Landau-Ginsburg equations for an anisotropic superfluid*

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A discussion of an anisotropic (p state) Fermi superfluid with a spatially varying order parameter is presented. We restrict ourself to the Landau-Ginzburg region and consider spatial variations governed by the linear gap equation. The kernel of the linearized gap equation is shown to be related to a current-current correlation function of quasiparticles in the normal state. This relation permits the calculation of the two coherence lengths ξ_L and ξ_T defined earlier in the framework of a phenomenological Landau-Ginzburg theory, and a discussion of boundary conditions for the order parameter at interfaces. A significant anisotropy in the coherence lengths is found ($\xi_L = \sqrt{3}\xi_T$), which reflects itself in the nature of the supercurrents. The boundary conditions at specularly and diffusely reflecting surfaces lead us to expect that the vector $\hat{l}$, which describes the orbital angular momentum of a Cooper pair in an "axial" state, is anchored normal to the walls. Some consequences of these results on various experimental problems (fourth sound, monodomain production, Josephson couplings, behavior in thin capillaries) are briefly discussed.

I. PHENOMENOLOGICAL CONSIDERATIONS

A. Introduction

The resonance data on the A phase of ^{3}He[1] strongly suggest that it is a superfluid phase with Cooper pairs in a triplet spin state.[2] The orbital pairing state must then correspond to L odd ($L = 1, 3, \ldots$). We assume in the present paper that $L = 1$ (p states) is the correct answer. Our aim is to clarify some macroscopic properties of such an anisotropic superfluid, through the Landau-Ginsburg equations. In the present section we shall first recall some basic phenomenological features, which have been already described briefly.[3] In Sec. II we present a microscopic analysis of some of the relevant parameters. In Sec. III we discuss the problem of boundary conditions, together with some applications to wall alignment, to the behavior of ^{3}He in capillaries and in porous materials, etc.

Our starting point is the condensation amplitude $\Delta_{\alpha\beta}(\hat{k})$, which is conveniently written as[4]

$$\Delta_{\alpha\beta}(\hat{k}) = \sum_{p,i} A_{pi}\, k_i\, w_p(\alpha\beta), \tag{1}$$

where k_i is a component of the wave vector $\vec{k}$. For instance, k_z corresponds to pairing in an orbital state Y_{10}, with $L_z = 0$. The spin-wave functions $w_p (p = 1, 2, 3)$ are defined symmetrically with the orbital functions: w_p is an eigenfunction of the pair spin operator S_p with eigenvalue zero:

$$S_p w_p = 0. \tag{2}$$

For instance w_3 has the explicit form

$$w_3 = \frac{1}{\sqrt{2}}[\uparrow\uparrow + \downarrow\downarrow]. \tag{3}$$

The nine complex coefficients A_{pi} involved in Eq. (1) define the order parameter of the system. Clearly, the description of the system in terms of 18 real numbers is exceedingly heavy, and one of our main aims will be to simplify it as much as possible.

B. Free energy, permanent currents

The Landau expansion of the free energy F in powers of the order parameter is particularly interesting.[4,5] The Landau assumptions concerning this expansion are in fact acceptable for ^{3}He (just as they are for superconductors) because the low-temperature coherence length

$$\xi_0 \cong v_F/2\pi T_c \tag{4}$$

is much larger than the interparticle distance. Neglecting dipolar forces and assuming no external field (for the moment), we observe that the free energy F must be invariant separately under rotations in spin space (involving the indices p) and in orbital space (involving the indices i). It is then of the form

$$F = \tfrac{1}{2}a(T)\sum_{p,i} A_{pi}A_{pi}^* + F_4 + \cdots, \tag{5}$$

where a is linear in temperature:

$$a = a'(T - T_c), \quad a' > 0 \tag{6}$$

while F_4 is the sum of all terms quartic in A, and is $F_4 \geq 0$. Below T_c the quadratic term is negative and favors all components A_{pi} equally. This

implies a considerable degeneracy, which is removed (partially) by F_4, as discussed in Refs. 4 and 6. The situation here may be compared to a Heisenberg ferromagnet, with magnetization $\vec{M}$, and with a weak cubic anisotropy, leading to an F of the form

$$F = \tfrac{1}{2}aM^2 + \tfrac{1}{4}bM^4 + \epsilon(M_x^4 + M_y^4 + M_z^4). \tag{7}$$

Depending on the sign of ϵ, one has either $\vec{M}$ along a [111] axis ($\epsilon > 0$) or $\vec{M}$ along a [001] axis ($\epsilon < 0$): the continuous rotational degeneracy of the Heisenberg ferromagnet is reduced to a discrete set of possibilities.

The ^{3}He case is conceptually similar, but practically more complex, because, after application of the F_4 terms, we are left with nonequivalent states, and also many of these states still retain a continuous degeneracy.

Let us now allow spatial variations of the order parameter A_{pi} (assuming these variations slow on the scale of ξ_0). Then the free energy contains extra terms of which the leading terms will usually be quadratic in the gradients, and have the form[3]

$$F_{g2} = \sum_p \tfrac{1}{2}K_L |\mathrm{div}\vec{A}_p|^2 + \tfrac{1}{2}K_T |\mathrm{curl}\vec{A}_p|^2, \tag{8}$$

where K_L and K_T are two positive constants and $\vec{A}_p$ is the vector of components A_{pi}. The constants K_L and K_T may be associated with two coherence lengths:

$$\xi_L^2 = K_L/|a| \quad \text{(longitudinal)},$$

$$\xi_T^2 = K_T/|a| \quad \text{(transverse)}.$$

Apart from the quadratic terms F_{g2}, we may also find (in some very special cases) terms which are *linear* in the gradients, namely

$$F_{g1} = \sum_p \tfrac{1}{2}Q(\vec{A}_p^* \cdot \mathrm{curl}\vec{A}_p + \mathrm{c.c.}). \tag{9}$$

F_{g1} is a pseudoscalar, and Q will be nonvanishing only if the system does not have mirror symmetry; a possible example is a porous medium, made with very small quartz particles, all of the same chirality, with ^{3}He inside the pores.

The current density J may be derived from F_g by the following operation: A fictitious electric charge e is attributed to the ^{3}He atoms, and an external vector potential $\vec{a}$ is applied. In the total free energy F the gradient terms must then be modified according to the rule

$$\nabla A_{pi} \to (\nabla + 2ie\vec{a})A_{pi},$$

where $2e$ is the charge of a Cooper pair. The particle current is the functional derivative

$$\vec{J} = e^{-1}\frac{\delta f}{\delta \vec{a}}\bigg|_{e=0}. \tag{10}$$

Carrying out these operations with the forms (8) and (9) for F_g one finds

$$\vec{J} = 2\,\mathrm{Im}\sum_p (K_L \vec{A}_p^* \,\mathrm{div}\vec{A}_p + K_T \vec{A}_p^* \times \mathrm{curl}\vec{A}_p - Q\vec{A}_p^* \times \vec{A}_p). \tag{11}$$

Note that, because $\vec{A}_p$ is a complex vector, the cross product $\vec{A}_p^* \times \vec{A}_p$ is nonvanishing.

C. London limit

In the following discussion we shall restrict our attention to a particular set of states, which is at present the most plausible candidate for the A phase of ^{3}He.

Both from the theoretical calculations of Anderson and Brinkman[6] and from the resonance data[1] as analyzed by Leggett,[2,7] we are led to postulate that the following structure for the A_{pi}'s corresponds to the minimum of the free energy[4] in the A phase:

$$A_{pi} = \mathrm{const} V_p \Delta_i, \tag{12}$$

where $\vec{V}$ is a real vector (the "spin quantization axis") and can be chosen to be of unit length ($\vec{V}^2 = 1$). The orbital part $\vec{\Delta}$ is a complex vector:

$$\vec{\Delta} = \vec{\Delta}' + i\vec{\Delta}''. \tag{13}$$

We shall call this state, which was first investigated by Anderson and Morel,[8] the "axial state."[7,9] It may be described by the condition

$$\vec{\Delta}^2 = 0 \tag{14}$$

or

$$|\Delta'| = |\Delta''| = \Delta, \quad \vec{\Delta}' \cdot \vec{\Delta}'' = 0.$$

This state carries an orbital angular momentum parallel to $\vec{\Delta}' \times \vec{\Delta}''$. The unit vector along this direction will be called $\vec{l}$. The orientation of the triad $\vec{\Delta}', \vec{\Delta}'', \vec{l}$ can be defined by $\vec{l}$ plus an angle φ specifying the orientation of $\vec{\Delta}'$ in the plane normal to $\vec{l}$. A multiplication of all the components A_{pi} by a phase factor $e^{i\varphi_0}$ is equivalent to a rotation in this plane $\varphi \to \varphi + \varphi_0$. Thus the angle φ may be called the *phase* of the order parameter.

To summarize: An ordered state of the axial type is characterized by one amplitude $\Delta = |\Delta'| = |\Delta''|$ and by *five* angular variables, namely, the polar angles of $\vec{V}$ and $\vec{l}$, plus the phase φ. Our next aim is to find what is the relation between these geometrical objects and physical observables such as the supercurrents.

The London limit corresponds to a situation where the currents are weak, and we have an axial phase with constant amplitudes Δ and angular

variables $(\vec{V}, \vec{l}, \varphi)$ which vary slowly from point to point. The explicit meaning of the word "slowly" is given by

$$\xi(T)\nabla\varphi \ll 1$$

(and similar inequalities for $\vec{V}$ and $\vec{l}$). In the London limit the structure of the current is comparatively simple. Using Eqs. (11) and (12), we find that $\vec{V}$ does not contribute and we are left with

$$\vec{J} = \hat{\rho}_s \,\mathrm{grad}\varphi + \hat{c}\,\mathrm{curl}\vec{l} - 4Q\Delta^2\vec{l}, \tag{15}$$

where $(\mathrm{grad}\varphi)\cdot d\vec{x}$ is the angle of the rotation around the local axis $\vec{l}(\vec{x})$ which maps $\vec{\Delta}'(\vec{x})$, $\vec{\Delta}''(\vec{x})$ into $\vec{\Delta}'(\vec{x}+d\vec{x})$, $\vec{\Delta}''(\vec{x}+d\vec{x})$. In general $\mathrm{curl}(\mathrm{grad}\varphi)$ need not be zero. The compact expression (15) for the current has been achieved at the expense of a generally valid global definition of a φ field. The first term in Eq. (15) is a natural generalization of the London equation, $\hat{\rho}_s$ being a tensor superfluid density, with different principal values along $\vec{l}\,(\rho_{s\parallel})$ and transverse to $\vec{l}\,(\rho_{s\perp})$. A simple calculation yields

$$\begin{aligned} \rho_{s\parallel} &= 4\Delta^2 K_T, \\ \rho_{s\perp} &= 2\Delta^2(K_L + K_T). \end{aligned} \tag{16}$$

The second term in (15) is the analog of the equation for the magnetization current in a ferromagnet ($\vec{J} = \mathrm{curl}\vec{M}$), but the coefficient C is also anisotropic:

$$C_\parallel = 2\Delta^2 K_T, \quad C_\perp = 2\Delta^2 K_L. \tag{17}$$

The last term in (15) shows that rotating pairs moving in a helical medium must also have a translational motion of well defined sign. This effect may be observable. However, for simplicity, we shall assume $Q = 0$ in all that follows.

One remarkable consequence of the anisotropy of $\hat{\rho}_s$ is the following: even if $\vec{l}$ is constant in space, the superfluid current $\vec{J}$ can have a *non-vanishing curl*. Defining axes (1,2,3) along $\vec{\Delta}'$, $\vec{\Delta}''$, and $\vec{l}$, we have in fact (for $\hat{\rho} = \mathrm{const}$)

$$\left.\begin{aligned} (\mathrm{curl}\vec{J})_1 &= 2\Delta^2(K_T - K_L)\frac{\partial^2\varphi}{\partial x_2 \partial x_3}, \\ (\mathrm{curl}\vec{J})_2 &= 2\Delta^2(K_L - K_T)\frac{\partial^2\varphi}{\partial x_1 \partial x_3}, \end{aligned}\right\} \tag{18}$$
$$(\mathrm{curl}\vec{J})_3 = 0.$$

Thus the anisotropic superfluid density has as a consequence the fact that one can have a distribution of vorticity from the first term of (15) without singularities in the order parameter.

By adding terms to the free energy density (8) which are proportional to total derivatives, one can generate through the prescription (10) an additional current density $\vec{J} \propto \mathrm{curl}\vec{A}_p^* \times \vec{A}_p$. It is, however, a consequence of the boundary condition derived in Sec. III ($\vec{A}_p$ parallel to walls) that there is no net current of this form through any physical channel.

II. LINEARIZED-GAP EQUATION

A. Weak-coupling limit

In this chapter we shall show how the nonhomogeneous states of a p-state Fermi superfluid, discussed in Sec. I, may be obtained from microscopic theory, and we shall calculate the two coherence lengths $\xi_L(T)$ and $\xi_T(T)$. In addition, the theory developed here will allow us in Sec. III to study situations in which translational invariance is broken by boundaries.

To begin with, we consider a BCS-type pairing theory. In the next subsection we give arguments which suggest that the inclusion of higher order effects, essential to explaining the stability of an anisotropic state,[6] does not change the essence of the results obtained here.

Our starting point is the linearized self-consistency equation for the pairing amplitude $\Delta_{\alpha\beta}(\vec{k}, \vec{q})$. Using Gor'kov's formulation[10] of the pairing theory, one obtains in the momentum representation

$$\begin{aligned} \Delta_{\alpha\beta}(\vec{k}, \vec{q}) = T\sum_n \sum_{\vec{k}', \vec{k}'', \vec{q}'} & \\ \times V(\vec{k}, \vec{k}')\, G^0_{\omega_n}(\vec{k}' + \tfrac{1}{2}\vec{q}\,, \vec{k}'' + \tfrac{1}{2}\vec{q}') & \\ \times G^0_{-\omega_n}(-\vec{k}' + \tfrac{1}{2}\vec{q}, -\vec{k}'' + \tfrac{1}{2}\vec{q}')\Delta_{\alpha\beta}(\vec{k}'', \vec{q}'). & \end{aligned} \tag{19}$$

In the weak-coupling approximation for p-state superfluids $V(\vec{k}, \vec{k}')$ is approximated by a point interaction of p symmetry[11]

$$V(\vec{k}, \vec{k}') = \frac{3g}{k_F^2}\,\vec{k}\cdot\vec{k}'. \tag{20}$$

$G^0_{\omega_n}(\vec{k}, \vec{k}')$ are the propagators for noninteracting fermions moving in a potential determined by the walls of the container; ω_n are the Matsubara frequencies $[\omega_n = (2n-1)\pi T, n = 0, \pm 1, \ldots]$.

Using definition (1) of the order parameters A_{pi}, and going over to the spatial representation, one obtains

$$A_{pi}(\vec{R}) = \int d^3R'\, K_{ij}(\vec{R}, \vec{R}')A_{pj}(\vec{R}'), \tag{21}$$

where

$$K_{ij}(\vec{R},\vec{R}') = 3gT\sum_n \lim_{\vec{R}_1\to\vec{R},\vec{R}_1'\to\vec{R}'} \frac{i}{4k_F^2}\left[\left(\frac{\partial}{\partial R_i}-\frac{\partial}{\partial R_{1i}}\right)\left(\frac{\partial}{\partial R_j'}-\frac{\partial}{\partial R_{1j}'}\right) G^0_{\omega_n}(\vec{R},\vec{R}')\, G^0_{-\omega_n}(\vec{R}_1,\vec{R}_1')\right]. \tag{22}$$

We shall now transform K_{ij} into a correlation function for fermions in the normal state. For this purpose we use the Lehmann representation of the propagators,

$$G^0_{\omega_n}(\vec{R}_1,\vec{R}_2) = \sum_\nu \frac{1}{i\omega_n-\epsilon_\nu} U_\nu^*(\vec{R})U_\nu(\vec{R}'). \tag{23}$$

$U_\nu(\vec{R})$ are the one-particle eigenfunctions defined by

$$\left(\frac{p^2}{2M}-E_F+V(\vec{R})\right)U_\nu(\vec{R}) = \epsilon_\nu U_\nu(\vec{R}). \tag{24}$$

The potential $V(\vec{R})$ contains all the effect of the walls. Since the Hamiltonian in Eq. (24) is invariant under time reversal, the $U_\nu(\vec{R})$ can be chosen to be real. Using this property K_{ij} can be written in terms of matrix elements of the current-density operators $j(\vec{R})$:

$$K_{ij}(\vec{R},\vec{R}') = 6gT\sum_{n>0}\sum_{\nu\mu}\frac{1}{i\omega_n-\epsilon_\nu}\,\frac{1}{-i\omega_n-\epsilon_\mu} \times\frac{M^2}{k_F^2}\langle\nu|j_i(\vec{R})|\mu\rangle\langle\mu|j_j(\vec{R}')|\nu\rangle. \tag{25}$$

The identity (for $\omega_n>0$)

$$\frac{1}{(i\omega_n-\epsilon_\nu)(-i\omega_n-\epsilon_\mu)} = i\int_0^\infty dt\,\frac{e^{-(|\omega_n|-i\epsilon_\nu)t}}{i\omega_n-\epsilon_\nu}\,e^{i(\epsilon_\mu-\epsilon_\nu)t} \tag{26}$$

allows one to rewrite Eq. (26) in the form

$$K_{ij}(\vec{R},\vec{R}') = 3gT\sum_n\int_0^\infty dt\int d\epsilon\, i\,\frac{e^{-(|\omega_n|-i\epsilon)t}}{i\omega_n-\epsilon} \times\frac{M^2}{k_F^2}\sum_\nu\langle\nu|j_i(\vec{R},0)j_j(\vec{R}',t)|\nu\rangle\delta(\epsilon-\epsilon_\nu). \tag{27}$$

Equation (27) is still exact within the framework of a weak coupling theory. It should be compared with the equivalent formula in an s-type superfluid.[12] The only (but significant) difference is that the *current densities* in Eq. (27) are replaced in the s-wave case by *particle densities*. Our subsequent approximations are analogous to those used in the s-wave problem, and may be justified in the same way.[12] The two essential steps are (a) in the relevant energy range $\sum_\nu\langle\nu|jj|\nu\rangle\delta(\epsilon-\epsilon_\nu)$ can be approximated by $N(0)\langle jj\rangle_{\epsilon=0}$, where $N(0)$ is the density of states at the Fermi energy and $\langle jj\rangle_{\epsilon=0}$ represents the current-current correlation function averaged over states at the Fermi energy. (b) This averaged correlation function can be replaced by the classical current-current correlation function calculated for a microcanonical ensemble of energy E_F. One then obtains the final result

$$K_{ij}(\vec{R},\vec{R}') = 6\pi N(0)gT\sum_n\int_0^\infty dt\, e^{-2|\omega_n|t} \times\frac{M^2}{k_F^2}\langle j_i(\vec{R},0)j_j(\vec{R}',t)\rangle_{\text{classical}}. \tag{28}$$

The correlation function $\langle jj\rangle$ in Eq. (28) is closely related to the nonlocal conductance used to discuss the skin effect in normal metals.

An evaluation of the correlation function in Eq. (28) for a bulk system leads to the result

$$K_{ij}(\vec{R},\vec{R}') = \frac{3N(0)gT}{2v_F}\sum_n\frac{(\vec{R}-\vec{R}')_i(\vec{R}-\vec{R}')_j}{|\vec{R}-\vec{R}'|^4} \times\exp\left(-\frac{2|\omega_n||\vec{R}-\vec{R}'|}{v_F}\right). \tag{29}$$

Substituting this form in (21) and developing the order parameter on the right-hand side around the point $\vec{R}$, one then easily obtains in leading order

$$\left(\ln\frac{T_c}{T}\right)A_{pi}(\vec{R}) + \tfrac{3}{5}\xi_s^2(\delta_{ij}\nabla^2+2\nabla_i\nabla_j)A_{pj}(\vec{R}) = 0. \tag{30}$$

Above, T_c is the transition temperature ($T_c=\omega_0\exp\{-[N(0)g]^{-1}\}$), and ξ_s is the quantity which occurs in the s-wave theory,[10] namely $\xi_s^2 = 7v_F^2\zeta(3)/48\pi^2T_c^2$. Comparing (30) with the equation one obtains by requiring that the phenomenological free energy Eq. (5) plus Eq. (8) be stationary with respect to variations of A_{pi}^*, one concludes that in the model of the present section

$$K_T = \tfrac{3}{5}\xi_s^2, \quad K_L - K_T = \tfrac{6}{5}\xi_s^2. \tag{31}$$

Thus the coherence lengths ξ_T and ξ_L defined below Eq. (8) are given by

$$\xi_L^2 = 3\xi_T^2 = \tfrac{9}{5}\xi_s^2(T_c/T_c - T).$$

(Recently Vuorio has obtained the result $\xi_L^2=3\xi_T^2$ by another method.[13])

B. Fermi-liquid effects

We shall now give a brief justification of the simple pairing model used in the rest of this paper. The arguments we give are very similar to those employed in the theory of superconduc-

tivity (see, e.g., McMillan and Rowell[14]).

Consider the linearized gap equation for a translationally invariant system in its general form

$$\Delta_{\alpha\beta}(\vec{k},\omega_n) = T\sum_{\omega_n'}\sum_{\vec{k}',\vec{q}} I_{\alpha\beta\gamma\delta}(\vec{k},\vec{k}',\vec{q};\omega_n,\omega_n') \times G_{\omega_n'}(\vec{k}'+\tfrac{1}{2}\vec{q}) \times G_{-\omega_n'}(-\vec{k}'+\tfrac{1}{2}\vec{q})\Delta_{\gamma\delta}(\vec{k}',\omega_n'), \quad (32)$$

where I represents the irreducible particle-particle vertex and the G's are the exact one-particle propagators.

That Eq. (32) is equivalent to the simplified form (19) follows from the following observations. The smallest momentum and frequency scales which enter I and the self-energy parts Σ (contained in the G's) are set by k_{SF} and ω_{SF}, the characteristic paramagnon wave number and frequency. These variations have to be compared with ξ_0^{-1} and T_c, which are characteristic of the peaked structure coming from the propagator product GG. Since in ^{3}He $k_{SF}\xi_0 \gg 1$ and $\omega_{SF}T_c \gg 1$, it is reasonable to make the replacement $I(\vec{k},\vec{k}',q;\omega_n,\omega_n') \approx I(k_F\hat{k},k_F\hat{k}',0;0,0)$ and at the same time to introduce a cutoff at $\omega_n,\omega_n' = \omega_{SF}$. Expanding I in terms of spherical harmonics and keeping only the p-wave part leads to the form of the pairing interaction used in Eq. (19). (We mention in passing that different angular momenta only decouple at $q=0$. The coupling to $l=3,5,\ldots$ pairs modifies the results only in orders higher than q^2 and is therefore neglected.)

The observations made above also provide a justification for making a quasiparticle approximation for the single-particle propagators in Eq. (32). One expands the self-energies occurring in the propagators about $k=k_F$ and $\omega_n=0$ as in Eqs. (7)–(9) of Ref. 14. The Fermi-liquid corrections are then contained in a wave-function renormalization constant Z and an effective mass correction M^*/M. The parameters $N(0)$, v_F, and g of Sec. II A have thus to be interpreted as renormalized quantities

$$N(0) \to N^*(0) = (M^*/M)N(0),$$
$$v_F \to v_F^* = (M/M^*)v_F, \quad (33)$$
$$g \to g^* = I/Z^2.$$

The current-current correlation function in Eq. (28) thus has to be understood as the correlation function of *noninteracting* quasiparticles moving with velocity v_F^*. We want to emphasize that this correlation function replaces a simple product of propagators and does not contain any vertex corrections to this product. Therefore, any type of collective mode need not and indeed must not be included in Eq. (28).

Some special remarks should be made with respect to boundary effects discussed in the following sections. In a small range of order $1/k_S$ near a boundary the self-energy and the vertex I are modified. This effect is not included in Eqs. (27) and (28). However, since we are not interested in calculating properties of this small surface layer, these effects will be absorbed into our phenomenological description of the scattering properties of *bulk* quasiparticles at a surface. It will be shown in the following sections that all interesting effects occur in a rather large region (of order ξ_0) at the surface. This justifies *a posteriori* the neglect of detailed structures very near the surface.

III. BOUNDARY EFFECTS

A. Specular reflection on solid wall

If it is clear from Eq. (28) that the kernel $K_{ij}(12)$ of the gap equation is very sensitive to the collisions of quasiparticles on the walls, when points 1 and 2 are within a distance ξ_0 from the wall. We shall now discuss this effect, for a planar wall lying in the (xy) plane, with the ^{3}He filling the upper-half space ($z>0$). We assume that all directions in the plane of the wall are equivalent, and also that the wall does not contain any chiral species.

We shall more or less follow the approach used long ago[15] to discuss a superconductor-insulator interface. As explained in this reference, it is enough for most boundary effects to consider a restricted problem, where (i) the order parameter depends only on z; (ii) the temperature T is set exactly equal to T_c. The most general solution for the A_{pi} is then easily shown to be asymptotically linear in z

$$A_{pi} = \text{const}(b_i + z) \quad (z \gg \xi_0). \quad (34)$$

The aim of the more detailed calculations to be carried out near $z=0$ is to find the correct value of the extrapolation length b_i.

Let us consider first the case of a wall which is an ideal mirror (specular reflection). Then the kernel $K_{ij}(12)$ (for two points 1 and 2 which are on the helium side) may be split into two parts: a direct part K^d coming from particles traveling on a straight line from 2 to 1, and a reflected part K^r coming from particles which start from 2, hit the wall, and then appear to come from the image point $\bar{2}$. Upon reflection, the velocity components v_x and v_y are unchanged, while v_z is reversed. This implies the following properties

$$K^r_{xx}(12) = K^d_{xx}(1\bar{2}),$$
$$K^r_{yy}(12) = K^d_{yy}(1\bar{2}), \tag{35}$$
$$K^r_{zz}(12) = -K^d_{zz}(1\bar{2}).$$

(The nondiagonal components K_{ij} do not enter in the final results when A_{pi} depends only on z.)

We shall discuss now more specifically the properties of a "transverse" component A_{px} (or A_{py}). The equation for A_{px} reads

$$A_{px}(1) = \int_{z_2>0} d2\,[K^d_{xx}(12) + K^d_{xx}(1\bar{2})]A_{px}(2). \tag{36}$$

We can transform this by extending the definition of A_{px} to the nonphysical region $z<0$. We choose the following convention:

$$A_{px}(\bar{2}) = A_{px}(2). \tag{37}$$

Then Eq. (36) takes a form which does not differ from the bulk

$$A_{px}(1) = \int_{\text{all } z_2} d2\,K^d_{xx}(12)A_{px}(2). \tag{38}$$

The solution of this equation which is compatible with the parity requirement (37) is simply

$$A_{px} = \text{const}. \tag{39}$$

Comparing with (34) we see that the extrapolation length b_x (or b_y) is infinite: The *transverse* components are *not affected by a specular boundary* (Fig. 1).

Consider now the longitudinal component A_{pz}, which satisfies the equation

$$A_{pz}(1) = \int_{a_2>0} d2\,[K^d_{zz}(12) - K^d_{zz}(1\bar{2})]A_{pz}(2). \tag{40}$$

Here the natural continuation is

$$A_{pz}(\bar{2}) = -A_{pz}(2), \tag{41}$$

and this again leads to a translationally invariant equation

$$A_{pz}(1) = \int_{\text{all } z_2} d2\,K^d_{zz}(12)A_{pz}(2). \tag{42}$$

Let us recall at this moment that we are just at T_c. This implies that

$$\int K^d_{zz}(12)\,d2 = 1.$$

Then, as already mentioned, the most general solution to an equation of the form (42), with K^d depending on the relative distance $(2-1)$, is simply a linear function of the coordinates (here, of z). The odd-parity condition (41) imposes in fact

$$A_{pz} = \text{const}\, z.$$

This coincides, as it should, with Eq. (34) at large distances ($z > \xi_0$), and we see that the extrapolation length $b_z = 0$. We conclude that the longitudinal components of the order parameter are strongly depressed by the presence of the wall. The aspect of $A_{pz}(z)$ at a temperature slightly below T_c is shown on Fig. (1).

Turning now to the more specific case of an axial phase, where A_{pi} has the structure described in Eq. (8), we see that the vectors $\vec{\Delta}'$ and $\vec{\Delta}''$ will tend to lie in the plane of the wall (to avoid the depletion associated with A_{pz}). Thus the vector $\vec{l}$ will be anchored *normal to the wall*.

Note, on the other hand, that the spin quantization axis $\vec{V}$ is *not* anchored to any particular direction (since the spin indices played no role in our discussion). This will be true for all walls where spin effects are not important (with specular or diffuse scattering). The resulting boundary condition is one of zero surface torque on $\vec{V}$,

$$\frac{d\vec{V}}{dz} = 0.$$

On the other hand, if the wall contained magnetic spins (such as in the case of cerium manganese nitrate), very different conditions would prevail.

B. Diffuse reflection

We consider again a planar wall at $z=0$, which confines the superfluid to $z>0$. Since the boundary condition corresponding to diffuse reflection is somewhat more complex than that due to specular reflection, we shall only derive the two main effects caused by such a wall. These are (a) the longitudinal component (A_{pz}) of the order parameter is again suppressed by the wall, and (b) in contrast to a specularly reflecting wall, the amplitudes of the favored components A_{px}, A_{py} are also

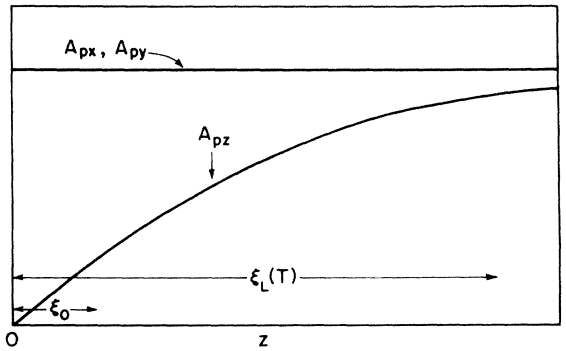

FIG. 1. Schematic drawing of transverse (A_{px}, A_{py}) and longitudinal (A_{pz}) components of the order parameter near a specularly reflecting wall.

reduced in the neighborhood of a diffusely reflecting wall. Owing to this effect, we predict a reduction of the transition temperature in small geometries (size effect).

In the case of diffuse reflection, the contribution to the kernel $K_{ij}(\vec{R},\vec{R}')$ coming from reflected paths can be obtained by inserting the classical correlation function

$$\langle j_i(\vec{R},0)j_j(\vec{R}',t)\rangle_{\text{reflected}} = \frac{1}{N(0)M^2}\int_{p_z<0}\frac{d^3p}{(2\pi)^3}\int_{p_z'>0}d^3p'\,\delta\left(\frac{p'^2}{2M}-E_F\right)\delta^3\left(\vec{R}'-\vec{R}-\vec{p}\frac{R_z}{|p_z'|}+\vec{p}'\frac{R_z}{|p_z'|}-\vec{p}'\frac{t}{M}\right)$$
$$\times\left(\frac{p_z'}{\pi p'^3}\right)p_i p_j'\,\delta(|\vec{p}|-|\vec{p}'|) \tag{43}$$

into Eq. (28). This has to be added to the contribution from the direct path which is just the bulk kernel given in Eq. (29). A useful function for a qualitative discussion is the space integral

$$C_{ij}(\vec{R}) = \int_{z'>0} d^3R'\,K_{ij}(\vec{R},\vec{R}'). \tag{44}$$

$C_{ij}(\vec{R})$ is the result of a first iteration of the gap equation if one starts with a constant order parameter. In the present case one obtains

$$C_{ij}(\vec{R}) = 0 \quad (i\neq j),$$
$$C_{zz}(\vec{R}) = N(0)gT\sum_n\left\{\frac{\pi}{|\omega_n|}\left[1-\exp\left(-\frac{2|\omega_n|R_z}{v_F}\right)\right]+\pi\frac{R_z}{v_F}\left[E_2\left(\frac{2|\omega_n|R_z}{v_F}\right)+E_3\left(\frac{2|\omega_n|R_z}{v_F}\right)\right]\right\},$$
$$C_{xx}(\vec{R}) = C_{yy}(\vec{R}) = N(0)gT\sum_n\left\{\frac{\pi}{|\omega_n|}\left[1-\tfrac{1}{2}\exp\left(-\frac{2|\omega_n|R_z}{v_F}\right)\right]+\pi\frac{R_z}{v_F}\left[\tfrac{3}{2}E_1\left(\frac{2|\omega_n|R_z}{v_F}\right)-\tfrac{1}{2}E_3\left(\frac{2|\omega_n|R_z}{v_F}\right)\right]\right\}, \tag{45}$$

where $E_i(R_z)$ are the exponential integrals. $C_{zz}(R_z)$ and $C_{xx}(R_z)[C_{yy}(R_z)]$ are shown in Fig. 2. Two conclusions can be drawn from these results: (a) In neither of the cases is a constant order parameter a solution of the gap equation. The longitudinal and the transverse order parameters are both strongly modified by the diffuse boundary. (b) The longitudinal component is more strongly reduced by the boundary than the transverse components. It in fact goes to zero at the surface.

The vanishing of the longitudinal component at a boundary is a general feature (not restricted to the first iteration) which occurs for any type and shape of boundary. It can be derived from the fact that the correlation function

$$\langle v_\perp(\vec{R}_b)v_j(\vec{R},t)\rangle$$

vanishes at a boundary point $\vec{R}_b$ if $v_\perp$ is the component of the velocity perpendicular to the surface.

Returning now to the nonlinear Landau-Ginsburg equations for T slightly smaller than T_c, one can show that the reduction of the order parameters for the longitudinal and transverse components heals in the lengths $\xi_L(T)$ and $\xi_T(T)$, respectively. The loss of condensation energy is thus of order

$$\Delta F \simeq (F_n - F_s)\,\xi_{L,T}(T),$$

where $F_n - F_s$ is the bulk condensation energy. Since the longitudinal component A_{pz} is more strongly depressed at the surface and in addition has a longer healing length $\xi_L(T)$ we conclude that this energetically unfavored component is suppressed by the wall. For the axial state this again means that $\vec{l}$ is anchored *normal to the wall*.

Let us now discuss in more detail the behavior of the transverse components A_{px}, A_{py} which are selected out by the wall. The physically most

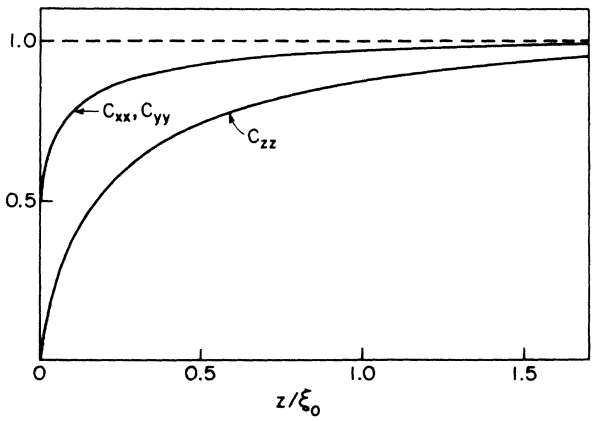

FIG. 2. Result of the first iteration of the gap equation for diffuse reflection as given in Eq. (45).

important information is contained in the extrapolation length b_T defined in Eq. (34). b_T can be obtained from the solution of the linearized gap equation at T_c (we omit for brevity the indices of the order parameters):

$$A(R_z) = \int_0^\infty dR_z' K(R_z, R_z') A(R_z'). \tag{46}$$

For the transverse components, $K(R_z, R_z')$ has the form

$$K(R_z, R_z') = \frac{3N(0)g\pi}{2v_F} T\sum_n \left[E_1\left(\frac{2|\omega_n||R_z - R_z'|}{v_F}\right) - E_3\left(\frac{2|\omega_n||R_z - R_z'|}{v_F}\right)\right]. \tag{47}$$

Equation (46) can be solved by the Wiener-Hopf method. One obtains for the Fourier transform $\tilde{A}(q)$ of $A(R_z)$,

$$\tilde{A}(q) = \text{const}\,\frac{q\xi_0 - i}{q^2} \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dk \frac{\ln[(k^2+1)\Phi(k)]}{k - q\xi_0}\right) \quad (\text{Im}q < 0), \tag{48}$$

where

$$\Phi(k) = \frac{1}{k^2}\left\{1 - \frac{3}{2} T\sum_n \frac{1}{|\omega_n|}\left(\frac{2n-1}{k}\right)^2\left[\left(\frac{2n-1}{k} + \frac{k}{2n-1}\right)\tan^{-1}\frac{k}{2n-1} - 1\right] \Big/ T\sum_n \frac{1}{|\omega_n|}\right\}, \tag{49}$$

where the sums are cut off at $|\omega_n| = \omega_0$.

From the small q behavior of the solution $\tilde{A}(q)$, one finds the extrapolation length b_T:

$$b_T = -\xi_0 \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dk}{k}\frac{\Phi'(k)}{\Phi(k)}. \tag{50}$$

A numerical evaluation of this integral gives

$$b_T = 0.54\xi_0 \quad (\xi_0 = v_F/2\pi T_c).$$

This rather small extrapolation length expresses the strong influence of the diffuse boundary even on the favored components of the order parameter. This leads us to expect observable size effects in thin geometries with such type of boundaries. A detailed calculation of these size effects will be given in a subsequent paper.

C. Junctions and channels

(a) An *ideal* tunneling junction, made of an ultrathin, flat, solid barrier is probably unfeasible with helium atoms as the tunneling species. However, it is of some conceptual interest to look at the structure of the Josephson coupling U_{12} which would be expected for such a case, with two vessels of order parameters $A_{pi}^{(1)}$ $A_{pi}^{(2)}$ coupled by the junction. Assuming that the tunneling layer is a achiral and isotropic in its plane, we would have a coupling of the form

$$U_{12} = -\tfrac{1}{2}G_T(A_{px}^{(1)}A_{px}^{(2)*} + A_{py}^{(1)}A_{py}^{(2)*}) - \tfrac{1}{2}G_L A_{pz}^{(1)}A_{pz}^{(2)*} + \text{c.c.}, \tag{52}$$

where z is always the normal to the walls. If we have axial phases in each vessel, each with $\vec{l}$ normal to the walls, and if $\vec{l}_2$ is *parallel* to $\vec{l}_1$, this gives a conventional Josephson energy

$$U_{12} = -2q_T\Delta^2\cos(\varphi_1 - \varphi_2). \tag{53}$$

But if the $\vec{l}$ values on both sides are *antiparallel*, we find $U_{12} = 0$! Thus the function would probe not only the relative phase $\varphi_1 - \varphi_2$, but also (to some extent) the relative orientation of the angular momenta. It is of interest to see if similar properties are also expected for some more practical weak links; we shall now discuss two types of such links, a thin channel and a "large" hole.

(b) A long, thin, *channel* of well-defined radius R, connecting two compartments, can be prepared by etching of a particle track the diameter $2R$ being adjustable between 100 Å and a few microns.[16] With mica as the substrate, the channel walls show noncircular cross sections and well-defined crystalline planes, possibly leading to a good specular reflection. For such a channel, depending on the temperature, we expect different types of superfluid ordering:

(i) Very near T_c the boundary conditions will accept only one orbital component A_{pz} (z being the channel axis); in this regime the angular momentum is quenched.

(ii) At low temperatures [i.e., when $\xi_T(T) \lesssim R$] the angular momentum should become unquenched. The vector $\vec{l}$ will tend to be radial (in agreement with the boundary conditions) and there will be a singular line (disgyration) on the channel axis (Fig. 3).

Can we use this sort of channel as a Josephson junction connecting two vessels 1 and 2? In regime (i) the effective coupling between 1 and 2 will be independent of the relative orientations of $\vec{l}_1$ and $\vec{l}_2$ (because $\vec{\Delta}$ has only one component in the chan-

nel). Thus regime (i) appears less interesting than the ideal junction described by Eq. (52).

(c) Regime (ii) would be obtained most easily with a "large" (10 μm) *hole* inside a wall, as shown in Fig. 4. Here, if the vectors $\vec{l}_1$ and $\vec{l}_2$ in both vessels are antiparallel [Fig. 4(a)] we expect to have a singular point (or a short portion of disgyration) at the *center* of the hole. On the other hand, if $\vec{l}_1$ and $\vec{l}_2$ are parallel [Fig. 4(b)] we shall have a disgyration running on (or close to) the *equatorial line*. This difference in configuration may be significant if we study superfluid flows through the hole which involve the nucleation of vortex loops. Because the vortex cores are large [$\sim\xi(T)\sim 1000$ Å] in ^{3}He, the vortices will not necessarily nucleate on point defects at the walls: They may in fact nucleate on the *disgyrations*, and the nucleation barriers could be quite different for Fig. 4(a) versus Fig. 4(b).

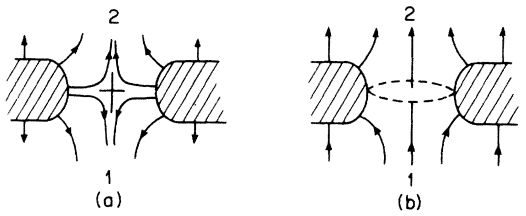

FIG. 4. Configurations of the $\vec{l}$ vector around a "large hole" separating two vessels 1 and 2: (a) $\vec{l}_1$ antiparallel to $\vec{l}_2$, (b) $\vec{l}_1$ parallel to $\vec{l}_2$.

IV. CONCLUDING REMARKS

We conclude with a brief discussion of various experimental effects which arise as a result of the boundary effects discussed in Sec. III.

(a) A thin slab of ^{3}He between two walls should tend to have its orbital momentum $\vec{l}$ normal to the walls. This does not grant that the sample will be single domain ($\vec{l}$ might be upwards in some part of the sample, and downwards in others, the matching being realized through a system of disgyrations). But, when employed in conjunction with other means,[3] this geometry may be of some help to reach a uniform alignment of $\vec{l}$.

(b) Conversely, if we do succeed (in the future) to have single domains in thin samples, it will be interesting to apply conflicting fields to this structure, and to detect the analog of the "Frederick's transitions" familiar in nematic liquid crystals.[17] This will be discussed in a separate publication.

(c) The measurements on *fourth sound* (oscillatory motions of the superfluid inside a porous medium), which traps the normal fluid component, are also very dependent on the boundary conditions at the solid-fluid interface, and on the detailed shape of the pores: (i) If the pores can be described as a net work of long capillaries, we shall probably find, inside each capillary, the arrangement shown in Fig. 3, with $\vec{l}$ normal to the walls. Then the superfluid current will flow normal to $\vec{l}$, and the fourth sound velocity will depend only on the quantity $\rho_{s\perp}$ [defined in Eq. (16)]. (ii) If the solid component is obtained by a regular stacking of spherical grains, all of the same size, the $\vec{l}$ field between the grains will be rather complex, and the supercurrent $\vec{J}$ will in general be oblique with respect to $\vec{l}$. The local relation between $\vec{J}$ and the "gradient" of the phase gradφ will thus involve a space dependent matrix ρ_s, and the flow regime is rather complex. (iii) If the grains have some randomness in their size or in their stacking, the situation will be still more complicated. However, after performing a coarse grained average over regions much larger than the pore size, it will be possible to define an effective superfluid density $\rho_{s\,\mathrm{eff}}$ by

$$\bar{J} = \rho_{s\,\mathrm{eff}}\,\overline{\mathrm{grad}\varphi}$$

(where f is the volume fraction of ^{3}He). Because $\rho_{s\,\mathrm{eff}}$ is well defined, it should still be possible to observe a long-wavelength fourth sound. But it will be extremely hard to relate $\rho_{s\,\mathrm{eff}}$ to the microscopic parameters $\rho_{s\perp}$ and $\rho_{s\parallel}$. (iv) A special mention should be given to the case where the grains carry electronic spins, for instance, when they are made with the traditional cooling agent cerium magnesium nitrate. Here, our discussion of boundary effect should be augmented to cover spin-dependent potentials. We hope to return to this question in a later publication.

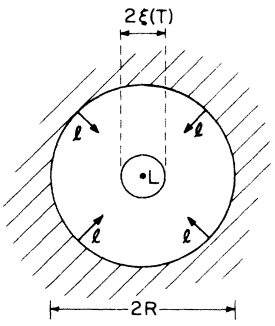

FIG. 3. Radial disgyration in a capillary with radius $R > \xi(T)$. The core of the disgyration L occupies a region of radius $\sim\xi(T)$ around the axis.

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[1]D. D. Osheroff, W. J. Gully, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. 29, 920 (1972).

[2]A. J. Leggett, Phys. Rev. Lett. 29, 1227 (1972).

[3]P. G. deGennes, Phys. Lett. 44A, 271 (1973); in *Proceedings of the Nobel Symposium 24*, 1973 (Academic Press, New York, 1974).

[4]N. D. Mermin and G. Stare, Phys. Rev. Lett. 30, 1135 (1973).

[5]V. Ambegaokar and N. D. Mermin, Phys. Rev. Lett. 30, 81 (1973); in *Proceedings of the Nobel Symposium 24*, 1973 (Academic Press, New York, 1974).

[6]P. W. Anderson and W. F. Brinkman, Phys. Rev. Lett. 30, 1108 (1973); W. F. Brinkman and J. Serene (unpublished).

[7]A. J. Leggett, Phys. Rev. Lett. 31, 352 (1973); and (unpublished).

[8]P. W. Anderson and P. Morel, Phys. Rev. 123, 1911 (1961).

[9]C. M. Varma and N. R. Werthamer, Phys. Rev. A 9, 1465 (1974).

[10]L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 34, 735 (1958) [Sov. Phys.—JETP 7, 505 (1958)].

[11]R. Balian and N. R. Werthamer, Phys. Rev. 131, 1553 (1963).

[12]For details of similar calculations, see, for instance, P. G. deGennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966) and G. Lüders and K. D. Usadel, *The Method of the Correlation Function in Superconductivity Theory* (Springer, Berlin, 1971).

[13]M. Vuorio (unpublished).

[14]W. L. McMillan and J. M. Rowell, in *Superconductivity*, edited by R. Parks (Marcel Dekker, New York, 1969), Vol. 1, p. 571.

[15]P. G. deGennes, Rev. Mod. Phys. 36, 225 (1964).

[16]R. L. Fleisher, P. B. Price, and R. M. Walker, Rev. Sci. Instrum. 34, 510 (1963).

[17]See, for instance, P. G. deGennes, *The Physics of Liquid Crystals* (Oxford U. P., London, to be published).