Modifications to the Scully-Lamb laser master equation

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A quantum theory of the laser is presented in which the coupling between atoms and field is retained in the treatment of the irreversible behavior of the atoms. A master equation for the optical field is obtained and compared with that resulting from the laser theory of Scully and Lamb, where atom-field correlations are neglected in describing the relaxation of the atoms. It is confirmed that for laser powers currently available the approximation employed by Scully and Lamb is valid.

I. INTRODUCTION

Since the laser was first proposed as an extension of the maser principle to the optical region by Schawlow and Townes[1] in 1958, it has attracted a considerable amount of attention from both theoreticians and experimentalists. In fact, more than 10 000 papers have been published on lasers and related topics.[2] The theoretical description of the laser can be approached in a variety of ways depending on the exact nature of the questions one wishes to ask about the system. A distinction may be drawn between three categories. First, we have the fully quantum-mechanical approach utilizing either the Langevin equations, the density matrix or master equation, or the Fokker-Planck equation. (For a review, see the articles by Haken,[2] Lax,[3] and Risken.[4]) Second, averaging over the quantum fluctuations leads to the semiclassical equations.[4-7] Finally, several of the gross features of the laser are accessible via the rate equations.[8-10] In this paper we are concerned solely with the fully quantum-mechanical approach.

Although the semiclassical theories of the laser were successful in deriving the most important properties of lasing systems, the need for a fully quantum-mechanical treatment was recognized if a complete understanding of line shape and photon statistics was to be attained. For a formalism to include quantum fluctuations, it is necessary to introduce loss and pumping in a nonphenomenological manner. This may be accomplished by coupling both the laser-active atoms and the field to thermal reservoirs, which accounts for the interaction of the atoms with lattice vibrations, nonlasing light modes, etc., and the field with the mirrors, scattering centers, etc. The quantum theory of dissipation from a single quantum system S coupled to a Markoffian reservoir R has been extensively studied.[11-15] However, it has been pointed out, initially by Walls[16] and subsequently by others,[17-19] that the conventional techniques for treating cases where S involves internal coupling are incorrect, in that they assume a factorization of the reduced-density operator in the derivation of the irreversible part of the master equation. This has rather dire consequences, since it leads to an incorrect canonical form for the stationary solution of the density operator, and in general destroys detailed balance.[19]

The strength of the internal interaction in the laser model depends directly on the strength of the optical field, and it has therefore been tacitly assumed in all laser theories to date that this field is sufficiently weak to justify the use of the conventional factorization ansatz in the treatment of all reservoir interactions (see, for example, Weidlich and Haake.[13] The limitations of this approximation are recognized by Haken,[2] where he states: "It should be noted, however, that very strong fields cause a partial quenching of the spontaneous-emission linewidth. In this region the incoherent terms of the density matrix should be derived by using the coupled system: field and atoms from the very beginning." The aim of the present paper is to remove the factorization ansatz and in doing so investigate the limits of validity of this approximation. We follow closely the analysis presented in the paper by Scully and Lamb (SL),[20] hereafter cited as I, since for reasons pointed out in an earlier publication[19] it is necessary for us to work with the matrix elements of the density operator rather than with the full operator equations. Following a short discussion of the model in Sec. II, we present in Sec. III a treatment of the relaxation of a single laser atom without neglecting the effects of the coupling to the field on the reservoir interaction. We then derive the master equation for the reduced-density operator of the field alone in Sec. IV, and conclude with a discus-

sion of our results in Sec. V. Throughout we emphasize the very close correspondence between our analysis and that of SL.

II. MODEL AND ANALYTICAL APPROACH

A diagrammatic representation of a physical model on which a complete quantum-mechanical description of the laser could be based is given in Fig. 1. Expressed within the formalism developed in our earlier paper,[19] hereafter cited as II, the Hamiltonian describing this system reads

$$H = H_S + H_R + H_{SR}, \tag{2.1}$$

where

$$H_S = H_F + \sum_\mu H_{A(\mu)} + \sum_\mu H_{FA(\mu)},$$

$$H_R = H_{R_F} + \sum_\mu H_{R_{A(\mu)}},$$

$$H_{SR} = H_{FR_F} + \sum_\mu H_{A(\mu)R_{A(\mu)}}. \tag{2.2}$$

Here H_F, $H_{A(\mu)}$ and $H_{FA(\mu)}$ are the Hamiltonians for the free field, the μth free atom and their interaction, H_{R_F} and $H_{R_{A(\mu)}}$ are the Hamiltonians for the thermal reservoirs associated with the field and the μth atom, and H_{FR_F} and $H_{A(\mu)R_{A(\mu)}}$ express the interactions between these reservoirs and the corresponding parts of the free lasing system. Now theoretically a master equation for the reduced-density operator ρ of the free system which does not depend on any factorization ansatz may be derived using the approach discussed in II. However, as was pointed out there, this requires knowledge of the solutions to the equations of motion

$$\frac{dH_{FR_F}}{dt} = \frac{1}{i\hbar}[H_{FR_F}, H_S + H_R],$$

$$\frac{dH_{A(\mu)R_{A(\mu)}}}{dt} = \frac{1}{i\hbar}[H_{A(\mu)R_{A(\mu)}}, H_S + H_R]. \tag{2.3}$$

Owing to the interaction term in H_S these are nonlinear operator equations which remain unsolved, and for this reason we must look to a laser model which works directly with matrix elements. Hence the adoption of the SL approach as outlined below.

We treat a single-mode laser which is resonant at a frequency ω_0 with the laser-active atoms, each of which is taken to be a four-level system as represented by Fig. 2. Laser transitions take place between the upper two states $|4\rangle$ and $|3\rangle$, each of which interacts with its respective reservoir R_a and R_b so that relaxation to the lower two states $|2\rangle$ and $|1\rangle$ occurs. These reservoirs are assumed to be at zero temperature, which means we are neglecting thermal fluctuations arising from the interaction of the atoms with their surroundings. Motions of the laser-active atoms are neglected and they are considered to be so confined that they all see essentially the same amplitude of the field.

We work in the representation for which a basis

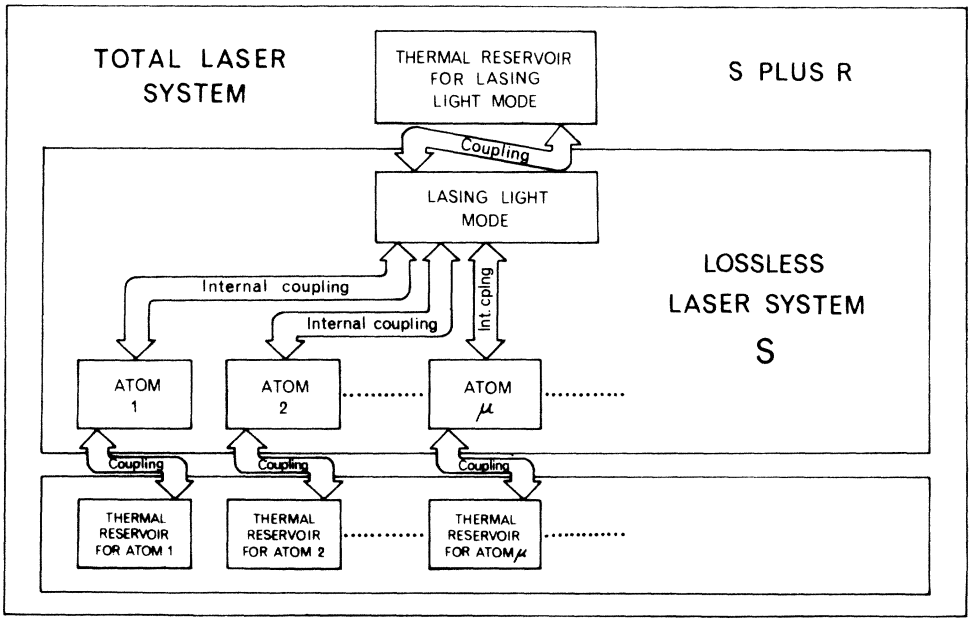

FIG. 1. Diagrammatic representation of the laser model.

is formed by the kets $|n, \xi, \ldots, \xi_\mu, \ldots\rangle$, where $n = 0, 1, 2, \ldots$ refers to the state of the field and $\xi_\mu = 1, 2, 3, 4$ refers to the state of the μth atom. If χ is the density operator for the complete system including all reservoirs, then the reduced density operator ρ for the free lasing system alone is defined by the matrix elements

$$\rho_{n,\xi_1,\ldots,\xi_\mu,\ldots;m,\eta_1,\ldots,\eta_\mu,\ldots} = \langle n, \xi_1, \ldots, \xi_\mu, \ldots | \operatorname{Tr}_R(\chi) | m, \eta_1, \ldots, \eta_\mu, \ldots \rangle . \tag{2.4}$$

We define the matrix elements $\rho_{n,\xi_\mu;m,\eta_\mu}$ by tracing over all of the atoms except the μth one,

$$\rho_{n,\xi_\mu;m,\eta_\mu} = \langle n, \xi_\mu | \left[\prod_{\nu \neq \mu} \operatorname{Tr}_{A(\nu)} (\operatorname{Tr}_R(\chi)) \right] | m, \eta_\mu \rangle , \tag{2.5}$$

and then the reduced-density operator for the optical field alone has matrix elements $\rho_{n,m}$ given by

$$\rho_{n,m} = \left[\prod_{\mu} \operatorname{Tr}_{A(\mu)} (\operatorname{Tr}_R(\chi)) \right] = \sum_{\xi_\mu=1}^{4} \rho_{n,\xi_\mu;m,\xi_\mu} . \tag{2.6}$$

Now our approach, adopted from SL, treats each atom separately in interaction with the net field and thus neglects atom-atom correlations. We consider the μth atom injected into the upper lasing state at a time t and then follow the time evolution of the system, considering only the interactions directly involving the μth atom (that defined by $H_{FA(\mu)}$ and that defined by $H_{A(\mu)R_{A(\mu)}}$), to a time $t + \Delta t$, observing the change $\delta\rho^{\mu}_{n,m}(t)$ in the matrix element $\rho_{n,m}$ produced by the μth atom during this interval; i.e., using the Hamiltonian

$$H = H_S + H_R + H_{SR} , \tag{2.7}$$

with

$$\begin{aligned} H_S &= H_F + H_{A(\mu)} + H_{FA(\mu)} , \\ H_R &= H_{R_{A(\mu)}} , \\ H_{SR} &= H_{A(\mu)R_{A(\mu)}} , \end{aligned} \tag{2.8}$$

we calculate

$$\delta\rho^{\mu}_{n,m}(t) = \sum_{\xi_\mu=1}^{4} \rho_{n,\xi_\mu;m,\xi_\mu}(t + \Delta t) - \rho_{n,m}(t) . \tag{2.9}$$

Since it is quite reasonable to assume that changes in the field occur on a time scale which is large compared with the relaxation time of the atoms, Δt is chosen to be much greater than these relaxation times. Hence we may set $\rho_{n,4;m,4}(t + \Delta t)$ and $\rho_{n,3;m,3}(t + \Delta t)$ equal to zero. Here we also find some justification for neglecting the coupling of the field to its reservoir during this stage of the analysis.

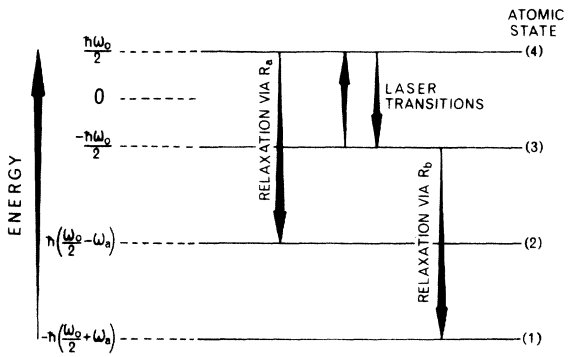

FIG. 2. Energy-level scheme for laser-active atoms.

From this one-atom interaction, we can obtain the terms in the master equation due to spontaneous and stimulated emission by summing over all of the atoms, under the assumption that $r_a \Delta t$ of them appear in their upper state and relax during any interval Δt, r_a being the pumping rate. This yields a macroscopic change $\Delta\rho_{n,m}(t)$ in the density matrix elements given by

$$\Delta\rho_{n,m}(t) = \sum_{\mu} \delta\rho^{\mu}_{n,m}(t) = r_a \Delta t \delta\rho_{n,m}(t) , \tag{2.10}$$

where the superscript μ can be dropped since all of the atoms are identical. The coarse-grain time derivative may then be defined to give

$$\left(\frac{\partial\rho_{n,m}(t)}{\partial t} \right)_{\text{emission}} = \frac{\Delta\rho_{n,m}(t)}{\Delta t} = r_a \delta\rho_{n,m}(t) . \tag{2.11}$$

The contribution $(\partial\rho_{n,m}(t)/\partial t)_{\text{dissipation}}$ due to the damping of the field is considered separately.

III. RELAXATION OF SINGLE LASER ATOM

In order to calculate the change $\delta\rho^{\mu}_{n,m}(t)$ defined by Eq. (2.9), it is necessary for us to derive explicit expressions for the matrix elements $\rho_{n,\xi_\mu;m,\xi_\mu}(t + \Delta t)$. Since all the laser-active atoms are identical we may drop the subscript μ and formulate this problem in the following manner.

The system whose behavior we must describe falls into the general classification of an open quantum system S involving internal coupling interacting with a Markoffian reservoir R (as discussed in II). In this particular case the system S consists of a four-level atom coupled at resonance via its upper two levels to a single mode of the electromagnetic field, the reservoir divides into two independent subsystems R_a and R_b, and the interaction between S and R involves separate terms expressing coupling between R_a and atomic

levels $|4\rangle$ and $|2\rangle$, and R_b and atomic levels $|3\rangle$ and $|1\rangle$. Hence the total system $S+R$ may be defined (apart from an explicit expression for H_R which need not be given) by the Hamiltonian

$$H = H_S + H_R + H_{SR}, \tag{3.1}$$

where

$$H_S = \hbar\omega_0 a^\dagger a + \sum_{\xi=1}^{4} E_\xi A_\xi^\dagger A_\xi + \hbar\kappa (A_4^\dagger A_3 a + a^\dagger A_3^\dagger A_4),$$

$$H_{SR} = H_{SR_a} + H_{SR_b} = A_2^\dagger A_4 \Gamma_{R_a} + \Gamma_{R_a}^\dagger A_4^\dagger A_2 + A_1^\dagger A_3 \Gamma_{R_b} + \Gamma_{R_b}^\dagger A_3^\dagger A_1 . \tag{3.2}$$

Here $a^\dagger$, a are boson creation and annihilation operators for the field mode, $A_\xi^\dagger$ and A_ξ create and annihilate the atom in the state $|\xi\rangle$, $\xi = 1, 2, 3, 4$, and $\Gamma_{R_a}^\dagger$, Γ_{R_a} and $\Gamma_{R_b}^\dagger$, Γ_{R_b} are appropriate reservoir operators. The atomic energy eigenvalues are given by

$$E_4 = \tfrac{1}{2}\hbar\omega_0, \quad E_2 = \tfrac{1}{2}\hbar\omega_0 - \hbar\omega_a, \quad E_3 = -\tfrac{1}{2}\hbar\omega_0, \quad E_1 = -\tfrac{1}{2}\hbar\omega_0 - \hbar\omega_b, \tag{3.3}$$

and κ is a coupling constant. Use has been made of the familiar rotating-wave approximation in expressing the internal interaction.

The problem now involves two distinct steps. We must first derive a set of coupled equations for the matrix elements $\rho_{n,\xi;m,\eta}(t)$, and then follow this by obtaining the solution of these equations for $t > t_0$, subject to the initial condition

$$\rho_{n,\xi;m,\eta}(t_0) = \delta_{\xi,4}\delta_{\eta,4}\rho_{n,m}(t_0) . \tag{3.4}$$

Formally, the solution to the first part of the problem is given by

$$\frac{\partial \rho_{n,\xi;m,\eta}}{\partial t} = \langle n, \xi \left| \left[\left(\frac{\partial \rho}{\partial t}\right)_{\text{rev}} + \left(\frac{\partial \rho}{\partial t}\right)_{\text{irrev}}^{a} + \left(\frac{\partial \rho}{\partial t}\right)_{\text{irrev}}^{b} \right] \right| m, \eta \rangle , \tag{3.5}$$

where

$$\left(\frac{\partial \rho}{\partial t}\right)_{\text{rev}} = (1/i\hbar)[H_S, \rho],$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{\text{irrev}}^{a,b} = e^{(-iH_S/\hbar)t} \left(\frac{\partial \tilde{\rho}}{\partial t}\right)^{a,b} e^{(iH_S/\hbar)t}, \tag{3.6}$$

$$\left(\frac{\partial \tilde{\rho}}{\partial t}\right)^{a,b} = (-1/\hbar^2) \int_0^t \mathrm{Tr}_{R_{a,b}} \times \{[\tilde{H}_{SR_{a,b}}(t), [\tilde{H}_{SR_{a,b}}(t'), \tilde{\rho} f_0(R_{a,b})]]\} dt' , \tag{3.7}$$

with

$$\tilde{\rho}(t) = e^{(iH_S/\hbar)t} \rho(t) e^{(-iH_S/\hbar)t},$$

$$\tilde{H}_{SR_{a,b}}(t) = e^{i(H_S + H_R/\hbar)t} H_{SR_{a,b}} e^{-i(H_S + H_R/\hbar)t} . \tag{3.8}$$

Here ρ is the reduced-density operator for the system S defined by the Hamiltonian H_S and

$$f_0(R) = f_0(R_a) f_0(R_b) = e^{-H_R/kT} / \mathrm{Tr}_R (e^{-H_R/kT}) \tag{3.9}$$

is the density operator for the thermal reservoir assumed to be in thermal equilibrium at a temperature T. Explicit evaluation of this formal solution may be carried out using the same method as was applied in II to the treatment of dissipation from a boson-field mode coupled to a two-level atom. This simply involves expansion of all operators in terms of the chosen representation and substitution for the transition amplitudes

$$\langle n, \xi | e^{(\pm iH_S/\hbar)(t-t')} | m, \eta \rangle$$

which arise. For our four-level system the nonzero amplitudes read

$$\langle n, 1 | e^{(\pm iH_S/\hbar)t} | m, 1 \rangle = \delta_{n,m} e^{\pm i[(n-1/2)\omega_0 - \omega_b]t},$$

$$\langle n, 2 | e^{(\pm iH_S/\hbar)t} | m, 2 \rangle = \delta_{n,m} e^{\pm i[(n+1/2)\omega_0 - \omega_a]t},$$

$$\langle n, 3 | e^{(\pm iH_S/\hbar)t} | m, 3 \rangle = \delta_{n,m} e^{\pm i\omega_0(n-1/2)t} \cos\kappa n^{1/2} t ,$$

$$\langle n, 4 | e^{(\pm iH_S/\hbar)t} | m, 4 \rangle = \delta_{n,m} e^{\pm i\omega_0(n+1/2)t} \cos\kappa (n+1)^{1/2} t , \tag{3.10}$$

$$\langle n, 3 | e^{(\pm iH_S/\hbar)t} | m, 4 \rangle = \pm i\delta_{n,m+1} e^{\pm i\omega_0(n-1/2)t} \sin\kappa n^{1/2} t ,$$

$$\langle n, 4 | e^{(\pm iH_S/\hbar)t} | m, 3 \rangle = \pm i\delta_{n,m-1} e^{\pm i\omega_0(n+1/2)t} \sin\kappa (n+1)^{1/2} t .$$

Hence after a certain amount of algebra we arrive at the coupled equations

$$\frac{\partial \rho_{n,4;m,4}}{\partial t} = -[\gamma_a^{(+)}(n) + \gamma_a^{(+)}(m)] \rho_{n,4;m,4} - i\{[\kappa (n+1)^{1/2} - i\gamma_a^{(-)}(n)] \rho_{n+1,3;m,4} - [\kappa (m+1)^{1/2} + i\gamma_a^{(-)}(m)] \rho_{n,4;m+1,3}\}, \tag{3.11}$$

$$\frac{\partial \rho_{n+1,3;m+1,3}}{\partial t} = -[\gamma_b^{(+)}(n) + \gamma_b^{(+)}(m)] \rho_{n+1,3;m+1,3} - i\{[\kappa (n+1)^{1/2} - i\gamma_b^{(-)}(n)] \rho_{n,4;m+1,3} - [\kappa (m+1)^{1/2} + i\gamma_b^{(-)}(m)] \rho_{n+1,3;m,4}\}, \tag{3.12}$$

$$\frac{\partial \rho_{n,4;m+1,3}}{\partial t} = -[\gamma_a^{(+)}(n)+\gamma_b^{(+)}(m)]\rho_{n,4;m+1,3} - i\{[\kappa(n+1)^{1/2} - i\gamma_a^{(-)}(n)]\rho_{n+1,3;m+1,3} - [\kappa(m+1)^{1/2}+i\gamma_b^{(-)}(m)]\rho_{n,4;m,4}\}, \tag{3.13}$$

$$\frac{\partial \rho_{n+1,3;m,4}}{\partial t} = -[\gamma_a^{(+)}(m)+\gamma_b^{(+)}(n)]\rho_{n+1,3;m,4} - i\{[\kappa(n+1)^{1/2} - i\gamma_b^{(-)}(n)]\rho_{n,4;m,4} - [\kappa(m+1)^{1/2}+i\gamma_a^{(-)}(m)]\rho_{n+1,3;m+1,3}\}, \tag{3.14}$$

$$\frac{\partial \rho_{n,2;m,2}}{\partial t} = [\gamma_a^{(+)}(n)+\gamma_a^{(+)}(m)]\rho_{n,4;m,4} + \gamma_a^{(-)}(n)\rho_{n+1,3;m,4} + \gamma_a^{(-)}(m)\rho_{n,4;m+1,3}, \tag{3.15}$$

$$\frac{\partial \rho_{n+1,1;m+1,1}}{\partial t} = [\gamma_b^{(+)}(n)+\gamma_b^{(+)}(m)]\rho_{n+1,3;m+1,3} + \gamma_b^{(-)}(n)\rho_{n,4;m+1,3} + \gamma_b^{(-)}(m)\rho_{n+1,3;m,4}, \tag{3.16}$$

where

$$\gamma_{a,b}^{(\pm)}(k) = \tfrac{1}{4}[\Gamma_{a,b}(\omega_{a,b}^{(1)}(k)) \pm \Gamma_{a,b}(\omega_{a,b}^{(2)}(k))],$$

$$\Gamma_{a,b}(\omega) = 2\pi g_{a,b}(\omega)|\kappa_{a,b}(\omega)|^2, \tag{3.17}$$

$$\omega_{a,b}^{(1)}(k) = \omega_{a,b} + \kappa(k+1)^{1/2},$$

$$\omega_{a,b}^{(2)}(k) = \omega_{a,b} - \kappa(k+1)^{1/2}. \tag{3.18}$$

The limit $T \to 0$ has been taken in deriving these equations so that thermal fluctuation effects are excluded, and for mathematical convenience ρ has been redefined according to the scheme

$$\rho^{\text{new}}_{n,\xi;m,\eta} = \rho^{\text{old}}_{n,\xi;m,\eta} e^{-i(n-m)\omega_0 t}. \tag{3.19}$$

A comparison of Eqs. (3.11)–(3.16) with Eqs. (67a)–(67d), (66e), and (66f) of I will now show the essential difference between our analysis and that of SL. The effect of the inclusion of the internal coupling in the treatment of the reservoir interactions has been to introduce a dependence on the strength of the optical field into the decay constants $\gamma_a^{(+)}(k)$ and $\gamma_b^{(+)}(k)$, and to include an additional coupling between the matrix elements exhibited by the terms $\gamma_a^{(-)}(k)$ and $\gamma_b^{(-)}(k)$. Our equations correspond exactly with those of SL only under the nonphysical conditions $n+1 = m+1 = 0$.

Now in seeking a solution to Eqs. (3.11)–(3.16) we observe that we need in fact only consider a set of four coupled equations, Eqs. (3.11)–(3.14). The solutions to the remaining two equations may be expressed as direct integrals once these have been solved. Thus we will have

$$\rho_{n,2;m,2}(t) = [\gamma_a^{(+)}(n)+\gamma_a^{(+)}(m)]\int_{t_0}^{t}\rho_{n,4;m,4}(t')dt' + \gamma_a^{(-)}(n)\int_{t_0}^{t}\rho_{n+1,3;m,4}(t')dt' + \gamma_a^{(-)}(m)\int_{t_0}^{t}\rho_{n,4;m+1,3}(t')dt' \tag{3.20}$$

and

$$\rho_{n+1,1;m+1,1}(t) = [\gamma_b^{(+)}(n)+\gamma_b^{(+)}(m)]\int_{t_0}^{t}\rho_{n+1,3;m+1,3}(t')dt' + \gamma_b^{(-)}(n)\int_{t_0}^{t}\rho_{n,4;m+1,3}(t')dt' + \gamma_b^{(-)}(m)\int_{t_0}^{t}\rho_{n+1,3;m,4}(t')dt'. \tag{3.21}$$

A relatively simple methodology for solving the coupled Eqs. (3.11)–(3.14) is available if we adopt a matrix formalism. Interpreting $\rho(t)$ now as the 2×2 matrix defined by

$$\rho(t) = \begin{pmatrix} \rho_{n,4;m,4}(t) & \rho_{n,4;m+1,3}(t) \\ \rho_{n+1,3;m,4}(t) & \rho_{n+1,3;m+1,3}(t) \end{pmatrix}, \tag{3.22}$$

these equations may be written as

$$\frac{\partial \rho}{\partial t} = -i(N\rho - \rho M), \tag{3.23}$$

where N and M are the 2×2 matrices

$$N = \begin{pmatrix} -i\gamma_a^{(+)}(n) & \kappa(n+1)^{1/2} - i\gamma_a^{(-)}(n) \\ \kappa(n+1)^{1/2} - i\gamma_b^{(-)}(n) & -i\gamma_b^{(+)}(n) \end{pmatrix},$$

$$M = \begin{pmatrix} i\gamma_a^{(+)}(m) & \kappa(m+1)^{1/2} + i\gamma_b^{(-)}(m) \\ \kappa(m+1)^{1/2} + i\gamma_a^{(-)}(m) & i\gamma_b^{(+)}(m) \end{pmatrix}. \tag{3.24}$$

If we then seek matrices P and Q such that

$$P^{-1}NP = \Lambda_N = \begin{pmatrix} \lambda_1^N(n) & 0 \\ 0 & \lambda_2^N(n) \end{pmatrix},$$

$$Q^{-1}MQ = \Lambda_M = \begin{pmatrix} \lambda_1^M(m) & 0 \\ 0 & \lambda_2^M(m) \end{pmatrix} \tag{3.25}$$

we may convert Eq. (2.23) into the form

$$\frac{\partial \sigma}{\partial t} = -i(\Lambda_N \sigma - \sigma \Lambda_M), \tag{3.26}$$

where

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = P^{-1}\rho Q, \tag{3.27}$$

and since Eq. (3.26) is now diagonal, it will have the obvious solution

$$\sigma_{11}(t)=\sigma_{11}(t_0)e^{-\lambda_{11}(n,m)(t-t_0)},$$
$$\sigma_{22}(t)=\sigma_{22}(t_0)e^{-\lambda_{22}(n,m)(t-t_0)},$$
$$\sigma_{12}(t)=\sigma_{12}(t_0)e^{-\lambda_{12}(n,m)(t-t_0)},$$
$$\sigma_{21}(t)=\sigma_{21}(t_0)e^{-\lambda_{21}(n,m)(t-t_0)}, \tag{3.28}$$

where

$$\lambda_{kj}(n,m)=i[\lambda_k^N(n)-\lambda_j^M(m)]. \tag{3.29}$$

Although this technique is straightforward, the algebra involved in its application is extremely laborious and would be presented here to no advantage. Therefore we merely state that P and Q and the eigenvalues $\lambda_{1,2}^N(n)$ and $\lambda_{1,2}^M(m)$ may be evaluated by conventional techniques[21] and that subsequent inversion of the transformation (3.27) then leads to the solution

$$\rho_{n,4;m,4}(t)=\frac{\rho_{n,m}(t_0)}{(1-p_1p_2)(1-q_1q_2)}\,(e^{-\lambda_{11}(n,m)(t-t_0)}-q_1q_2e^{-\lambda_{12}(n,m)(t-t_0)}$$
$$-p_1p_2e^{-\lambda_{21}(n,m)(t-t_0)}+p_1p_2q_1q_2e^{-\lambda_{22}(n,m)(t-t_0)}), \tag{3.30}$$

$$\rho_{n,4;m+1,3}(t)=\frac{\rho_{n,m}(t_0)}{(1-p_1p_2)(1-q_1q_2)}\,(-q_2e^{-\lambda_{11}(n,m)(t-t_0)}+q_2e^{-\lambda_{12}(n,m)(t-t_0)}$$
$$+p_1p_2q_2e^{-\lambda_{21}(n,m)(t-t_0)}-p_1p_2q_2e^{-\lambda_{22}(n,m)(t-t_0)}), \tag{3.31}$$

$$\rho_{n+1,3;m,4}(t)=\frac{\rho_{n,m}(t_0)}{(1-p_1p_2)(1-q_1q_2)}\,(p_1e^{-\lambda_{11}(n,m)(t-t_0)}-p_1q_1q_2e^{-\lambda_{12}(n,m)(t-t_0)}$$
$$-p_1e^{-\lambda_{21}(n,m)(t-t_0)}+p_1q_1q_2e^{-\lambda_{22}(n,m)(t-t_0)}), \tag{3.32}$$

$$\rho_{n+1,3;m+1,3}(t)=\frac{\rho_{n,m}(t_0)}{(1-p_1p_2)(1-q_1q_2)}\,(-p_1q_2e^{-\lambda_{11}(n,m)(t-t_0)}+p_1q_2e^{-\lambda_{12}(n,m)(t-t_0)}$$
$$+p_1q_2e^{-\lambda_{21}(n,m)(t-t_0)}-p_1q_2e^{-\lambda_{22}(n,m)(t-t_0)}), \tag{3.33}$$

where p_1 and p_2 are functions of n defined by

$$p_1=\frac{\kappa(n+1)^{1/2}-i\gamma_b^{(-)}(n)}{\lambda_1^N(n)+i\gamma_b^{(+)}(n)}=\frac{\lambda_1^N(n)+i\gamma_a^{(+)}(n)}{\kappa(n+1)^{1/2}-i\gamma_a^{(-)}(n)}, \tag{3.34}$$

$$p_2=\frac{\kappa(n+1)^{1/2}-i\gamma_a^{(-)}(n)}{\lambda_2^N(n)+i\gamma_a^{(+)}(n)}=\frac{\lambda_2^N(n)+i\gamma_b^{(+)}(n)}{\kappa(n+1)^{1/2}-i\gamma_b^{(-)}(n)}, \tag{3.35}$$

and q_1 and q_2 are functions of m defined by

$$q_1=\frac{\kappa(m+1)^{1/2}+i\gamma_a^{(-)}(m)}{\lambda_1^M(m)-i\gamma_b^{(+)}(m)}=\frac{\lambda_1^M(m)-i\gamma_a^{(+)}(m)}{\kappa(m+1)^{1/2}+i\gamma_b^{(-)}(m)}, \tag{3.36}$$

$$q_2=\frac{\kappa(m+1)^{1/2}+i\gamma_b^{(-)}(m)}{\lambda_2^M(m)-i\gamma_a^{(+)}(m)}=\frac{\lambda_2^M(m)-i\gamma_b^{(+)}(m)}{\kappa(m+1)^{1/2}+i\gamma_a^{(-)}(m)}. \tag{3.37}$$

The eigenvalues $\lambda_{1,2}^N(n)$ and $\lambda_{1,2}^M(m)$ are given by

$$\lambda_{1,2}^N(n)=-i\tfrac{1}{2}[\gamma_a^{(+)}(n)+\gamma_b^{(+)}(n)]\pm\{[\kappa(n+1)^{1/2}-i\gamma_a^{(-)}(n)][\kappa(n+1)^{1/2}-i\gamma_b^{(-)}(n)]-\{\tfrac{1}{2}[\gamma_a^{(+)}(n)-\gamma_b^{(+)}(n)]\}^2\}^{1/2}, \tag{3.38}$$

$$\lambda_{1,2}^M(m)=i\tfrac{1}{2}[\gamma_a^{(+)}(m)+\gamma_b^{(+)}(m)]\pm\{[\kappa(m+1)^{1/2}+i\gamma_a^{(-)}(m)][\kappa(m+1)^{1/2}+i\gamma_b^{(-)}(m)]-\{\tfrac{1}{2}[\gamma_a^{(+)}(m)-\gamma_b^{(+)}(m)]\}^2\}^{1/2}, \tag{3.39}$$

Equations (3.30)–(3.39) together with Eqs. (3.20) and (3.21) now constitute the complete solution for the reduced-density matrix of our relaxing system. Although this solution may appear too complicated to provide much physical insight, it will be used in Sec. IV to produce results bearing a very close relationship to those of SL. In a future publication we hope to investigate these solutions within their own right, paying particular attention to the use of the rotating-wave approximation.

IV. MASTER EQUATION FOR OPTICAL FIELD

Having treated the single-atom interaction, we may now turn to the problem of deriving the master equation for the optical field in the manner discussed in Sec. II. The term arising from spontaneous and stimulated emission follows from Eqs. (2.11) and (2.9) and reads

$$\left(\frac{\partial \rho_{n,m}(t)}{\partial t}\right)_{\text{emission}} = \gamma_a \left(\sum_{\xi=1}^{4} \rho_{n,\xi;m,\xi}(t+\Delta t) - \rho_{n,m}(t) \right) . \tag{4.1}$$

Now recalling the assumption that Δt is large compared with atomic relaxation times, we are in a position to simplify the forthcoming calculations considerably. Under this assumption we may set $\rho_{n,4;m,4}(t+\Delta t)$ and $\rho_{n+1,3;m+1,3}(t+\Delta t)$ equal to zero, and in the light of the solutions (3.30)–(3.33) this implies that all of the decaying functions $\exp[-\lambda_{kj}(n,m)\Delta t]$ vanish. Hence in addition to the obvious simplification

$$\left(\frac{\partial \rho_{n,m}(t)}{\partial t}\right)_{\text{emission}} = \gamma_a \left(\sum_{\xi=1}^{2} \rho_{n,\xi;m,\xi}(t+\Delta t) - \rho_{n,m}(t) \right) \tag{4.2}$$

we need only retain the contribution from the lower integration limit (now t) in the evaluation of $\rho_{n,2;m,2}(t+\Delta t)$ and $\rho_{n+1,1;m+1,1}(t+\Delta t)$ from Eqs. (3.20) and (3.21), since the terms arising from the upper limit (now $t+\Delta t$) involve linear combinations of the functions $\exp[-\lambda_{kj}(n,m)\Delta t]$.

Thus from Eqs. (3.30)–(3.33), replacing t_0 by t and t by $t+\Delta t$, the four integrals required for the evaluation of the two nonzero diagonal matrix elements read

$$\int_t^{t+\Delta t} \rho_{n,4;m,4}(t')dt' = \frac{\rho_{n,m}(t)}{(1-p_1p_2)(1-q_1q_2)} \left(\frac{1}{\lambda_{11}(n,m)} - \frac{q_1q_2}{\lambda_{12}(n,m)} - \frac{p_1p_2}{\lambda_{21}(n,m)} + \frac{p_1p_2q_1q_2}{\lambda_{22}(n,m)} \right), \tag{4.3}$$

$$\int_t^{t+\Delta t} \rho_{n,4;m+1,3}(t')dt' = \frac{\rho_{n,m}(t)}{(1-p_1p_2)(1-q_1q_2)} \left(-\frac{q_2}{\lambda_{11}(n,m)} + \frac{q_2}{\lambda_{12}(n,m)} + \frac{q_2p_1p_2}{\lambda_{21}(n,m)} - \frac{q_2p_1p_2}{\lambda_{22}(n,m)} \right), \tag{4.4}$$

$$\int_t^{t+\Delta t} \rho_{n+1,3;m,4}(t')dt' = \frac{\rho_{n,m}(t)}{(1-p_1p_2)(1-q_1q_2)} \left(\frac{p_1}{\lambda_{11}(n,m)} - \frac{p_1q_1q_2}{\lambda_{12}(n,m)} - \frac{p_1}{\lambda_{21}(n,m)} + \frac{p_1q_1q_2}{\lambda_{22}(n,m)} \right), \tag{4.5}$$

$$\int_t^{t+\Delta t} \rho_{n+1,3;m+1,3}(t')dt' = \frac{\rho_{n,m}(t)}{(1-p_1p_2)(1-q_1q_2)} \left(-\frac{p_1q_2}{\lambda_{11}(n,m)} + \frac{p_1q_2}{\lambda_{12}(n,m)} + \frac{p_1q_2}{\lambda_{21}(n,m)} - \frac{p_1q_2}{\lambda_{22}(n,m)} \right). \tag{4.6}$$

Now it is desirable to cast these results into a form which bears direct comparison with Eqs. (78a) and (78b) of I. Thus with our sights set in this direction we turn to a simplification using the various definitions presented in Eqs. (3.29) and (3.34)–(3.39). After a considerable amount of algebraic manipulation we arrive at the following:

$$\gamma_a(n,m) \int_t^{t+\Delta t} \rho_{n,4;m,4}(t')dt' = \rho_{n,m}(t) \left(1 - \frac{K_a(n)K_b(n)}{\kappa^2} \Re(n,m) - \frac{K_a^*(m)K_b^*(m)}{\kappa^2} \Re^*(m,n) \right), \tag{4.7}$$

$$\gamma_b(n,m) \int_t^{t+\Delta t} \rho_{n+1,3;m+1,3}(t')dt' = \rho_{n,m}(t) \frac{K_b(n)K_b^*(m)}{\kappa^2} [\Re(n,m) + \Re^*(m,n)], \tag{4.8}$$

$$\int_t^{t+\Delta t} \rho_{n+1,3;m,4}(t')dt' = -i\rho_{n,m}(t) \frac{K_b(n)}{\kappa^2} \Re(n,m), \tag{4.9}$$

$$\int_t^{t+\Delta t} \rho_{n,4;m+1,3}(t')dt' = i\rho_{n,m}(t) \frac{K_b^*(m)}{\kappa^2} \Re^*(m,n), \tag{4.10}$$

where we have defined

$$\begin{aligned} \gamma_a(n,m) &= \gamma_a^{(+)}(n) + \gamma_a^{(+)}(m), \\ \gamma_b(n,m) &= \gamma_b^{(+)}(n) + \gamma_b^{(+)}(m), \\ \gamma_{ab}(n,m) &= \gamma_a^{(+)}(n) + \gamma_b^{(+)}(m); \end{aligned} \tag{4.11}$$

$$\begin{aligned} K_a(k) &= \kappa(k+1)^{1/2} - i\gamma_a^{(-)}(k), \\ K_b(k) &= \kappa(k+1)^{1/2} - i\gamma_b^{(-)}(k); \end{aligned} \tag{4.12}$$

and

$$\mathcal{R}(n,m)=\kappa^2\,\frac{\gamma_b(n,m)\gamma_{ab}(n,m)+[K_a(n)K_b(n)-K_a^*(m)K_b^*(m)]}{\Lambda(n,m)}, \tag{4.13}$$

with

$$\Lambda(n,m)=\gamma_a(n,m)\gamma_b(n,m)\gamma_{ab}(n,m)\gamma_{ab}(m,n)+K_a(n)K_b(n)[\gamma_a(n,m)\gamma_{ab}(m,n)+\gamma_b(n,m)\gamma_{ab}(n,m)]$$

$$+K_a^*(m)K_b^*(n)[\gamma_a(n,m)\gamma_{ab}(n,m)+\gamma_b(n,m)\gamma_{ab}(m,n)]+[K_a(n)K_b(n)-K_a^*(m)K_b^*(m)]^2. \tag{4.14}$$

We are now in a position to evaluate the matrix elements $\rho_{n,2;m,2}(t+\Delta t)$ and $\rho_{n+1,1;m+1,1}(t+\Delta t)$ using the formulas (3.20) and (3.21). Substitution of the results (4.7)–(4.10) into these equations yields

$$\rho_{n,2;m,2}(t+\Delta t)=\rho_{n,m}(t)\{1-\mathcal{R}(n,m)[K_b(n)/\kappa](n+1)^{1/2}-\mathcal{R}^*(m,n)[K_b^*(m)/\kappa](m+1)^{1/2}\}, \tag{4.15}$$

$$\rho_{n+1,1;m+1,1}(t+\Delta t)=\rho_{n,m}(t)\{\mathcal{R}(n,m)[K_b(n)/\kappa](m+1)^{1/2}+\mathcal{R}^*(m,n)[K_b^*(m)/\kappa](n+1)^{1/2}\}. \tag{4.16}$$

As we would expect, from these equations we may establish the relationship

$$\rho_{n,2;n,2}(t+\Delta t)+\rho_{n+1,1;n+1,1}(t+\Delta t)=\rho_{n,n}(t). \tag{4.17}$$

The final step in the derivation of $[\partial\rho_{n,m}(t)/\partial t]_{\text{emission}}$ now simply involves the substitution of the results (4.15) and (4.16) into Eq. (4.2). By defining $R(n,m)$ by

$$R(n,m)=\gamma_a\mathcal{R}(n,m) \tag{4.18}$$

this yields

$$\left(\frac{\partial\rho_{n,m}(t)}{\partial t}\right)_{\text{emission}}=-\{(n+1)^{1/2}[K_b(n)/\kappa]R(n,m)+(m+1)^{1/2}[K_b^*(m)/\kappa]R^*(m,n)\}\rho_{n,m}(t)$$

$$+\{m^{1/2}[K_b(n-1)/\kappa]R(n-1,m-1)+n^{1/2}[K_b^*(m-1)/\kappa]R^*(m-1,n-1)\}\rho_{n-1,m-1}(t), \tag{4.19}$$

which may be compared with the equivalent result, Eq. (83) in I. This comparison shows that the two results will be exactly the same only if we set $\kappa=0$ in definition (3.18)

The derivation of the required master equation will now be completed when we include the term describing the damping of the optical field. In II we considered dissipation from a single mode of the radiation field coupled to a two-level atom. In the derivation of the master equations for the diagonal elements of the reduced-density operator the internal coupling was retained throughout, and as a consequence, the resulting equations differed from those obtained by conventional techniques. It is easily checked, however, that if the photon number is much greater than unity (as in the laser) the differences are negligible and the usual irreversible term for a damped-oscillator system results. This seems quite reasonable since we would expect a strong field to be largely unaffected by its coupling to a single atom. However, in the laser we have something a little different. Here there are many atoms coupled to the one-field mode and their cooperative behavior could conceivably have a noticeable effect on the nature of the dissipation. In contrast, however, the damping of the field which results largely from loss through the end mirrors is essentially classical in nature, and therefore the usual linear damping term

$$\left(\frac{\partial\rho_{n,n}(t)}{\partial t}\right)_{\text{dissipation}}=-C[n\rho_{n,n}(t)-(n+1)\rho_{n+1,n+1}(t)], \tag{4.20}$$

where C is the cavity bandwidth, seems perfectly in order irrespective of the strength of the internal laser coupling. In any case, treatment of this problem by the technique used in II is unthinkable because of the number of degrees of freedom in S and so we must be content with retaining SL's term

$$\left(\frac{\partial\rho_{n,m}(t)}{\partial t}\right)_{\text{dissipation}}=-C\{\tfrac{1}{2}(n+m)\rho_{n,m}(t)-(n+1)^{1/2}(m+1)^{1/2}\rho_{n+1,m+1}(t)\}. \tag{4.21}$$

Now combining Eqs. (4.19) and (4.21) the full laser master equation reads

$$\frac{\partial \rho_{n,m}(t)}{\partial t} = -\{(n+1)^{1/2}[K_b(n)/\kappa]R(n,m) + (m+1)^{1/2}K_b^*(m)R^*(m,n)\}\rho_{n,m}(t)$$
$$+\{m^{1/2}[K_b(n-1)/\kappa]R(n-1,m-1) + n^{1/2}[K_b^*(m-1)/\kappa]R^*(m-1,n-1)\}\rho_{n-1,m-1}(t)$$
$$-C\{\tfrac{1}{2}(n+m)\rho_{n,m}(t) - (n+1)^{1/2}(m+1)^{1/2}\rho_{n+1,m+1}(t)\}. \tag{4.22}$$

If we focus our attention on the diagonal matrix elements alone, we can simplify Eq. (4.22) to obtain the form

$$\frac{\partial \rho_{n,n}(t)}{\partial t} = -(n+1)A(n)\{1+(n+1)[B(n)/A(n)]\}^{-1}\rho_{n,n}(t) + nA(n-1)\{1+n[B(n-1)/A(n-1)]\}^{-1}\rho_{n-1,n-1}(t)$$
$$-Cn\rho_{n,n}(t) + C(n+1)\rho_{n+1,n+1}(t), \tag{4.23}$$

where

$$A(n) = 2\kappa^2\gamma_a\frac{\gamma_b(n)\gamma_{ab}(n) - \delta_b(n)\delta_{ab}(n)}{\gamma_{ab}^2(n)[\gamma_a(n)\gamma_b(n) - \delta_a(n)\delta_b(n)]}, \tag{4.24}$$

$$B(n) = 4\kappa^2\frac{\gamma_{ab}^2(n) - \delta_{ab}^2(n)}{\gamma_{ab}^2(n)[\gamma_a(n)\gamma_b(n) - \delta_a(n)\delta_b(n)]}A(n), \tag{4.25}$$

with

$$\gamma_a(n) = \gamma_a(n,n) = 2\gamma_a^{(+)}(n),$$
$$\gamma_b(n) = \gamma_b(n,n) = 2\gamma_b^{(+)}(n), \tag{4.26}$$
$$\gamma_{ab}(n) = \gamma_{ab}(n,n) = \gamma_a^{(+)}(n) + \gamma_b^{(+)}(n) = \tfrac{1}{2}[\gamma_a(n) + \gamma_b(n)]$$

and

$$\delta_a(n) = 2\gamma_a^{(-)}(n),$$
$$\delta_b(n) = 2\gamma_b^{(-)}(n), \tag{4.27}$$
$$\delta_{ab}(n) = \gamma_a^{(-)}(n) + \gamma_b^{(-)}(n) = \tfrac{1}{2}[\delta_a(n) + \delta_b(n)].$$

This master equation is of exactly the same form as Eq. (86) in I except for the n dependence which now appears in $A(n)$ and $B(n)$.

V. DISCUSSION AND CONCLUSIONS

In this final section we discuss the results of the preceding analysis with a mind to clarifying the differences between them and those of SL. In particular the significance of these differences will be considered from the point of view of justifying the normal use of the factorization ansatz. To this end, we commence by pointing out that it is clear, without any further investigation of the master equation (4.23), just what condition determines the accuracy of the conventional approach. As has been apparent in all earlier treatments of open systems with internal coupling,[16-19] the effect of retaining the complete system Hamiltonian when deriving the irreversible term in the master equation has been to split the frequency at which the system and reservoir are resonant. Thus here we have resonant frequencies $\omega_a^{(1)}(n)$, $\omega_a^{(2)}(n)$, $\omega_b^{(1)}(n)$, and $\omega_b^{(2)}(n)$ defined by Eqs. (3.18), while conventional techniques would simply give ω_a and ω_b. Fundamentally this is the only difference between our analysis and that of SL. If we were to set $\omega_a^{(1)}(n) = \omega_a^{(2)}(n) = \omega_a$ and $\omega_b^{(1)}(n) = \omega_b^{(2)}(n) = \omega_b$ the two treatments would become identical. It is therefore clear from Eqs. (3.18), that if $\kappa(n_p+1)^{1/2} \ll \omega_{a,b}$, where n_p is the photon number around which the laser distribution is peaked, the corrections arising from our approach would be completely insignificant. In actual fact, as will become apparent further on, the condition is less stringent that this, reading

$$\kappa^2(n_p+1) \ll \omega_a\omega_b. \tag{5.1}$$

This condition is certainly fulfilled for current continuous-output lasers and the only possibility of finding sufficiently intense fields to violate this inequality would be to look among the large pulsed systems which are currently being developed. For a neodymium glass laser where $\tau \sim 0.2$ msec and $\lambda_0 \sim 1.06$ μm, we find for a cavity of length 20 cm and cross-sectional area 25 mm^2 $|\kappa|^2 = 1.7 \times 10^5$ sec^{-2}. Hence for this example the usual factorization ansatz would become invalid as the peak photon number approached the value $n_p \sim 10^{24}$. This corresponds to an energy of $\sim 10^5$ J. This condition could possibly be violated in the microwave region, where for a cavity volume of 10^2 cm^3 and an atomic dipole moment of $e \times 10^{-8}$ cm angular frequencies of $\sim 10^{10}$ sec^{-1} result in the failure of the above condition for $n \sim 10^{11}$.

We now proceed to a more detailed discussion of various aspects of our solution, seeking to understand just what modifications we should expect to arise if the condition (5.1) fails to hold. A certain amount of care is required here, since recognizing the use of the rotating-wave approxi-

mation (RWA) in expressing the Hamiltonian (3.2), we cannot expect to be able to extend our results to the region $\kappa(n_p+1)^{1/2}\sim\omega_{a,b}$ and obtain reliable predictions. We make this statement both in the light of the attention recently paid to the RWA[22-27] and in the knowledge of the fact that in this region, definite anomalies arise in the results of Sec. III which we believe appear solely as a consequence of the RWA. We have strong evidence to support this belief but will pursue the matter no further here, leaving a detailed exposition to a future publication.

Now as we mentioned above, the fundamental difference in our analysis from that of SL is the splitting of the resonant frequencies ω_a and ω_b giving the two pairs $\omega_a^{(1)}(n)$, $\omega_a^{(2)}(n)$ and $\omega_b^{(1)}(n)$, $\omega_b^{(2)}(n)$. This difference manifests itself via the definition of $\gamma_{a,b}^{(\pm)}(n)$, or more precisely by the exact form of the strength function $\Gamma_{a,b}(\omega)$. Before discussing the solution to the master equation (4.23) we will therefore take a closer look at the "decay constants" $\gamma_{a,b}^{(\pm)}(n)$. In order to clarify to some extent their significance, we find it expedient to expand the strength function $\Gamma_a(\omega)$ and $\Gamma_b(\omega)$ in Taylor series about ω_a and ω_b, respectively. Thus we have

$$\Gamma_a(\omega)=\Gamma_a(\omega_a)\sum_{k=0}^{\infty}a_k\left(\frac{\omega-\omega_a}{\omega_a}\right)^k,$$
$$\Gamma_b(\omega)=\Gamma_b(\omega_b)\sum_{k=0}^{\infty}b_k\left(\frac{\omega-\omega_b}{\omega_b}\right)^k, \tag{5.2}$$

where a_k and b_k are dimensionless constants defined by

$$a_k=\frac{1}{k!}\frac{\omega_a^k}{\Gamma_a(\omega_a)}\frac{d^k\Gamma_a(\omega)}{d\omega^k}\bigg|_{\omega_a},$$
$$b_k=\frac{1}{k!}\frac{\omega_b^k}{\Gamma_b(\omega_b)}\frac{d^k\Gamma_b(\omega)}{d\omega^k}\bigg|_{\omega_b}. \tag{5.3}$$

Invoking these expansions in Eq. (3.17) and making use of Eqs. (4.26) and (4.27) we then arrive at the following expansions for the functions $\gamma_{a,b}(n)$ and $\delta_{a,b}(n)$:

$$\gamma_a(n)=\frac{\Gamma_a(\omega_a)}{2}\sum_{k=0}^{\infty}a_{2k}\left(\frac{\kappa(n+1)^{1/2}}{\omega_a}\right)^{2k},$$
$$\gamma_b(n)=\frac{\Gamma_b(\omega_b)}{2}\sum_{k=0}^{\infty}b_{2k}\left(\frac{\kappa(n+1)^{1/2}}{\omega_b}\right)^{2k} \tag{5.4}$$

and

$$\delta_a(n)=\frac{\Gamma_a(\omega_a)}{2}\sum_{k=0}^{\infty}a_{2k+1}\left(\frac{\kappa(n+1)^{1/2}}{\omega_a}\right)^{2k+1},$$
$$\delta_b(n)=\frac{\Gamma_b(\omega_b)}{2}\sum_{k=0}^{\infty}b_{2k+1}\left(\frac{\kappa(n+1)^{1/2}}{\omega_b}\right)^{2k+1}. \tag{5.5}$$

Now the interesting feature of these results lies in the fact that $\gamma_a(n)$ and $\gamma_b(n)$ depend only on the even powers of $\kappa(n+1)^{1/2}/\omega_{a,b}$, while $\delta_a(n)$ and $\delta_b(n)$ depend only on the odd powers. This would seem to support the interpretation that $\gamma_a(n)$ and $\gamma_b(n)$ include the effects that all the even-order interactions between the atom and the field have on the irreversible behavior of the atom, and $\delta_a(n)$ and $\delta_b(n)$ account for the effects of all the odd interactions. A distinction may be drawn between these two categories since the even interactions have no direct effect on the state of the field, although their inclusion can alter it indirectly by modifying the atomic lifetime. The odd-order interactions change the state of the field directly, both an increase and a decrease in photon number being possibilities. It does appear that these two possibilities occur in opposition, since if we take the case where the two lasing levels are identically damped,

$$\gamma_a(n)=\gamma_b(n)=\gamma_{ab}(n)=\gamma(n),$$
$$\delta_a(n)=\delta_b(n)=\delta_{ab}(n)=\delta(n), \tag{5.6}$$

then from Eqs. (4.24) and (4.25), $A(n)$ and $B(n)$ are given by

$$A(n)=2\kappa^2\gamma_a\frac{1}{\gamma^2(n)},$$
$$B(n)=4\kappa^2\frac{1}{\gamma^2(n)}A(n). \tag{5.7}$$

These expressions are both independent of $\delta(n)$, which implies that here the optical field is completely unaltered by the inclusion of the odd-order terms. We propose the interpretation that this indicates a cancelation between the effects of those odd-order interactions which give rise to an increase in photon number, and those which give rise to a decrease. Thus for a laser where both lasing levels have similar lifetimes, the only modification we expect is a change in effective lifetime exhibited by the variation in $\gamma(n)$. For example, if we propose a flat frequency spectrum for the strength function around and above the frequencies $\omega_{a,b}$, and a fairly rapid fall-off at low frequencies to $\Gamma(0)=0$, according to our theory we would observe a decrease in the "decay constant" as the strength of the optical field increased, until for $\kappa(n+1)^{1/2}\geq\omega_{a,b}$, $\gamma(n)$ would have the value $\frac{1}{2}\gamma(0)$. Thus we obtain a quenching of the spontaneous emission linewidth corresponding to the prediction of Haken quoted in Sec. I.

To conclude let us turn our attention to the master equation (4.23) and its steady-state solution. This follows from detailed balance considerations in exactly the same manner as in I and reads

$$\rho_{n,n}=\rho_{0,0}\prod_{k=1}^{n}\frac{A(k-1)}{C}\left(1+k\frac{B(k-1)}{A(k-1)}\right)^{-1}. \quad (5.8)$$

Before any comparison can be made between this and Eq. (92) of I, a model must be adopted for the strength functions $\Gamma_a(\omega)$ and $\Gamma_b(\omega)$ so that the n dependence of $A(n)$ and $B(n)$ becomes explicit. The specific differences observed will therefore depend directly on the particular model chosen. As an example we take the model which was used by Haake[28] when he treated non-Markoffian effects in the damped oscillator. Under the assumption that both lasing levels see reservoirs with the same strength functions, this model is defined by

$$\Gamma_{a,b}(\omega)=\frac{\gamma\omega}{\Gamma^2+\omega^2}, \quad (5.9)$$

where γ and Γ are appropriately chosen constants, $\Gamma\gg\omega_{a,b}$. From Eqs. (4.26), (4.27), (3.17), and (3.18), it then follows for $\kappa(n+1)^{1/2}<\omega_{a,b}$, that

$$\gamma_a(n)=\gamma_a=(\gamma/\Gamma^2)\omega_a, \quad \delta_a(n)=\gamma_a(n+1)^{1/2}(\kappa/\omega_a),$$

$$\gamma_b(n)=\gamma_b=(\gamma/\Gamma^2)\omega_b, \quad \delta_b(n)=\gamma_b(n+1)^{1/2}(\kappa/\omega_b),$$

$$\gamma_{ab}(n)=\gamma_{ab}=(\gamma/\Gamma^2)\omega_{ab}, \delta_{ab}(n)=\gamma_{ab}(n+1)^{1/2}(\kappa/\omega_{ab}), \quad (5.10)$$

where

$$\omega_{ab}=\tfrac{1}{2}(\omega_a+\omega_b). \quad (5.11)$$

Here, since the model is linear in ω around the region of interest, the n dependence has dropped out of $\gamma_a(n)$ and $\gamma_b(n)$, and we can therefore isolate the influence of the odd-order interactions for a case where $\gamma_a\neq\gamma_b$. Using the results (5.10) in Eqs. (4.25) and substituting in Eq. (5.8), we obtain

$$\rho_{n,n}=\rho_{0,0}\prod_{k=1}^{n}\left\{\frac{A}{C}\left(1-k\frac{\kappa^2}{\omega_a\omega_b}(1+\Delta\omega)\right)\Big/\left[1-k\frac{\kappa^2}{\omega_a\omega_b}+k\frac{B}{A}\left(1-k\frac{\kappa^2}{\omega_a\omega_b}(1-\Delta\omega^2)\right)\right]\right\}, \quad (5.12)$$

where we define $\Delta\omega$ by

$$\Delta\omega=(\omega_a-\omega_b)/(\omega_a+\omega_b) \quad (5.13)$$

and A and B have the same form as in I:

$$A=2\kappa^2\gamma_a\frac{1}{\gamma_a\gamma_{ab}}, \quad B=4\kappa^2\frac{1}{\gamma_a\gamma_b}A. \quad (5.14)$$

If we now assume $\Delta\omega$ to be sufficiently small to neglect the second-order term, we find we may write this in the form

$$\rho_{n,n}=\frac{\rho_{0,0}}{\rho^{SL}_{0,0}}\rho^{SL}_{n,n}\prod_{k=1}^{n}\left[\left(1-k\frac{\kappa^2}{\omega_a\omega_b}(1+\Delta\omega)\right)\Big/\left(1-k\frac{\kappa^2}{\omega_a\omega_b}\right)\right], \quad (5.15)$$

$\rho^{SL}_{n,n}$ being the SL result for the nth diagonal matrix element. In this expression we find direct verification of the condition (5.1), since if the inequality (5.1) holds, then we retain the SL distribution.

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