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Brownian motion of atomic systems: Fokker-Planck limit of the transport equation

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A comparison is made between the Fokker–Planck equation (FPE) and the transport equation (TE) solutions of the problem of the collisional relaxation of an atomic or molecular system undergoing Brownian motion. It is shown that, for times $\lesssim$ (inverse collision rate), the FPE does not provide an appropriate approximation to the TE if the velocity bandwidth of the initially prepared sample is less than the root-mean-square change in velocity per collision. Possible experimental verification of the theory using laser-pump sources to provide narrow initial-velocity distributions is discussed.

I. INTRODUCTION

The linearized Boltzmann or transport equation (TE) is commonly used to trace the time evolution of the velocity distribution function associated with a classical system of particles from some initial value to a final equilibrium value. The changes in the distribution function are caused by collisions of the particles with some reservoir of colliding particles, and it is implicitly assumed that the reservoir's distribution function is approximately unchanged by the collisions. This latter assumption enables one to linearize the Boltzmann equation and is valid, for example, in a case where the system particles are distinguishable from and much less numerous than the reservoir particles, as might be the case if the "system" consisted of excited-state atoms and the reservoir of ground-state atoms.

The dynamics of the collisions is usually complex enough to lead one to seek out approximate forms for the TE. In the case where small-angle scattering is the dominant collision mechanism ("weak" collisions), a Brownian-motion-type model can be adopted. In this limit, the TE is generally thought to go over into the Fokker-Planck equation (FPE), for which solutions are readily available.[1-3] The validity conditions and applicability of the transformation from the TE to the FPE, written symbolically as TE $\rightarrow$ FPE, remain somewhat obscure and are generally stated in mathematical rather than physical terms. In this paper, we shall examine in detail the transformation TE $\rightarrow$ FPE as applied to an atomic or molecular system in order to gain a better physical understanding of the range of validity of the transformation. In this manner, one will be in a position to judge whether or not the FPE is an appropriate equation to use to describe various experimental situations involving systems of particles undergoing Brownian motion. In addition, this paper is intended to lay the groundwork for future articles that will deal with more complex transport equations than the one to be considered here.

II. GENERAL REMARKS

If we are to examine the transformation TE $\rightarrow$ FPE, we must assume for the present discussion that our system is describable by a classical-type Brownian-motion transport equation. An atomic or molecular system can be characterized by a classical-type transport equation only if (a) the system remains in a given eigenstate for the entire experiment or (b) the collisional interaction is the same for all the internal states involved in the experiment. If the collisional interaction is state dependent (analogous to the interaction of a spin system with a Stern-Gerlach magnet), then it is impossible to ascribe a classical trajectory to the atom or molecule following a collision and, consequently, a quantum-mechanical transport equation is needed.[4] As far as the Brownian-motion assumption is concerned, its ultimate justification will usually rest on a comparison between theory and experiment.

Accepting the validity of the Brownian-motion TF, we can show that the transformation TE $\rightarrow$ FPE will be valid provided that a characteristic collisional change in velocity Δu is small compared to some important physical parameter of the problem. For example, in atomic radiation problems, TE $\rightarrow$ FPE might be valid if the acquired collision-induced change in the Doppler phase factor $k\Delta ut$ is much less than unity for all times t of interest ($\vec{k}$ is a radiation propagation vector). For the problem of particle diffusion in velocity space to be considered here, TE $\rightarrow$ FPE will be shown to be valid if Δu is much less than the velocity bandwidth of the initially prepared sample. One must expect significant differences between TE and FPE solutions if Δu does not satisfy the above requirements.

This subject matter takes on added importance with the recent use of laser-pump sources to excite very narrow velocity subgroups of atoms or molecules. In tracing the relaxation of these subgroups back to equilibrium, one may be able to experimentally compare the TE and FPE predictions. An experiment to test the theory will be proposed.

III. TE→FPE

The TE for the distribution function $\rho(v, t)$ of a spatially isotropic system is given by[1,3]

$$\frac{\partial\rho(v,t)}{\partial t} = -\Gamma(v)\rho(v,t) + \int W(v'\to v)\rho(v',t)dv', \tag{1}$$

where the kernel $W(v'\to v)$ is the probability per unit time for a particle to undergo the collisional velocity change $v'\to v$ and $\Gamma(v) = \int W(v\to v')dv'$ is the rate of collisions. For mathematical simplicity we have chosen a one-dimensional problem, but the results to be derived are quite general.

The transformation of Eq. (1) to the corresponding FPE in the Brownian-motion limit is well represented in the literature.[1,3] A simple way to obtain the result is to realize that, for Brownian motion, $W(v'\to v)$ is a highly peaked function about $v' = v$ (i.e., the collisional velocity changes are small). In that case, one can expand $\rho(v', t)$ in the integral of Eq. (1) about $v' = v$ to obtain

$$\frac{\partial\rho(v,t)}{\partial t} = -\Gamma(v)\rho(v,t) + \sum_0^\infty A_n(v)\frac{\partial^n\rho(v,t)}{\partial v^n}, \tag{2}$$

where

$$A_n(v) = \frac{1}{n!}\int W(v'\to v)(v'-v)^n dv'. \tag{3}$$

If one cuts off the sum at $n = 2$, one arrives at a slightly more general[5] form of the Fokker-Planck equation:

$$\frac{\partial\rho(v,t)}{\partial t} = [-\Gamma(v) + A_0(v)]\rho(v,t) + A_1(v)\frac{\partial\rho(v,t)}{\partial v} + A_2(v)\frac{\partial^2\rho(v,t)}{\partial v^2}. \tag{4}$$

Equation (4) will provide a good approximation to (2) if

$$A_n\frac{\partial^n\rho}{\partial v^n} \ll A_{n-2}\frac{\partial^{n-2}\rho}{\partial v^{n-2}} \tag{5}$$

for $n > 2$. Typically this requirement will be met if the velocity width of the kernel $W(v'\to v)$ is much less than the velocity width of the distribution function $\rho(v, t)$. To see this more clearly, let us choose a phenomenological collision kernel suggested by Keilson and Storer,[3]

$$W(v'\to v) = \Gamma[\pi(\Delta u)^2]^{-1/2}\exp[(v-\alpha v')^2/(\Delta u)^2], \tag{6}$$

where Γ is the (velocity-independent) rate of collisions and Δu is root 2 times the root-mean-square change in velocity per collision and is related to the most probable speed u of the thermal equilibrium distribution by

$$(\Delta u)^2 = (1-\alpha^2)u^2. \tag{7}$$

The kernel (6), while not derived from first principles, does seem to provide reasonable agreement with computer simulations of collisions[6] and also yields the Brownian-motion results for $\alpha\approx 1$ (corresponding to small Δu). In what follows we shall assume (6) is applicable with $\alpha\approx 1$ so that $1-\alpha^2 \approx 2(1-\alpha)$.

The moments A_n are easily calculated using (3) and (6) to be

$$A_n = \Gamma\frac{1\times 3\times 5\times\cdots\times(n-1)}{\alpha^{n+1}2^{n/2}n!}(\Delta u)^{2n}, \quad n \text{ even}$$
$$= \Gamma\frac{1\times 3\times 5\times\cdots\times(n-2)}{\alpha^{n+1}(n-1)!2^{(n-1)/2}}(1-\alpha)v(\Delta u)^{n-1}, \quad n \text{ odd} \tag{8}$$

where higher-order terms in $(1-\alpha)$ have been neglected. The validity condition (5) for the FPE, given a smooth distribution function $\rho(v, t)$ with width u_t at time t, becomes

$$(\Delta u/u_t)^2 \ll 1 \tag{9}$$

for all t, in agreement with the statement of Sec. II. Thus, the physical values of Δu and u_t will determine whether or not the FPE is applicable to the study of Brownian motion under consideration. Typical values for Δu and u_t will be discussed in Sec. VI, but it might be noted here that Δu is determined by the dynamics of the collisions.

If condition (9) holds, then the FPE [Eq. (4)] is valid and may be written (using the notation of Chandresekhar[2])

$$\frac{\partial\rho(v,t)}{\partial t} = \beta\rho(v,t) + \beta v\frac{\partial\rho(v,t)}{\partial v} + q\frac{\partial^2\rho(v,t)}{\partial v^2}, \tag{10}$$

where

$$\beta = \Gamma(1-\alpha) \tag{11a}$$

is the effective decay rate for the average velocity of the system and

$$q = \Gamma(\Delta u)^2/4 \tag{11b}$$

is the diffusion constant. The approximation $\alpha\approx 1$ is implicitly contained in Eqs. (10) and (11).

IV. SOLUTION OF FPE AND TE

We seek solutions of Eqs. (1) and (10) with the kernel $W(v' \to v)$ given by Eq. (6). For our initial condition, we take the Gaussian

$$\rho(v, 0) = (\pi u_0^2)^{-1/2} \exp[-(v - v_c)^2/u_0^2] \tag{12}$$

with $u_0 \leq u$. This initial distribution is mathematically easy to deal with and should be representative of any smooth distribution centered at $v = v_c$ with width u_0. From (9), we immediately see that FPE will be valid if $\Delta u/u_0 \ll 1$. Depending on the velocity selectivity of the excitation and the collision dynamics, $\Delta u/u_0$ will typically range from 1.0×10^{-3} to 1.0×10^{3} (see Sec. VI).

A. FPE

To solve the FPE [Eq. (10)], it is convenient to try a solution

$$\rho(v, t) = \int dv_0 \, G(v_0 \to v, t) \rho(v_0, 0) . \tag{13}$$

Substituting this into Eq. (10), one finds that the propagator $G(v_0 \to v, t)$ also satisfies the FPE with initial condition $G(v_0 \to v, 0) = \delta(v - v_0)$. The solution of the FPE for G is [1,3]

$$G(v_0 \to v, t; \text{FPE}) = \left(\frac{\beta}{2\pi q(1 - e^{-2\beta t})}\right)^{1/2} \times \exp\left(-\frac{\beta(v - v_0 e^{-\beta t})^2}{2q(1 - e^{-2\beta t})}\right), \tag{14}$$

which, together with Eqs. (13), (11), and (7), gives

$$\rho(v, t; \text{FPE}) = (\pi s^2)^{-1/2} \exp[-(v - e^{-\beta t} v_c)^2/s^2], \tag{15a}$$

where

$$s^2 = [u^2(1 - e^{-2\beta t}) + u_0^2 e^{-2\beta t}] . \tag{15b}$$

Thus, the FPE solution is a Gaussian that decays towards the equilibrium distribution $\rho(v, t >> \beta^{-1}) \sim (\pi u^2)^{-1/2} e^{-v^2/u^2}$ with an effective decay rate $\beta = \Gamma(1 - \alpha) \ll \Gamma$, the actual collision rate. The quantity s is a measure of the velocity width of the distribution and increases from the initial value u_0 to the equilibrium value u. If $u_0 = u$, $\rho(v, t)$ remains unchanged, as it must.

For $\beta t \ll 1$,

$$\rho(v, t; \text{FPE}) \sim \{\pi[\Gamma t(\Delta u)^2 + u_0^2]\}^{-1/2} \times \exp\{-(v - v_c)^2/[\Gamma t(\Delta u)^2 + u_0^2]\}, \tag{16}$$

where Eq. (7) has been used. Thus, it takes a time $\tau \approx \Gamma^{-1}(u_0/\Delta u)^2$ for any significant change to occur in the initial distribution. If $(u_0/\Delta u)^2 > 1$ (validity requirement for FPE), $\tau > \Gamma^{-1}$; that is, it takes many collisions to effect significant changes in the system. In some sense, this last statement can be used as a validity criterion for the FPE.

B. TE

Keilson and Storer[3] present an analytic solution for the TE with the kernel (6). Their result for the propagator $G(v_0 \to v, t)$ is

$$G(v_0 \to v, t; \text{TE}) = e^{-\Gamma t}\left(\delta(v - v_0) + \pi^{-1/2} \sum_1^\infty (n!)^{-1} (\Gamma t)^n [(1 - \alpha^{2n})u^2]^{-1/2} \exp[-(v - \alpha^n v_0)^2/(1 - \alpha^{2n})u^2]\right), \tag{17}$$

which, when used with Eq. (13), yields

$$\rho(v, t; \text{TE}) = e^{-\Gamma t} \sum_0^\infty (n!)^{-1} (\Gamma t)^n (\pi \omega_n^2)^{-1/2} \times \exp[-(v - \alpha^n v_c)^2/\omega_n^2], \tag{18a}$$

where

$$\omega_n^2 = u^2(1 - \alpha^{2n}) + u_0^2 \alpha^{2n} . \tag{18b}$$

The relationship of Eq. (18) to the corresponding FPE solution (15) is discussed in Appendix A. Equation (18) is valid for arbitrary α and for all times $t \geq 0$. We are interested in a much more restricted problem. As shall be shown below, the TE and FPE solutions approach each other for $\Gamma t \gg 1$. Consequently, to compare the TE and FPE we need only solve the TE for those times $\Gamma t \lesssim 1$, or, alternatively, for times $\beta t = \Gamma(1 - \alpha)t \ll 1$. In addition, we wish to use the approximation $\alpha \approx 1$. With these limiting assumptions, the Keilson and Storer result may be reduced to a more tractable form. Taking the appropriate limit of Eq. (18) or directly solving the TE as is done in Appendix B, one obtains

$$\rho(v, t; \text{TE}) = \sum_0^\infty e^{-\Gamma t} (\Gamma t)^n (n!)^{-1} \{\pi[u_0^2 + n(\Delta u)^2]\}^{-1/2} \times \exp\{-(v - v_c)^2/[u_0^2 + n(\Delta u)^2]\}, \tag{19}$$

valid for $\beta t = \Gamma(1 - \alpha)t \ll 1$ and $\alpha \approx 1$. The solution (19) does not approach equilibrium at $t \to \infty$ [the exact solution (17) does approach equilibrium] since it is valid only for $\beta t \ll 1$, but (19) has the needed property that $\rho(v, t)$ remains unchanged (to order βt) if one starts with an equilibrium distribution $u_0 = u$.

V. COMPARISON OF TE AND FPE

Equations (16) and (19) may now be compared for various values of u_0, Δu, $1 - \alpha$, and Γt, with the

restrictions $\Gamma(1-\alpha)t \ll 1$ and $1-\alpha \approx 0$. The sum in Eq. (19) has its major contribution from $n \approx \Gamma t$.[7] Hence, if $\Gamma t \gg 1$, one can replace $u_n^2 + n(\Delta u)^2$ by $u_0^2 + \Gamma t(\Delta u)^2$ for all n of importance and then perform the sum over n to find that $\rho(v, t; \text{TE}) \approx \rho(v, t; \text{FPE})$ for $\Gamma t \gg 1$. (The details of the calculation are given in Appendix A.) Physically, this result reflects the fact that subsequent collisions do not affect the system significantly once the system has undergone many collisions ($\Gamma t \gg 1$), regardless of the ratio of Δu and u_0; the relative insignificance of a single collision is, in effect, the validity requirement for the FPE. Thus, the TE and FPE may differ markedly only for $\Gamma t \lesssim 1$, as claimed in Sec. IV. Of course, if $\Delta u \ll u_0$, the TE once again goes over into the FPE solution for all t since the FPE validity requirement (9) [or (5)] is well satisfied.

A numerical comparison of Eqs. (16) and (19) is depicted graphically in Figs. 1–3. For $(\Delta u/u_0) = 0.5$, the TE and FPE solutions nearly coincide, but for $\Delta u/u_0 \gtrsim 1$ the TE and FPE solutions can differ significantly, as is seen for the cases $\Delta u/u_0 = 2$ and 10 shown in Figs. 2 and 3, respectively. In all cases, the TE and FPE solutions agree for $\Gamma t \gg 1$, as was predicted above. Figures 1–3, therefore, represent a graphic proof that the FPE is inadequate to describe velocity relaxation in a sample with $\Delta u/u_0 \gtrsim 1$ for times $\Gamma t \lesssim 1$. In such cases, the more general TE must be used.

VI. DISCUSSION

The time evolution of the distribution function of a spatially isotropic system of atoms or molecules whose center-of-mass motion could be given a classical Brownian-motion description has been considered. It has been shown that the FPE will *not* provide a good approximation to the TE for times $\Gamma t \lesssim 1$ (Γ is the collision rate) when the initial-velocity width u_0 of the distribution function is less than the characteristic collisional velocity change Δu.

Some estimate of characteristic values for u, u_0, and Δu may be obtained from a recent photon-echo experiment of Schmidt and Brewer.[8] Their data yielded a value $\Delta u \approx 200$ cm/sec, while the thermal velocity was $u \approx 4.0 \times 10^4$ cm/sec. Combining these values in Eq. (7) yields $(1-\alpha) \approx 1.0 \times 10^{-5}$, indicating that the Brownian-motion picture has some validity. The minimum value of u_0 possible in their experiment was limited by the pulse width

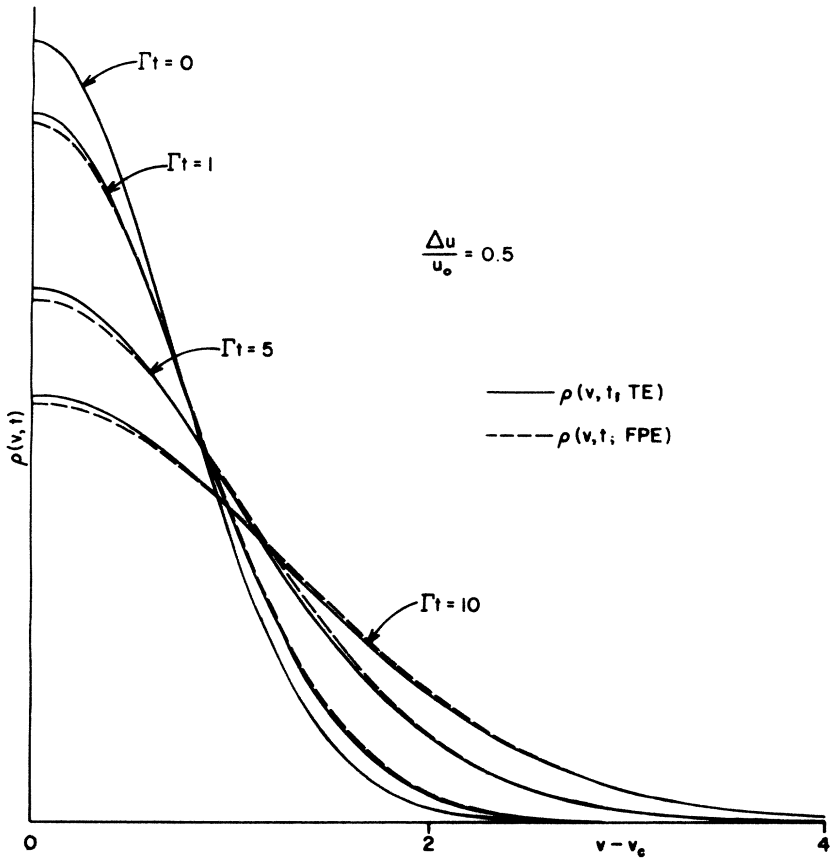

FIG. 1. Graph of the FPE and TE solutions for the distribution function for several values of Γt. The ratio $\Delta u/u_0 = 0.5$ in this case with u_0 having an arbitrary value of unity.

of the laser excitation and corresponded to a value $u_0 \approx 10^3$ cm/sec, giving $\Delta u/u_0 < 1$, so that the *velocity diffusion* could be correctly described by the FPE (one should note that the coupling of velocity diffusion with the *radiation processes* involved in echo formation *cannot* be correctly accounted for by the FPE—the complete TE is needed[8]). By going to longer pulse widths, the value of u_0 can be decreased to a limiting value $u_0(\min) \sim \gamma/k$, where γ is the natural width of the transition and $\vec{k}$ the propagation vector of the laser pulse. For vibrational transitions this value of $u_0(\min)$ might be as low as 1.0×10^{-1} cm/sec giving $\Delta u/u_0 \approx 1.0 \times 10^3$ and making it necessary to use the TE to describe velocity diffusion.

Experimentally, velocity diffusion might be studied using saturated absorption techniques similar to those of Hänsch *et al.*[9] An initial pulse selectively depletes the ground-state atoms or molecules. By introducing a delay in the return probe pulse and monitoring its absorption profile, one can indirectly measure the velocity width of these depleted atoms as a function of time. In this manner, the collisional relaxation of the sample is determined. It seems that the initial pulse width (in time) should be short enough to ensure that the excited atoms or molecules do not decay before being subjected to the probe pulse, but long enough to lead to a reasonably small value of u_0. It will probably be impossible to satisfy both these conditions for electronic transitions, but, for a 10-μm molecular transition with 100-μsec lifetime, a 1.0-μsec pulse satisfies the "short" restriction and corresponds to a u_0 of ~150 cm/sec, which may be comparable to Δu. Quantitative experiments may help to test the validity of the kernel (6).

The validity of the kernel (6) does remain an open question. It can lead to the FPE for $\alpha \approx 1$, which is certainly a necessary criterion. With a different value of α, it might still be a fair approximation in situations where important collisions occur with small impact parameter (i.e., hard-sphere collisions) and the Brownian-motion picture fails.[10] An exact quantum-mechanical kernel is available,[4] but, to date, little or no progress in its evaluation has been made.

In future papers, the comparison of TE and FPE solutions will be extended to cover cases where the system under consideration is also interacting with radiation fields. In addition to condition (9), the validity of the TE → FPE transformation will depend on the requirement that the collision-in-

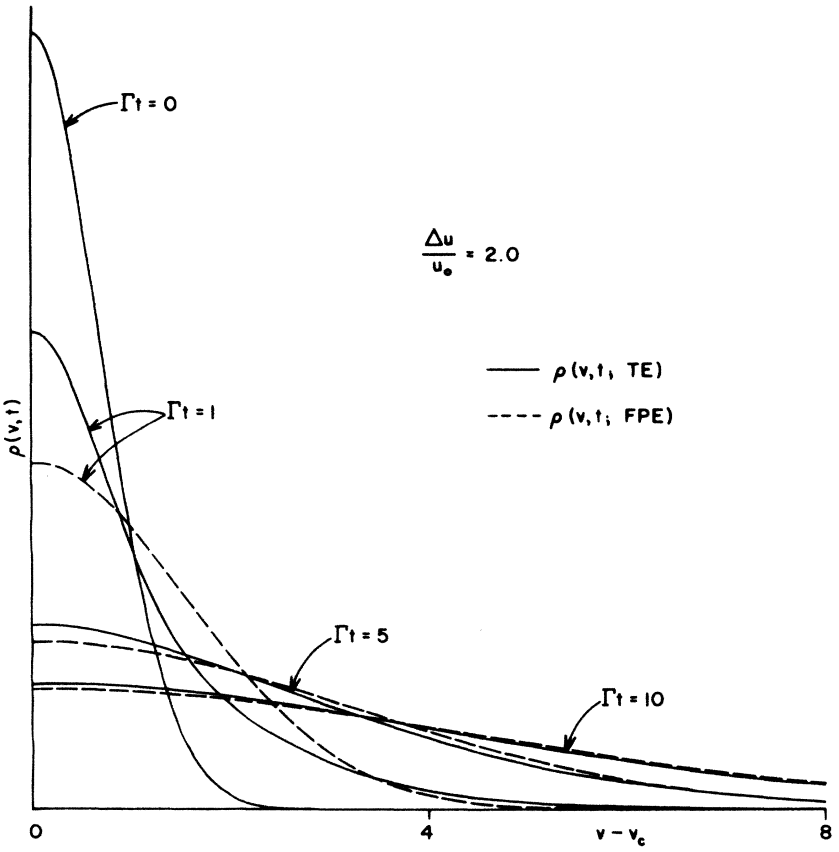

FIG. 2. Same as Fig. 1 except $\Delta u/u_0 = 2.0$.

duced change $k\Delta ut$ in the Doppler phase factor be small for all t of interest. While this requirement will be met in standard absorption and emission experiments, it will not be satisfied in photon-echo experiments, where the use of the complete TE is needed.[11] Moreover, the use of a classical TE in radiation problems is valid only if the collisional interaction is the same for all the levels involved in the radiative transitions. While this might be the case, to a first approximation, for rotational or vibrational transitions, it most certainly will not be true for electronic transitions. Such situations must be treated by a more general quantum-mechanical transport equation.[4]

ACKNOWLEDGMENT

I would like to thank R. Kariotis for performing the computer calculations needed in this work.

APPENDIX A

We wish to determine those conditions under which the TE solution (18) reduces to the FPE solution (15), using the assumption that $\alpha \approx 1$. Equation (18) is of the form

$$f(x) = \sum_0^\infty \frac{x^n}{n!} g(n), \tag{A1}$$

with

$$x = \Gamma t, \tag{A2a}$$

$$g(n) = e^{-x} (\pi \omega_n^2)^{-1/2} \exp[-(v - \alpha^n v_c)^2/\omega_n^2], \tag{A2b}$$

and ω_n given by Eq. (18b). The exponential sum $\sum x^n/n!$ has its major contribution about $n \approx x$. Thus, if $g(n)$ is a slowly varying function of n in the neighborhood of $n \approx x$, it makes sense to expand $g(n)$ as

$$g(n) \approx g(x) + \frac{\partial g(x)}{\partial x}(n - x) + \frac{1}{2}\frac{\partial^2 g(x)}{\partial x^2}(n - x)^2 + \cdots . \tag{A3}$$

Inserting this in (A1) and performing the sums over n gives

$$f(x) \approx e^x g(x) + \tfrac{1}{2} x e^x \frac{\partial^2 g(x)}{\partial x^2} + \cdots . \tag{A4}$$

Using Eqs. (A2) and the fact that $\alpha^{\Gamma t} \approx e^{-\Gamma(1-\alpha)t} = e^{-\beta t}$ for $\alpha \approx 1$, one finds the lead term of Eq. (A4),

$$e^x g(x) = (\pi \omega^2)^{1/2} \exp[-(v - e^{-\beta t} v_c)^2/\omega^2], \tag{A5a}$$

with

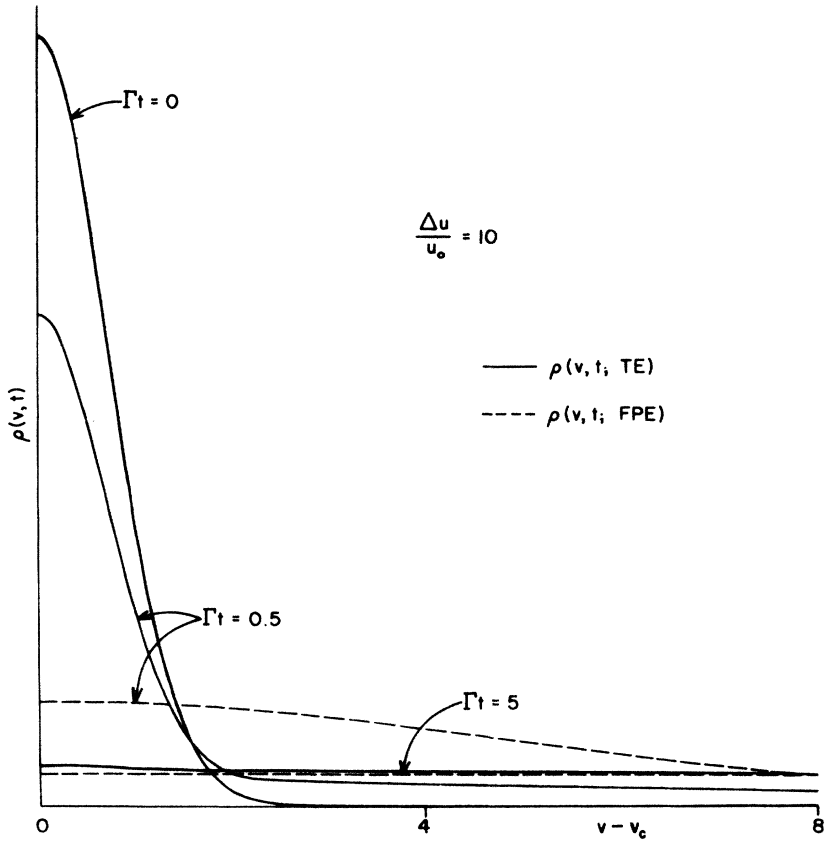

FIG. 3. Same as Fig. 1 except $\Delta u/u_0 = 10$.

$$\omega^2 = u^2(1 - e^{-2\beta t}) + u_0^2 e^{-2\beta t}, \tag{A5b}$$

which is precisely the FPE solution (15). Consequently, in terms of (A4), we see immediately that the FPE will be a good approximation to the TE provided

$$\left| \tfrac{1}{2} \times \left(\frac{\partial^2 g}{\partial x^2} \right) g^{-1} \right| \ll 1 . \tag{A6}$$

Using Eq. (A2b) for $g(x)$, one finds that (A6) is satisfied for all t if $\Delta u^2 = (1 - \alpha^2)u^2 \ll u_0^2$ and for $\Gamma t \gg 1$ regardless of the ratio of Δu to u_0. Actually, this statement is not strictly true—if $(v - e^{-\beta t} v_c)^2 \gg \omega^2$, then (A6) does not hold for $\beta t \lesssim 1$ regardless of the values of the parameters. However, in this region the distribution function itself is negligibly small. Hence, the discussion of the text is restricted to the region $(v - e^{-\beta t} v_c)^2 \lesssim \omega^2$.

APPENDIX B

Equation (19) may be derived simply from Eq. (1) in the limit $\beta t \ll 1$ and $\alpha \approx 1$. A solution of the form

$$\rho(v, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \rho(\kappa, t) e^{-i\kappa v} d\kappa \tag{B1}$$

when substituted into Eq. (1), using Eq. (6) for $W(v' \to v)$, yields an equation for $\rho(\kappa, t)$,

$$\frac{\partial \rho(\kappa, t)}{\partial t} = -\Gamma \rho(\kappa, t) + \Gamma [2\pi^2 (\Delta u)^2]^{-1/2} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dv' \, e^{-i\kappa v} \exp[-(v - \alpha v')^2/(\Delta u)^2] \rho(v', t)$$

$$= -\Gamma \rho(\kappa, t) + \Gamma e^{-\kappa^2 (\Delta u)^2/4} \rho(\alpha\kappa, t) . \tag{B2}$$

Taking $\alpha \approx 1$ to solve (B2), one finds

$$\rho(\kappa, t) \approx \rho(\kappa, 0) \exp[-\Gamma t (1 - e^{-\kappa^2 (\Delta u)^2/4})] . \tag{B3}$$

With $\rho(v, 0)$ given by Eq. (12), $\rho(\kappa, 0)$ may be obtained by inverting Eq. (B1) as

$$\rho(\kappa, 0) = (2\pi)^{-1/2} e^{i\kappa v_c} e^{-\kappa^2 (\Delta u)^2/4} . \tag{B4}$$

Inserting (B4) and (B3) into (B1), one finally arrives at

$$\rho(v, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\kappa \exp[i\kappa (v_c - v) - \Gamma t (1 - e^{-\kappa^2 (\Delta u)^2/4}) - \kappa^2 u_0^2] . \tag{B5}$$

If one expands

$$\exp(\Gamma t e^{-\kappa^2 \Delta u^2/4}) = \sum_0^{\infty} (\Gamma t)^n (n!)^{-1} e^{-n\kappa^2 (\Delta u)^2/4}$$

and does the κ integration in (B5), one arrives at Eq. (19) of the text. This solution is valid only if $\Gamma(1 - \alpha)t = \beta t \ll 1$, as can be verified by substituting the approximate solution (B3) into the exact equation (B2) and looking at the correction terms.

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[5] An expansion of $W(v' \to v)$ about $v' = v$ used by many authors (see Refs. 1 and 2) in deriving the FPE from the TE has not been made in Eq. (4).

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[7] This statement is true only if $v - v_c < u_t$, the width of the distribution at time t. For large $v - v_c$ the TE and FPE solutions will differ—see Appendix A for a discussion of this point.

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[9] T. W. Hänsch, I. S. Shahin, and A. L. Schawlow, Phys. Rev. Lett. 27, 707 (1971).

[10] There is no sound theoretical reason to believe that the kernel (6) should work for anything but Brownian motion. However, as a two-parameter kernel, it may have a wide range of validity.

[11]An article has recently appeared {A. P. Kolchenko, A. A. Pukhov, S. G. Rautian, and A. M. Shalagin, Zh. Eksp. Teor. Fiz. 63, 1173 (1972) [Sov. Phys.—JETP 36, 619 (1973)]} that discusses the effects of weak collisions on the "holes" burned into the velocity distribution of atoms by an electromagnetic field. These authors use the TE rather than FPE approach to the problem and their results, in addition to those of Ref. 8, indicate the importance of using the TE to describe weak collisions in certain cases of experimental interest.