) = \frac{3-\lambda}{4} \sum_{n=0}^{\infty} \frac{4n+5}{(n+1)(2n+3)[(n+1)(2n+3)-\lambda]} \tag{15}$$

while

$$\lambda_D = 1 - \frac{\langle W_{\uparrow\downarrow}(\theta, \varphi) \sin^2\tfrac{1}{2}\theta \, (1-\cos\varphi)/\cos\tfrac{1}{2}\theta \rangle}{\langle W(\theta, \varphi)/\cos\tfrac{1}{2}\theta \rangle} \ , \tag{16}$$

$$\lambda_K = 1 + 2 \frac{\langle W(\theta, \varphi) \cos\theta/\cos\tfrac{1}{2}\theta \rangle}{\langle W(\theta, \varphi)/\cos\tfrac{1}{2}\theta \rangle} \ , \tag{17}$$

$$\lambda_\eta = 1 - 3 \frac{\langle W(\theta, \varphi) \sin^4\tfrac{1}{2}\theta \sin^2\varphi/\cos\tfrac{1}{2}\theta \rangle}{\langle W(\theta, \varphi)/\cos\tfrac{1}{2}\theta \rangle} \ . \tag{18}$$

Finally, $1 + F_0^a$ is the magnetic susceptibility enhancement factor in the Landau Fermi-liquid theory.[16]

The high- and low-temperature expressions for the transport coefficients are not equivalent in the sense that they are not equally accurate solutions of the Boltzmann equation. The high-temperature solution may be obtained from the usual variational method[17] in which the entropy production is maximized at a fixed value of the appropriate current. Equations (7) and (8) result when the trial function for the quasiparticle distribution is taken to be

$$f_\sigma(\vec{p}) = f_\sigma^0 + \Phi_\sigma(\vec{p}) f_\sigma^0(p)[1 - f_\sigma^0(p)]/k_B T \ ,$$

with $\Phi_\sigma(\vec{p}) \sim p_x p_z$ for viscosity and $\Phi_{\uparrow\downarrow}(\vec{p}) \sim \pm p_z$ for spin diffusion; $f_\sigma^0(p)$ is the equilibrium Fermi distribution function $(\exp\{[\epsilon_\sigma(p) - \mu]/k_B T\} + 1) \rightarrow \exp\{[\mu - \epsilon_\sigma(p)]/k_B T\}$ in the classical regime; $\epsilon_\sigma(p)$ is the quasiparticle energy and μ is the ^{3}He chemical potential.

If the same trial functions are used in the degenerate regime, Eqs. (10) and (12) are the result except that $C(\lambda)$ is replaced by $\frac{3}{4}$. Similarly, if $\Phi_\sigma(\vec{p})$ is taken to be $\sim p_z[\epsilon_\sigma(p) - \mu]$, one finds Eq. (11) with $H(\lambda)$ replaced by $\frac{5}{12}$. In practice, the variational solutions differ by no more than a few per cent from the exact expressions, at least for those scattering amplitudes considered here. Thus the variational calculation leads to a very good approximate solution in the degenerate regime and we expect that the same is true at high temperatures, i.e., that Eqs. (7) and (8) are quite adequate for our purposes.

B. Scattering amplitudes

In Sec. I, we have discussed the various considerations that go into choosing $A_{\sigma\sigma'}(\vec{p}, \vec{p}\,'; \vec{q})$. The main point is that there is not enough information available from experiments to uniquely determine the amplitudes phenomenologically. For this reason they are assumed to be independent of X and T and are given simple functional forms containing several variable parameters. In addition, it is assumed that there is no intrinsic dependence of the scattering on the spin state of the quasiparticles, so that $A_{\uparrow\uparrow}$ is just a properly symmetrized version of $A_{\uparrow\downarrow}$; that is, parallel-spin quasiparticles are indistinguishable whereas anti-parallel-spin quasiparticles are not, provided the interaction cannot flip spins.

The functional forms of the two amplitudes that we have investigated are given in Sec. I; they are (i) modified power-series amplitude:

$$A_{\uparrow\downarrow}(\vec{p}, \vec{p}\,'; \vec{q}) \rightarrow A(q) = -(V_c/V) \sum_{n=0}^{N} a_n (q/q_0)^{2n} e^{-B(q/q_0)^2}$$

and (ii) modified dipolar amplitude:

$$A_{\uparrow\downarrow}(\vec{p}, \vec{p}\,'; \vec{q}) = -(1/V)[A_0 V_c + A_d(\vec{p} \cdot \vec{q}) \times (\vec{p}\,' \cdot \vec{q}) v_{40}/m_4 q^2] e^{-B(q/q_0)^2}$$

In case (i) we have used $N = 4$ so that there are six variable parameters $a_0, \ldots, a_4$, and B; in case (ii) there are three parameters, A_0, A_d, and B. The constant q_0 is taken to be such that $q_0/\hbar = 0.753$ Å^{-1}; it simply sets the scale of the momentum.

There is no possibility of ascribing any physical significance to the parameters in amplitude (i) or to its functional form; its over-all shape is all that can be considered of importance. Amplitude (ii), on the other hand, can be at least partially derived theoretically as described in Sec. I; thus both A_0 and A_d have some physical content and we would expect the empirically determined values to be close to the theoretical predictions of 0.081

and 1.7, respectively. Similarly, there is reason to expect that the dipolar amplitude is reduced by screening effects for large q. McMillan[13] has discussed this, and he points out that the screening would be represented by a factor such as $1-S^2(q)$, where $S(q)$ is the liquid-structure factor of ^{4}He in its ground state and is $\sim q$ at small q. The Gaussian cutoff we have used has this form for q not too large, since $e^{-B(q/q_0)^2}=1-B(q/q_0)^2+\cdots$. Thus, we expect it to be a reasonable representation of the screening.

III. RESULTS

The parameters in the amplitude were determined by calculating the transport coefficients from Eqs. (7), (8), and (10)–(12) and then minimizing the function $\sum_i[(T^i_{\text{exp}}-T^i_{\text{calc}})/T^i_{\text{exp}}]^2$. Here T^i_{exp} is the ith measured transport coefficient and T^i_{calc} is the corresponding calculated coefficient. In the numerical work we have taken m and F^a_0 from experiment; specifically, we use $F^a_0=0.08$ and $m=2.46m_3$ at $X=0.05$; $F^a_0=0.09$ and $m=2.37m_3$ at $X=0.013$ (Ref. 3); and $F^a_0=0$ and $m=2.28m_3$ in the limit of $X\rightarrow 0$.[18]

Note that all terms in the sum above are given equal weight. We also tried weighing each term according to the stated accuracy of the experiment and find that this makes almost no difference in the results.

A total of 23 distinct measurements has been used. They are summarized in Table I along with the calculated (fit) coefficients using amplitudes (i) and (ii). Table II contains the values of the parameters in the two amplitudes while Fig. 1 shows amplitude (i) (solid line) as a function of q.

For low-temperature calculations, only values of $q\leq 2k_F(5\%)\cong 0.636$ Å^{-1} are important since $2k_F$ is the maximum momentum transfer. In fact, at small values of q, the amplitude (i) which we have determined is quantitatively quite similar to both the BBP amplitude, also given in Fig. 1, and the amplitude deduced by Ebner and Edwards. This is shown in Fig. 1. For high-temperature calcu-

TABLE I. The transport coefficients used to find the scattering amplitudes. The experimental coefficients and the results of the fits for both amplitudes are given. Here, T is in units of degrees Kelvin, η is in units of micropoise, and K is in units of (ergs/cm sec K); D is in units of (10^{-3} cm^2/sec) in part A of the table, and in units of (10^{-6} cm^2/sec) in part B. The values of D in part A are appropriate for a 5% solution of ^{3}He in ^{4}He. They may be scaled as $1/X$ to obtain numbers appropriate for other concentrations.

A. High-temperature results

Temp.	η_B(expt.)	η_B(series)	η_B(dipole)	D (expt.)	D (series)	D (dipole)
0.04	9.0	9.34	10.2		0.46	0.50
0.06	12.5	13.3	13.8		0.68	0.71
0.08	16.0	17.4	17.0		0.96	0.92
0.10	19.0	21.3	19.6		1.28	1.14
0.15	27.0	29.1	23.3		2.22	1.64
0.20	35.0	32.9	23.7		2.97	1.94
0.25	33.0	33.3	22.7		3.20	2.04
0.30	29.0	31.7	21.4		3.00	2.01
0.40	24.5	26.9	19.1		2.29	1.83
0.50	21.6	23.0	17.6	1.98	1.79	1.67
0.60	20.2	20.7	16.7	1.65	1.50	1.56
0.70	19.0	19.4	16.2	1.42	1.33	1.49
0.80	18.2	18.9	15.9	1.28	1.24	1.45
0.90	17.5	18.8	15.7	1.14	1.19	1.42

B. Low-temperature results

	Expt.	Series	Dipole
KT (1.3%)	11.0	9.6	11.3
KT (5.0%)	24.0	28.0	35.0
DT^2 (1.3%)	17.2	13.5	16.3
DT^2 (5.0%)	90.0	68.0	85.0
ηT^2 (1.3%)		0.029	0.033
ηT^2 (5.0%)		0.28	0.36

TABLE II. The parameters contained in the two scattering amplitudes.

Series amplitude		Dipolar amplitude
$a_0 = 0.071$	$a_1 = 0.002\,854$	$A_0 = 0.068$
$B = 1.52$	$a_2 = -0.5985$	$A_D = 2.33$
	$a_3 = 0.6101$	$B = 0.467$
	$a_4 = -0.2941$	
For both amplitudes:	$q_0/\hbar = 0.753$ Å^{-1}	
	$V_c = 1.73 \times 10^{-37}$ erg cm^3	

lations, however, the previous forms of scattering amplitude are clearly unsatisfactory, since they do not converge at large values of momentum transfer. Of course, this was one motivation for introducing into the scattering amplitude the exponential factor which dominates it at large q. The behavior of the modified-power-series amplitude at all q is shown in Fig. 2.

As discussed earlier, the theories of both BBP and McMillan predict values for the quantities A_0 and A_d of 0.081 and approximately 1.7, respectively. The value of A_d extracted in our fits cannot be compared with this theoretical value, since we have used a form of the scattering amplitude which contains an exponential cutoff. The scattering amplitude at $q = 0$ can, however, be compared with the theory for both amplitudes (i) and (ii); in both cases, the phenomenological value falls about 14% below the theoretical value of 0.081, indicating a less attractive interaction than predicted for small values of q.

The agreement of the calculated low-temperature transport coefficients with experiment can be seen in Table I. Here the fit to the thermal conductivity and diffusion coefficient is better at 1.3% than at 5%. The discrepancies of the calculated values from the measured values in the present calculation, where both the high- and low-temperature data have been fitted, are of about the same magnitude as previous calculations which fit only the low-temperature data.

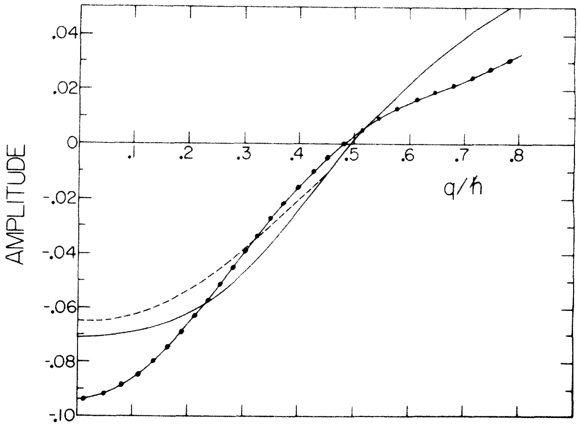

FIG. 1. Small-momentum-transfer scattering amplitudes in units of V_c/V vs $q/\hbar$ in units of Å^{-1}: solid line, modified power series; dashed line, $V_{\text{BBP}}(q)/V$; dot-dashed line, modified power series fitted to 5% low-temperature data alone.

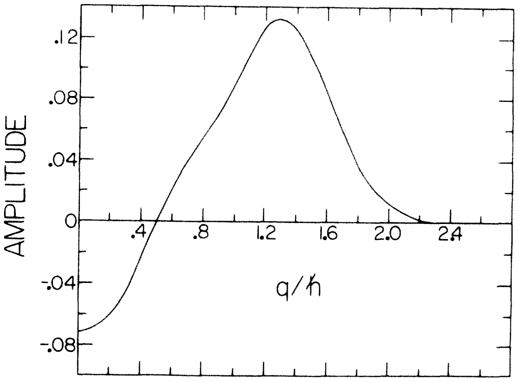

FIG. 2. Modified power-series scattering amplitude in units of V_c/V vs $q/\hbar$(Å^{-1}).

The high-temperature viscosity is also listed in Table I and plotted in Fig. 3. Here, both forms of the scattering amplitude give agreement to the general shape of the data, with the series amplitude producing a better fit. Of course, this was expected since there are six parameters in the series amplitude while there are only three in the dipolar case. Although the dipolar amplitude does not reproduce the Boltzmann viscosity data as well, we nevertheless feel that this amplitude is more physically meaningful.

For the high-temperature diffusion coefficient,

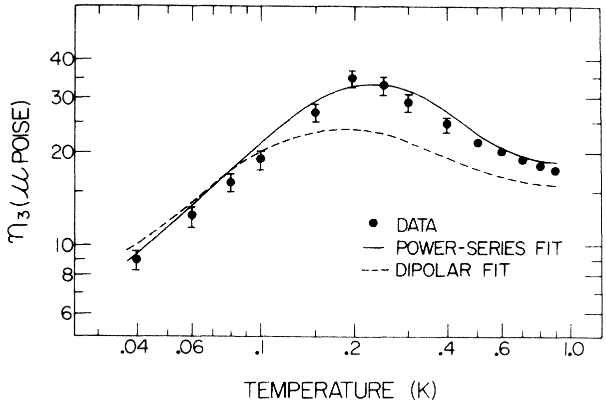

FIG. 3. High-temperature viscosity vs temperature. The data are from Ref. 6 while the solid (dashed) line is the viscosity calculated using the modified power-series (dipolar) amplitude.

once again both amplitudes reproduce the general shape of the data with the series amplitude giving the better agreement. The experimental values of the diffusion coefficient which we used in the fit were read off a smooth curve through the 5% data of Anderson *et al.*[3] Since there is only one data point (at 0.8K) at a temperature above 0.45 K, our interpolation may place too much emphasis on this single high-temperature point. Also, no attempt has been made to extract the portion of the diffusion coefficient which is limited by ^{3}He-^{3}He scattering as was done in the case of the viscosity coefficient. At the relatively high temperature of 0.8 K, for example, it is possible that rotons may play a role in determining D. As a practical matter, however, the actual values of D used in our fitting program do not have a marked effect on the shape of the scattering amplitude. That is, even if we are using values of D which may be somewhat in error, the net modification in the shape and magnitude of the scattering amplitude would be small.

In summary, we have found, first, that previously used forms of the scattering amplitude will not fit the high-temperature spin-diffusion and viscosity data unless a factor which limits the amplitudes at large values of q is included. Second, we have shown that these new forms of the scattering amplitude, in addition to fitting the high-temperature data, also fit the low-temperature data as accurately as previously determined scattering amplitudes. One fact stands out in all this work, however—that a given scattering amplitude is not really successful in fitting both low- and high-*concentration* data. This may imply quite strong concentration dependence of the amplitude.

We investigated this possibility by fitting the scattering amplitude to only the low-temperature 5% transport-coefficient measurements. It became immediately evident that for both forms of the scattering amplitude, the best fit was produced with a much deeper amplitude—that is, for a more attractive interaction. Specifically, whereas the results of fitting the high-temperature data and/or the low-concentration data clearly gave a value of a_0 or A_0 equal to 0.07, the high-concentration fit yielded a value of 0.093. These drastically different values indicate a stronger dependence of the amplitude on concentration than might have been guessed.

TABLE III. The low-temperature transport coefficients at $X = 0.05$. The experimental coefficients as well as the results from fitting these data alone are given. Here, KT is in units of (ergs/cm sec), DT^2 is in units of (10^{-6} cm^2 K^2/sec), and ηT^2 is in units of (10^{-6} poise K^2).

	Expt.	Series	Dipolar
KT	24.0	23.0	23.0
DT^2	90.0	86.0	86.0
ηT^2	0.28	0.29	0.29

The results of this high-concentration fit are shown in Table III, and the corresponding series amplitude is shown in Fig. 1 (dashed-dotted line). Again, for this low-temperature calculation, only values of $q \lesssim 0.64$ Å^{-1} are important. The depression in amplitude for small q is clear in Fig. 1 for values of $q < 0.2$ Å^{-1}. Furthermore, Table III shows that this depressed amplitude produces a significant improvement in fitting the 5% transport coefficients.

We can only speculate on the possible physical origin of concentration dependence in the quasiparticle scattering amplitude. One possibility is the screening of the interaction between two ^{3}He quasiparticles by other quasiparticles. In the simplest (linear) approximation, this would enhance the attractive interaction between two quasiparticles at small q, and it is, of course, a concentration-dependent effect. This mechanism can explain qualitatively our phenomenological findings, although a detailed calculation would be necessary to determine whether quantitative agreement is possible.

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