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# Van der Waals forces and zero-point energy for dielectric and permeable materials

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It is pointed out that symmetries of Maxwell's equations under interchange of electric and magnetic fields can be exploited to convert calculations of van der Waals forces between electrically polarizable particles and dielectric materials into results for magnetically polarizable particles and permeable materials. In particular, the forces between perfectly conducting materials calculated from ideas of zero-point radiation are the same as the forces between infinitely permeable materials. Combinations of dielectric materials and permeable materials can lead to repulsive van der Waals forces. For example, two infinitely permeable parallel plates are attracted together with exactly the same force as obtained by Casimir for the van der Waals attraction between two perfectly conducting plates. On the other hand, two parallel plates of area $A$ and separation $d$, one of which is a perfect conductor and one of which is infinitely permeable, are repelled by a force $F=\frac{7}{8}\pi^2\hbar cA/240d^4$, differing in magnitude by a factor of $\frac{7}{8}$ from Casimir's attractive force. A calculation of the repulsive force is given based on ideas of classical electromagnetic zero-point radiation.

## INTRODUCTION

Casimir[1] first pointed out that two uncharged conducting parallel plates should be attracted together by electromagnetic forces associated with the zero-point radiation field. In this paper we note that due to these same zero-point fluctuations, two uncharged permeable parallel plates are attracted, but two parallel plates, one of which is a conductor and the other of which is a permeable material, are repelled. We carry out a calculation for this last situation, in the limit of perfect conductivity and infinite permeability. The final part of our calculation is analogous to Casimir's original work on zero-point energy.

The van der Waals forces between macroscopic objects have been explored extensively theoretically and experimentally since Casimir's original calculation of the attraction of two conducting parallel plates. Lifshitz[2] extended the work to dielectric objects. Also the work has been connected with the van der Waals forces between polarizable particles,[2-5] between polarizable particles and conducting walls,[3,4,6] and between polarizable particles and dielectric walls.[2,7] The calculations refer to dielectric behavior of the materials, but make virtually no mention of the permeability. Of course, it is right that this should be so, since for most materials of interest, van der Waals forces are dominated by the dielectric behavior. Nevertheless, it is at least a curiosity that permeable materials give effects analogous to those of dielectric materials. Moreover, combinations of dielectric and permeable materials can lead to repulsive van der Waals forces.

## ATTRACTIVE AND REPULSIVE Van der WAALS FORCES

The original Casimir-Polder calculation[3] for the van der Waals force between two electrically polarizable particles was extended by Feinberg and Sucher[5] to include terms in both electric and magnetic polarizabilities, and by Lubkin[8] to include a crossed polarizability term. The full formula for the potential between particles $A$ and $B$, holding asymptotically at large separations $r$, is

$$U_{2P}(r)=(\hbar c/4\pi r^7)[-23(\alpha_E^A\alpha_E^B+\alpha_M^A\alpha_M^B)+7(\alpha_E^A\alpha_M^B+\alpha_M^A\alpha_E^B)-60\alpha_X^A\alpha_X^B], \quad (1)$$

where $\alpha_E^A$, $\alpha_E^B$ refer to the electric polarizabilities; $\alpha_M^A$, $\alpha_M^B$ refer to the magnetic polarizabilities; and $\alpha_X^A$, $\alpha_X^B$ to the crossed polarizabilities. We note that if all of the polarizabilities are positive, then two purely electrically polarizable particles are attracted together, as are two purely magnetically polarizable particles. However, a purely electrically polarizable particle will repel a purely magnetically polarizable particle.

The Casimir-Polder formula[3] for the attraction of an electrically polarizable particle to a perfectly conducting wall has also been extended to include magnetic polarizability.[6] The potential holding asymptotically at large separations is

$$U_{PW\epsilon}(r)=-(3\hbar c/8\pi r^4)(\alpha_E-\alpha_M). \quad (2)$$

Here again if $\alpha_E$ and $\alpha_M$ are both positive, then an electrically polarizable particle is attracted to the conducting wall but a magnetically polarizable particle is repelled. If we regard the perfectly conducting wall as obtained from the limit of a dielectric wall where the dielectric constant $\epsilon$ becomes large $\epsilon\to\infty$, then the attraction and repulsion obtained here fit qualitatively with those just noted above for polarizable particles.

## SYMMETRY OF MAXWELL'S EQUATIONS

In the absence of free charges and free currents, Maxwell's equations for the electromagnetic fields in matter are invariant under a transformation which carries

$$\vec{D}\to\vec{B},\quad \vec{E}\to\vec{H},\quad \vec{B}\to-\vec{D},\quad \vec{H}\to-\vec{E}. \tag{3}$$

Indeed the equations are invariant under the more general rotation

$$\begin{aligned}\vec{D}' &= \vec{D}\cos\beta+\vec{B}\sin\beta, \quad \vec{E}'=\vec{E}\cos\beta+\vec{H}\sin\beta,\\ \vec{B}' &= -\vec{D}\sin\beta+\vec{B}\cos\beta,\quad \vec{H}'=-\vec{E}\sin\beta+\vec{H}\cos\beta.\end{aligned} \tag{4}$$

Since van der Waals forces can be derived from ideas of macroscopic electrodynamics, we expect there to be a close connection between forces involving dielectric materials and those involving permeable materials. This symmetry can already be seen in Eq. (1) for the forces between two polarizable particles. The formula is invariant under the transformation (4). Indeed Lubkin[8] added the final term in $\alpha_X^A\alpha_X^B$ based upon this symmetry.

It is interesting to note that this symmetry can also be applied to a polarizable particle outside a conducting wall to obtain a polarizable particle outside a permeable wall. In the limit of infinite permeability $\mu\to\infty$ for the wall, we have from Eq. (2) interchanging the electric and magnetic polarizabilities

$$U_{PW_\mu}(r) = -(3\hbar c/8\pi r^4)(\alpha_M-\alpha_E). \tag{5}$$

Hence we expect that if all the polarizabilities are positive, a magnetically polarizable particle will be attracted to a permeable wall while an electrically polarizable particle will be repelled. Again this result for attractions and repulsions is in qualitative agreement with Eq. (1) for the forces between polarizable particles.

The calculations involving forces between macroscopic dielectric objects can also be converted to expressions involving permeable objects based upon the interchange symmetry of the electric and magnetic fields. We merely replace the dielectric constants $\epsilon_i$ by the permeabilities $\mu_i$ of the corresponding objects. Also the limiting situations $\epsilon\to\infty$, $\mu\to\infty$ involving perfect conductors and infinitely permeable objects can be obtained. In the limit $\epsilon\to\infty$, Lifshitz's force between parallel dielectric plates becomes Casimir's attraction between two perfectly conducting parallel plates of area $A$ and separation $d$, given by the potential

$$U_{\epsilon\epsilon}(d) = -\pi^2\hbar cA/720d^3. \tag{6}$$

This same potential $U_{\mu\mu}(d)$ holds between two infinitely permeable parallel plates.

## FORCES BASED ON ZERO-POINT ENERGY

Now Casimir's original calculation[1] for the force between two uncharged perfectly conducting parallel plates is based upon the change in electromagnetic zero-point energy due to the presence of the conducting plates. The zero-point energy is $\sum \frac{1}{2}\hbar\omega$, where the frequencies $\omega$ are the classical radiation normal modes imposed by the conducting plates. Because of the symmetry (3) for the interchange of elective and magnetic fields, it is easy to see that the frequencies $\omega$ of the radiation normal modes are unchanged when all of the perfectly conducting surfaces are converted to infinitely permeable surfaces. For the conducting surfaces, the boundary conditions require that the tangential components of $\vec{E}$ vanish at the surfaces; for the infinitely permeable surfaces, the tangential components of $\vec{H}$ must vanish. There have been a number of calculations[9] of van der Waals forces based upon zero-point energy changes due to various geometrics of perfectly conducting surfaces. The results apply immediately to infinitely permeable surfaces.

## REPULSIVE FORCE BETWEEN PARALLEL PLATES

The use of previous calculations together with the interchange symmetry (3) is limited. For example, the force between one permeable and one dielectric plate cannot be found from the interchange symmetry plus a calculation for two dielectric plates. Rather this force must be obtained by a full recalculation which meets the appropriate electric and magnetic boundary conditions.

Now the force between a dielectric surface and a permeable parallel surface seems a simple and intriguing problem. Hence, we will perform the calculation in the limit $\epsilon\to\infty$, $\mu\to\infty$. In this case the problem becomes analogous to that originally considered by Casimir.[1] The calculation can be carried out using the ideas of zero-point energy. Since the radiation normal modes are different from the case of perfect conductors considered by Casimir, the zero-point energy and hence the force is different. It turns out that the force between a perfectly conducting plate and an infinitely permeable plate is repulsive. This is in qualitative agreement with the remarks above involving combinations of electrically and magnetically polarizable particles and also involving polarizable particles outside conducting and permeable walls.

## CLASSICAL ZERO-POINT RADIATION

In work published recently[10] on "random electrodynamics," referring to the theory of classical electrodynamics with classical electromagnetic zero-point radiation, it has been emphasized that there is nothing inherently quantum mechanical in the idea of zero-point radiation. If we change the traditional boundary condition on classical electron theory so as to include random classical electromagnetic radiation with a Lorentz invariant spectrum, then a number of phenomena usually thought to require explanations in terms of quanta, are found to be easily comprehensible in classical terms.

As an illustration of this point of view, we will proceed to calculate the force between a perfectly conducting plate and an infinitely permeable parallel plate starting from the assumed presence of classical electromagnetic zero-point radiation. Part way through the calculation, we will recognize that we have arrived at the starting point of Casimir's procedure[1] which merely assigns an energy $\frac{1}{2}\hbar\omega$ per normal mode of the classical electromagnetic field. We will complete the calculation in analogy with Fierz's recalculation[11] of the Casimir force between conducting plates.

In random electrodynamics, we assume that the fundamental homogeneous boundary condition on Maxwell's equations consists of random classical electromagnetic radiation of the form[10]

$$\vec{E}_{ZP}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\,\mathfrak{h}(\vec{k},\lambda)\times\cos[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)], \quad (7)$$

$$\vec{B}_{ZP}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\frac{\vec{k}\times\hat{\epsilon}(\vec{k},\lambda)}{k}\,\mathfrak{h}(\vec{k},\lambda)\times\cos[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)], \quad (8)$$

with the familiar notation for plane waves

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\vec{k}=0,\quad \hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda')=\delta_{\lambda\lambda'}, \quad (9)$$

$$\sum_{\lambda=1}^{2}\epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda)=\delta_{ij}-k_ik_j/k^2,\quad \omega=ck, \quad (10)$$

and the scale of the spectrum given by

$$\pi^2\mathfrak{h}^2(\vec{k},\lambda)=\tfrac{1}{2}\hbar\omega. \quad (11)$$

Here we have expanded the random radiation in terms of transverse plane waves with the random character of the radiation described by the random phases $\theta(\vec{k},\lambda)$ independently and uniformly distributed on $[0,\ 2\pi]$. The one multiplicative constant for the Lorentz invariant radiation spectrum has been chosen to correspond to an energy of $\frac{1}{2}\hbar\omega$ per normal mode.

If we assume that an infinitely permeable plate is located in the $yz$ plane, then the random radiation pattern must meet the boundary conditions at this plate. Hence the random radiation $\vec{E}_{ZPR}$ must include the reflected waves, giving, in the region $x>0$,

$$\vec{E}_{ZPR}(\vec{r},t)=\sum_{\lambda=1}^{2}\int_{k_x<0}d^3k\,\mathfrak{h}\,2[-\hat{i}\epsilon_x\sin k_xx\sin(k_yy+k_zz-\omega t+\theta)+(\hat{j}\epsilon_y+\hat{k}\epsilon_z)\cos k_xx\cos(k_yy+k_zz-\omega t+\theta)], \quad (12)$$

$$\vec{B}_{ZPR}(\vec{r},t)=\sum_{\lambda=1}^{2}\int_{k_x<0}d^3k\,\mathfrak{h}\,2\left\{\hat{i}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_x\cos k_xx\cos(k_yy+k_zz-\omega t+\theta)-\left[\hat{j}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_y+\hat{k}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_z\right]\sin k_xx\times\sin(k_yy+k_zz-\omega t+\theta)\right\}. \quad (13)$$

The expression for $x<0$ is found by changing the integrals to run over values $k_x>0$. The expressions here meet the boundary conditions that the tangential components of the magnetic field $\vec{H}$ vanish at the infinitely permeable wall

$$\hat{i}\times\vec{B}_{ZPR}(0,y,z,t)=0 \quad \text{with } \vec{B}=\vec{H} \text{ in vacuum.} \quad (14)$$

If a perfectly conducting wall is located in the plane $x=d$, then the tangential components of the electric field must vanish at $x=d$. From expressions (13) holding in vacuum, we see that this restricts the value of $k_x$ to

$$k_x=(n+\tfrac{1}{2})\pi/d,\quad n=0,1,2,\ldots. \quad (15)$$

Thus in the region between the plates, the expression for $\vec{E}$ in Eq. (12) becomes a sum over modes in $k_x$,

$$\vec{E}_{ZPP}(\vec{r},t)=\sum_{\lambda=1}^{2}\frac{\pi}{d}\sum_{n=0}^{\infty}\int dk_y\int dk_z\,\mathfrak{h}\,2[-\hat{i}\epsilon_x\sin k_x x\sin(k_y y+k_z z-\omega t+\theta)$$
$$+(\hat{j}\epsilon_y+\hat{K}\epsilon_z)\cos k_x x\cos(k_y y+k_z z-\omega t+\theta)], \tag{16}$$

with $k_x$ given by Eq. (15), and the expression for $\vec{B}$ in Eq. (13) is modified in analogous fashion. The constant $\pi/d$ arises because of the change from the integral involving $k_x$ over to a sum on $n_x$, where the appropriate differential connection corresponds to

$$dk_x=(\pi/d)dn_x. \tag{17}$$

## CONNECTION WITH THE IDEA OF ZERO-POINT ENERGY

The force on the conducting plate can be found by evaluating the Maxwell stress-energy tensor[12] over a closed surface S surrounding the plate. By symmetry, the average force will be along the $x$ direction. Then choosing the surface $S$ to consist of two plane surfaces each of area $A$ at $z=d_+$, $z=d_-$, just outside the conductor at $z=d$, we have

$$\langle\vec{F}\rangle=\left\langle\oint_s da\,\hat{n}\cdot\vec{\mathrm{T}}\right\rangle$$
$$=\hat{i}\left(\int dy\int dz\,\langle T_{xx}(d_+,y,z,t)\rangle-\int dy\int dz\,\langle T_{xx}(d_-,y,z,t)\rangle\right). \tag{18}$$

The average is over the random phases $\theta(\vec{k},\lambda)$. The two integrals in Eq. (18) have the same basic structure and hence it is sufficient to consider in detail the form of one of these. Thus,

$$\int dy\int dz\langle T_{xx}(d_-,y,z,t)\rangle=\frac{1}{8\pi}\int dy\int dz\,\langle E_{xx}{}^2-E_{yy}{}^2-E_{zz}{}^2+B_{xx}{}^2-B_{yy}{}^2-B_{zz}{}^2\rangle, \tag{19}$$

with $\vec{E}$ and $\vec{B}$ as given in Eqs. (12), (13), and (16), evaluated at $x=d$.

The term involving $\langle E_{xx}{}^2\rangle$, for example, is of the form

$$\frac{1}{8\pi}\int dy\int dz\langle E_{xx}{}^2\rangle=\frac{1}{8\pi}\int dy\int dz\sum_{\lambda=1}^{2}\frac{\pi}{d}\sum_{n=0}^{\infty}\int dk_y\int dk_z\sum_{\lambda'=1}^{2}\frac{\pi}{d}\sum_{n'=0}^{\infty}\int dk_y'\int dk_z'$$
$$\times\epsilon_x\epsilon_x'\mathfrak{h}\mathfrak{h}'4\langle\cos(k_y y+k_z z-\omega t+\theta)\cos(k_y' y+k_z' z-\omega' t+\theta')\rangle$$
$$=\frac{1}{8\pi}\int dy\int dz\frac{\pi}{d}\sum_{\lambda=1}^{2}\sum_{n=0}^{\infty}\int dk_x\int dk_y\,\epsilon_x{}^2\frac{\hbar\omega}{2\pi^2}4\times\tfrac{1}{2}$$
$$=\sum_{n=0}^{\infty}\int_0^{\infty}dk_x\int_0^{\infty}dk_y\frac{A}{d}\frac{\hbar\omega}{2\pi^2}\left(1-\frac{k_x^2}{k^2}\right). \tag{20}$$

Here we have averaged over the random phases $\theta(\vec{k},\lambda)$, $\theta(\vec{k}',\lambda')$,

$$\langle\cos\theta(\vec{k},\lambda)\cos\theta(\vec{k}',\lambda')\rangle=\langle\sin\theta(\vec{k},\lambda)\sin\theta(\vec{k}',\lambda')\rangle$$
$$=\tfrac{1}{2}(d/\pi)\delta_{\lambda\lambda'}\delta_{nn'}\delta(k_y-k_y')\times\delta(k_z-k_z'), \tag{21}$$

$$\langle\cos\theta(\vec{k},\lambda)\sin\theta(\vec{k}',\lambda')\rangle=0. \tag{22}$$

In free space the averaging gives

$$\langle\cos\theta(\vec{k},\lambda)\cos\theta(\vec{k},\lambda')\rangle=\tfrac{1}{2}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'); \tag{23}$$

the change to the discrete index requires

$$\delta(k_x-k_x')\to\delta(n\pi/d-n'\pi/d)=\frac{d}{\pi}\delta_{nn'}. \tag{24}$$

Evaluating each of the terms in (19) in analogous fashion, we find

$$\int dy\int dz\,\langle T_{xx}(d_-,y,z,t)\rangle=\sum_{n=0}^{\infty}\int_0^{\infty}dk_y\int_0^{\infty}dk_z\frac{A\hbar c}{\pi^2}\frac{k_x^2}{kd}$$
$$=-\frac{\partial}{\partial d}\sum_{n_x=0}^{\infty}\int_0^{\infty}dn_y\int_0^{\infty}dn_z\,\tfrac{1}{2}\hbar\omega, \tag{25}$$

where

$$\omega = c\left[\left(\frac{(n_x+\frac{1}{2})\pi}{d}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2\right]^{1/2}, \quad L^2 = A. \tag{26}$$

Thus, the term corresponds to the derivative with respect to the separation $d$ of the zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode contained in the volume $d\times L\times L$.

The second term involving $\langle T_{xx}(d_+, y, z, t)\rangle$ can be found easily from the electric fields reflected from the conducting plane at $x=d$,

$$\vec{E}_{ZPR}(\vec{r},t) = \sum_{\lambda=1}^{2}\int_{k_x<0} d^3k\, \mathfrak{h}(\vec{k},\lambda)\, 2\lfloor \hat{i}\epsilon_x \cos k_x(x-d)\cos(k_y y + k_z z - \omega t + \theta) - (\hat{j}\epsilon_y + \hat{K}\epsilon_z)\sin k_x(x-d) \times \sin(k_y y + k_z z - \omega t + \theta)\rfloor, \tag{27}$$

$$\vec{B}_{ZPR}(\vec{r},t) = \sum_{\lambda=1}^{2}\int_{k_x<0} d^3k\, \mathfrak{h}(\vec{k},\lambda)\, 2\left\{-\hat{i}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_x \sin k_x(x-d)\sin(k_y y + k_z z - \omega t + \theta) + \left[\hat{j}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_y + \hat{K}\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)_z\right] \times \cos k_x(x-d)\cos(k_y y + k_z z - \omega t + \theta)\right\}. \tag{28}$$

The result is

$$\int dy \int dz \langle T_{xx}(d_+, y, z, t)\rangle = \int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z \frac{A\hbar c}{\pi^3}\frac{k_x^2}{k} = -\frac{\partial}{\partial d}\int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \times \tfrac{1}{2}\hbar\omega, \tag{29}$$

where here

$$\omega = c\left[\left(\frac{(n_x+\frac{1}{2})\pi}{R-d}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2\right]^{1/2}, \tag{30}$$

$A=L^2$, $R\gg d$, and we neglect terms in $R^{-1}$. This corresponds to the zero-point energy of a volume $(R-d)\times L\times L$.

Thus the force on the conducting plate can be regarded as given by a potential function which is the zero-point energy of a large box $R\times L\times L$ with infinitely permeable walls and a movable perfectly conducting partition located a distance $d$, with $d\gg R$ from one end. This is analogous to the idea of zero-point energy which Casimir[1] used as his starting point in calculating the force between two conducting parallel plates. Here we have been led to the same view by the idea of classical electromagnetic zero-point radiation.

## CALCULATION OF THE REPULSIVE FORCE BETWEEN A CONDUCTING AND A PERMEABLE PLATE

The calculation for the force between a perfectly conducting wall and a parallel infinitely permeable sheet can be completed easily in analogy with the method of Fierz[11] for conducting plates. We will take the potential function as the change in zero-point energy relative to the value when the conducting partition is halfway across the box of length $R$,

$$U_{\epsilon\mu}(d) = \mathcal{E}(d) + \mathcal{E}(R-d) - 2\mathcal{E}(\tfrac{1}{2}R), \tag{31}$$

where

$$\mathcal{E}(d) = \sum \tfrac{1}{2}\hbar\omega \tag{32}$$

is the zero-point energy of a box $d\times L\times L$ when the conducting end is separated by a distance $d$ from the opposite permeable end. We will introduce a temporary cutoff parameter $\lambda$ depending upon the wavelength of the radiation, and at the end of the calculation will go to the no-cutoff limit $\lambda\to 0_+$. Thus here we will work with $\mathcal{E}(d,\lambda)$,

$$\mathcal{E}(d,\lambda) = \tfrac{1}{2}\hbar c 2\sum_{n=0}^{\infty}\int_0^\infty dn_y \int_0^\infty dn_z \omega e^{-\lambda\omega/\pi c} \tag{33}$$

with

$$\omega = c\left[\left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2\right]^{1/2}. \tag{34}$$

The factor of 2 in Eq. (33) comes from the two independent polarizations.

In evaluating $\mathcal{E}(d,\lambda)$, we first change the variables of integration to $Y=n_y/L$, $Z=n_z/L$, and then introduce polar coordinates

$$\mathcal{E}(d,\lambda) = \pi L^2\hbar c\sum_{n=0}^{\infty}\int_0^\infty dY\int_0^\infty dZ\left[\left(\frac{n+\frac{1}{2}}{d}\right)^2 + Y^2 + Z^2\right]^{1/2}\exp\left\{-\lambda\left[\left(\frac{n+\frac{1}{2}}{d}\right)^2 + Y^2 + Z^2\right]^{1/2}\right\}$$
$$= \frac{2\pi^2}{4}L^2\hbar c\sum_{n=0}^{\infty}\int_0^\infty dr\, r\left[\left(\frac{n+\frac{1}{2}}{d}\right)^2 + r^2\right]^{1/2}\exp\left\{-\lambda\left[\left(\frac{n+\frac{1}{2}}{d}\right)^2 + r^2\right]^{1/2}\right\}. \tag{35}$$

Now setting

$$z=[d/(n+\tfrac{1}{2})]^2r^2\,, \tag{36}$$

$$\begin{aligned}\mathcal{E}(d,\lambda)&=\frac{\pi^2}{2}L^2\hbar c\sum_{n=0}^{\infty}\int_0^{\infty}\frac{dz}{2}\left(\frac{n+\frac{1}{2}}{d}\right)^3(1+z)^{1/2}\exp\left(-\lambda\frac{n+\frac{1}{2}}{d}(1+z)^{1/2}\right)\\&=-\frac{\pi^2}{4}L^2\hbar c\frac{\partial^3}{\partial\lambda^3}\int_0^{\infty}\frac{dz}{1+z}\sum_{n=0}^{\infty}\exp\left(-(n+\tfrac{1}{2})\frac{\lambda(1+z)^{1/2}}{d}\right).\end{aligned} \tag{37}$$

The sum over $n$ involves a geometrical series giving

$$\begin{aligned}\sum_{n=0}^{\infty}\exp\left[-(n+\tfrac{1}{2})\frac{\lambda(1+z)^{1/2}}{d}\right]&=\frac{\exp[-(\lambda/2d)(1+z)^{1/2}]}{1-\exp[-(\lambda/d)(1+z)^{1/2}]}\\&=\tfrac{1}{2}\,\mathrm{csch}\left((\lambda/2d)(1+z)^{1/2}\right).\end{aligned} \tag{38}$$

Now taking one of the derivatives in $\lambda$ into the integrand,

$$\mathcal{E}(d,\lambda)=\frac{-\pi^2}{4}L^2\hbar c\frac{\partial^2}{\partial\lambda^2}\int_0^{\infty}\frac{dz}{2(1+z)^{1/2}}\times\frac{\partial}{\partial[\lambda(1+z)^{1/2}/2d]}\,\mathrm{csch}\left(\frac{\lambda(1+z)^{1/2}}{2d}\right). \tag{39}$$

If we set

$$u=\lambda(1+z)^{1/2}/2d\,, \tag{40}$$

then

$$\begin{aligned}\mathcal{E}(d,\lambda)&=-\frac{\pi^2}{4}L^2\hbar c\frac{\partial^2}{\partial\lambda^2}\int_{u=\lambda/2d}^{\infty}\frac{du}{\lambda}\frac{d}{du}\mathrm{csch}\,u\\&=\frac{\pi^2}{4}\hbar c\frac{\partial^2}{\partial\lambda^2}\left(\frac{1}{\lambda}\mathrm{csch}\frac{\lambda}{2d}\right).\end{aligned} \tag{41}$$

We now write

$$\mathrm{csch}\,z=1/z-\tfrac{1}{6}z+\tfrac{7}{360}z^3-\tfrac{31}{15120}z^5+\cdots\,. \tag{42}$$

Thus, we arrive at the expansion

$$\mathcal{E}(d,\lambda)=\pi^2L^2\hbar c\left(\frac{3d}{\lambda^4}+\frac{7}{8}\frac{1}{720d^3}+O(\lambda^2)\right)\,. \tag{43}$$

If we now substitute this expression for $\mathcal{E}(d,\lambda)$ into Eq. (31), we find that the terms in $\lambda^{-4}$ cancel as

$$d+(R-d)-2R/2=0\,, \tag{44}$$

while the terms of order $\lambda^2$ and higher vanish as $\lambda\to 0$. In the limit in which the box enclosing the system is large, $R\to\infty$, we have

$$U_{\epsilon\mu}(d)=\tfrac{7}{8}\pi^2\hbar cA/720d^3\,, \tag{45}$$

writing the area $A$ as $L^2=A$. This is to be compared with Casimir's potential between two perfectly conducting plates given in Eq. (6). Thus the repulsive force between a perfectly conducting plate and an infinitely permeable parallel plate is $\frac{7}{8}$ of the magnitude of the attractive force between two perfectly conducting plates or between two infinitely permeable plates.

## REMARK ON THE EXPERIMENTAL SITUATION

van der Waals forces are extremely small for most laboratory situations, and hence are difficult to measure. However, the attractive forces between conducting plates and between dielectric surfaces have indeed been measured directly.[13] It seems interesting that the unexplored repulsive aspect of van der Waals forces may also be accessible to measurements in terms of the forces between macroscopic combinations of dielectric and permeable materials.

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