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Stability boundaries for two-mode lasers in the strong-coupling limit*

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We present here, for the first time, the criteria for determining what field fluctuations are *required* to cause a bistable two-mode laser to flip from one mode to the other. Results (for a particular two-mode model) generated numerically are presented. Extensions of the method to more-refined two-mode models and multimode cases are discussed.

Mode hopping in multimode lasers is a phenomenon which has been examined theoretically[1-3] and observed experimentally[4-6] by many investigators. The mode-hopping effect, for example in the context of Lamb's rate equations[1] for the modal electric field amplitudes, is fundamentally a *nonlinear* effect resulting from self-saturation of each mode and cross-saturation between modes. An interesting application of this effect would be the coding of information[6] on the transmitted beam by controlled selection of various mode configurations.

According to Lamb's theory of optical masers,[1] bistability may arise (e.g., for two-mode operation) in the strong-coupling limit ($\theta_{12}\theta_{21} > \beta_1\beta_2$); one or the other mode is completely suppressed in steady-state operation. The past history determines in this case which of the modes is ultimately suppressed and which attains a state of stable (finite) oscillation. It was noted by Lamb[1] that many gas-laser amplifiers are inhomogeneously broadened and, hence, the modes are usually (but not always!) weakly coupled ($\beta_1\beta_2 > \theta_{12}\theta_{21}$). However, it is possible to construct He-Ne lasers, for example, which display strong coupling.[3,6] We address ourselves to finding necessary and sufficient conditions for the two-mode strong-coupled system to evolve into each specific steady-state case. Such a consideration will be of utmost importance in the design of bistable lasers as coded information carriers. Our analysis derives from the theory of differential equations of Lyapunov's theory of stability of motion.[7,8] We treat here in detail the case of a particular two-mode bistable system.[1] Extensions of the theory to more elaborate two-mode models and multimode lasers are discussed.

Following Lamb[1] we consider the two-mode model in terms of the field intensities $X \equiv E_1^2$ and $Y \equiv E_2^2$ where

$$\dot{X} = 2X(\alpha_1 - \beta_1 X - \theta_{12} Y),$$
$$\dot{Y} = 2Y(\alpha_2 - \theta_{21} X - \beta_2 Y), \tag{1}$$

the α's are the unsaturated gain coefficients, and the β's and θ's are the respective self- and cross-saturation parameters.[1,4,6,9] We note that this model is restricted[6] to the case where

$$\left|\frac{\text{field intensity}}{\text{saturation field}}\right| \ll 1 .$$

For notational convenience we define[10] a coupling parameter

$$\zeta \equiv \beta_1\beta_2 - \theta_{12}\theta_{21} .$$

In the strong-coupling limit ($\zeta < 0$), the model (1) admits three nonvanishing critical points: (i) $X = \alpha_1/\beta_1$, $Y = 0$; (ii) $X = 0$, $Y = \alpha_2/\beta_2$; (iii) $X = (\alpha_1\beta_2 - \alpha_2\theta_{12})/\zeta$, $Y = (\alpha_2\beta_1 - \alpha_1\theta_{21})/\zeta$. The first two critical points are asymptotically stable (hence the bistability) and the third is unstable.[11] We are thus faced with the problem of determining the stability boundary in the positive XY quadrant past which *fluctuations* from one of the stable critical points will *relax* to the other. The problem is transformed by defining new variables:

$$x \equiv X - \alpha_1/\beta_1 \quad \text{and} \quad y \equiv Y .$$

In these new variables, Eqs. (1) become

$$\dot{x} = -2\alpha_1 x - 2\theta_{12} y - 2\alpha_1 x^2 - 2\theta_{12} xy ,$$
$$\dot{y} = 2(\alpha_2 - \theta_{21})y - 2\theta_{21} xy - 2\beta_2 y^2 . \tag{2}$$

To restate the problem then, which fluctuations $\delta x (> -\alpha_1/\beta_1)$ and $\delta y (>0)$ from the new origin of (2) will relax back to this origin, and which will evolve to the other stable critical point?

The solution is defined with the aid of an addendum to Lyapunov's stability theory provided by Zubov[12,13] requiring the solution of

$$\dot{x}(x,y)\frac{\partial v(x,y)}{\partial x}+\dot{y}(x,y)\frac{\partial v(x,y)}{\partial y} = -\phi(x,y)[1-v(x,y)], \quad (3)$$

where $\phi(x,y)$ is an arbitrary, positive definite, continuous, scalar function. The set $\{(x,y)|v(x,y)=1\}$ defines the desired boundary. As the required integrals in solving (3) for the case of (2) cannot be obtained analytically, one must resort to numerical solutions. Power-series solutions of (3) are straightforward but converge slowly.[14] Exact (numerical) solutions of (3) for two-dimensional systems may be quickly obtained by an algorithm developed recently by one of the authors.[8] The basis for the algorithm is that the stability boundary is piecewise an integral curve[15] of the system under investigation.[16] Segments of the boundary may thus be generated by numerical integration once a *point* of the boundary is known. A point of the boundary may be obtained by a number of methods[8]; in the case of (2) such a starting point is found by inspection to be the third (unstable) critical point: $x=\theta_{12}(\alpha_1\theta_{21}-\alpha_2\beta_1)/\zeta\beta_1$, $y=(\alpha_2\beta_1-\alpha_1\theta_{21})/\zeta$. Figure 1 illustrates the resulting family of boundaries for (2) where α_1 is varied over $[0.6, 1.9]$ with the other parameters fixed:

$$\alpha_2=\beta_1=\beta_2=1 \quad \text{and} \quad \theta_{12}=\theta_{21}=2\,.$$

As the numerical method used here is completely general for two-dimensional systems, the above analysis may be immediately extended to more refined models than (1) such as the fifth-order expansion in electric field of Uehara and Shimoda,[17,9(a)] for understanding systems with higher field intensities than admitted in (1),[6,18] or for the similar models of Bambini and Burlamacchi.[2] The method is also easily extended to the "free-run"[3] three-mode case. In this case the three-dimensional boundary surface is generated via a *sequence* of two-dimensional cross sections by parametrizing one of the coordinates.[14] In like manner, *partial* boundary surfaces may be generated for even more complex systems. Another interesting application might be a similar analysis of *polarization* mode hopping as experimentally observed by Isenor.[5] The general theory may be further extended by considering the *inverse* problem whereby constraints on the required gain and saturation coefficients are determined such that a *given* stability

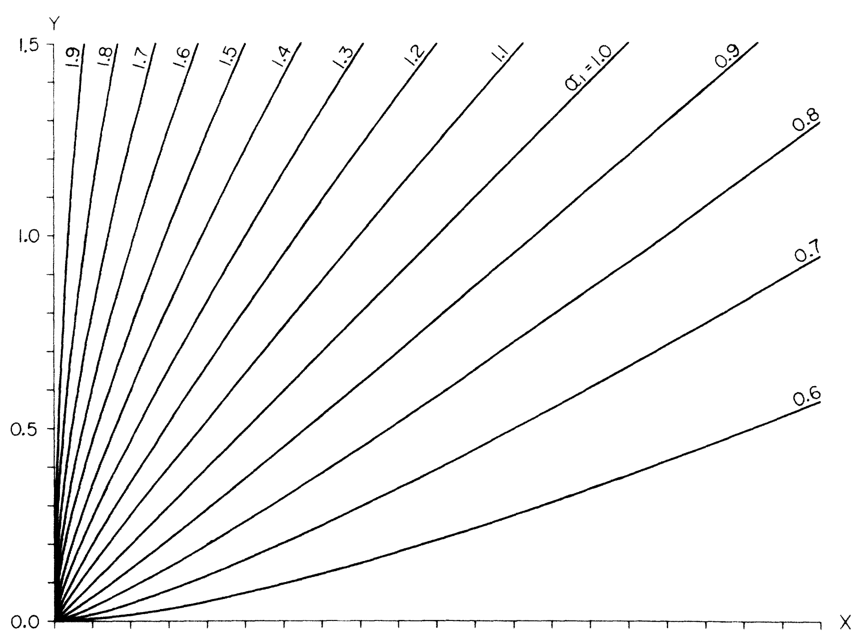

FIG. 1. Stability boundaries for the two-mode laser with strong coupling described by Lamb, Ref. 1, with $\alpha_2=\beta_1=\beta_2=1$ and $\theta_{12}=\theta_{21}=2$. Note that α_1 also describes the stable critical point $(\alpha_1, 0)$ along the $X=E_1^2$ axis while the other stable critical point (0, 1) does not vary with α_1. The case $\alpha_1=1$ corresponds to the symmetrical example illustrated by Lamb (Fig. 5 of Ref. 1). The concavity of the boundaries with respect to the symmetrical case illustrates an interesting perturbative effect on the symmetry induced by variations in unsaturated gain. It is apparent that as $\alpha_1\to 2$ or $\alpha_1\to 0.5$ we eventually obtained the situation where for all practical purposes only a *single* mode is allowed to operate. This is the typical behavior obtained for certain cases of weak coupling $(\theta_{12}\theta_{21}<\beta_1\beta_2)$.

boundary is obtained. Work on this latter problem is currently in progress[19] and may be particularly tractable in view of the experimental capability of *externally* modulating such systems demonstrated recently by Kobayashi *et al.*[20]

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