PHYSICAL REVIEW A VOLUME 9, NUMBER 5 MAY 1974

Analysis of scattering and excitation amplitudes in polarized-electron-atom collisions. II. Scattering on two-electron atoms

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(Received 12 November 1973)

A general analysis of polarization effects, resulting from exchange collisions between electrons and two-electron atoms, is given using density-matrix techniques. The formalism is presented in some detail. In particular, collisions on spin-1 atoms are discussed. General equations are derived which relate the quantities, characterizing the polarization state of the final beam, to their counterparts before scattering. Both sets of quantities are connected by functions of the scattering amplitudes. Thus, from these general equations we can read off what information can be extracted from experiments with particles in any given initial polarization state. Furthermore, some of the obtained expressions may be useful in discussions of other reactions, for example, processes in which spin-orbit and tensor interactions are important.

I. INTRODUCTION

A general analysis of spin-polarization effects in collisions between electrons and spin-$\frac{1}{2}$ atoms has been given by Burke and Schey[1] and was further discussed in a previous paper[2] (references to this will be designated by I). It is the purpose of this paper to generalize these results to the case of two-electron atoms.[3] Spin-orbit coupling, hyperfine interaction, and nuclear spin are neglected. Under these conditions a description of the electron-atom system does not involve any explicit spin-dependent force, and polarization phenomena arise from implications of the exclusion principle.

Most of the paper deals with electron collisions on spin-1 atoms. Singlet-triplet and triplet-singlet transitions are only considered in Sec. VI. Although most of the quantities arising in this discussion might be beyond the reach of present experimental techniques, we present the formalism in some detail, because the techniques used and some of the equations obtained are of interest for studying other reactions (which may include explicit spin-dependent forces).

It is our main aim to derive general equations in which the quantities, characterizing the polarization state of the electrons and atoms, are related to their counterparts before scattering. Both sets of quantities are connected by functions of the scattering amplitudes. Thus dynamical and geometrical elements are separated and from these general equations one can read off what information can be obtained from polarization measurements with an arbitrary polarized initial beam.

We proceed as follows. In Sec. II we give a rather pedestrian excursion on how to characterize the polarization state of the initial beam in terms of the density matrix ρ_{in}. We derive expressions for the density matrix useful not only in our case of interest, but in other reactions in which spin-$\frac{1}{2}$ and spin-1 particles participate.

We then investigate the scattering process of electrons and spin-1 atoms. We study the algebraic properties of the spin scattering matrix M, which transforms the initial spinor into the final one. This is done in some detail in Sec. III. Starting with the definitions of the M-matrix elements in terms of some simple "basic reactions," we derive the most general form M can have in terms of the Pauli and spin-1 matrices.

From the obtained expressions of ρ_{in} and M, we derive in Sec. IV equations expressing polarization vectors and tensors of the final beam in terms of the initial one. A discussion of these results and some applications follow in Sec. V. Inelastic reactions are discussed in Sec. VI.

II. CHARACTERIZATION OF THE POLARIZATION STATE OF AN ENSEMBLE

The spin state of an ensemble of spin-S particles in general must be described as a statistical mixture of the pure spin states in which the particle may be found. It is convenient to describe this mixture in terms of the density matrix ρ.[4] Representing any pure state of the beam in the form of a column vector $a_i^{(n)}$ (where i runs from 1 to $2S+1$ and n denotes the nth pure state of the mixture) the elements of the $(2S+1)$-dimensional density matrix are given by

$$\rho_{ij} = \sum_n p_n a_i^{(n)} a_j^{(n)*}, \tag{1}$$

where p_n is the probability of finding the nth pure state in the given ensemble and the asterisk denotes the complex conjugate.

We remember that the expectation value of any operator in spin space is given by the following trace:

$$\langle\Omega\rangle = \mathrm{tr}\rho\Omega, \tag{2}$$

where the brackets denote the expectation value.

The complex numbers ρ_{ij} characterize a given ensemble. Because ρ is Hermitian, there are $(2S+1)^2$ independent parameters. Normalizing the density matrix to

$$\mathrm{tr}\rho = 1, \tag{3}$$

there remains $(2S+1)^2 - 1$ independent parameters containing all the information of the polarization state. Instead of assigning the elements (1), we can expand the density matrix in terms of $(2S+1)^2$ Hermitian "basis matrices" M_i which form a complete set in spin space; that is, there is no operator in this space which is not a linear combination of these matrices.

In the space of these Hermitian matrices we define an inner product $(M_i, M_j) = \mathrm{tr}(M_i, M_j) = C\delta_{ij}$, where C is a normalization constant. Thus the basis set is orthogonal in the sense that the product of any two of these matrices has trace zero. It is convenient that one of these basis elements be the identity matrix. In this case the orthogonality relation requires all other basis matrices to be traceless.

In terms of this set, the density matrix can be written as $\rho = \sum_i a_i M_i$ where $a_i = \mathrm{tr}\rho M_i / \mathrm{tr}M_i^2 = \langle M_i\rangle / \mathrm{tr}M_i^2$. Thus the state characterized by ρ is represented by the expectation values of the basis elements. The expectation value of the identity matrix specifies the normalization, the remaining expectation values determine the polarization state of the system.

For example, in the case of spin-$\frac{1}{2}$ particles a basis set is used which consists of the 2×2 identity matrix ϵ and the Pauli matrices σ_i $(i = x, y, z)$. The orthogonality of the Pauli matrices follows from the well-known relation

$$\sigma_i\sigma_j = \delta_{ij}\epsilon + i\epsilon_{ijk}\sigma_k, \tag{4a}$$

and because the σ_i are traceless they are orthogonal to ϵ. ρ is given in terms of this set by the familiar equation

$$\rho = \frac{1}{2}\left(\epsilon + \sum_i P_i\sigma_i\right). \tag{4b}$$

P_i is a component of the polarization vector, defined as $\langle\sigma_i\rangle$. Thus ρ is characterized by the expectation values of the basis set.

The density matrix characterizing an ensemble of spin-1 particles is a 3×3 matrix; thus nine basis matrices are needed to expand ρ. Besides the three-dimensional identity matrix E and the three spin-1 matrices S_i, we use the following set S_{ij}:

$$S_{ij} = \tfrac{3}{2}(S_iS_j + S_jS_i) - 2\delta_{ij}E \tag{5}$$

$(i, j = x, y, z)$. We are using the standard representation of these matrices. Definition (5) is in accordance with the "Madison convention"[5] in nuclear physics.

These matrices have simple properties under rotation. The S_i transform as the components of a vector (like the Pauli matrices); the S_{ij} transform as the components of a symmetric second-rank tensor.

This set of ten matrices span the spin space; however, since only nine are needed this set is overcomplete. Remembering the relation $\sum_i S_i^2 = S(S+1)E$ for spin-S particles, we get

$$\sum_i S_{ii} = 0. \tag{6}$$

Note that owing to this overcompleteness this set of matrices is not orthogonal.

This overcompleteness can be avoided by using irreducible tensors. However, the use of Cartesian tensors simplifies the description of experiments and because the overcompleteness causes little trouble, we prefer to use this set.

The properties of the S_i and the S_{ij}, required to go through the details of the calculations in this paper, may be condensed to the form:

$$S_iS_jS_k = \tfrac{1}{2}(\delta_{jk}S_i + \delta_{ij}S_k + i\epsilon_{ijm}S_kS_m + i\epsilon_{ikm}S_mS_j + i\epsilon_{jkm}S_iS_m), \tag{7}$$

which governs the algebra of these matrices. Equation (7) is analogous to Eq. (4a) which summarizes the algebraic properties of the Pauli matrices. ϵ_{ijk} is the standard Kronecker tensor.

For spin-$\frac{1}{2}$ particles, a quadratic combination of spin operators reduces to a linear combination of the Pauli matrices as shown by Eq. (4a). In the case of spin-1 particles, however, the expectation values $\langle S_{ij}\rangle$ give new information. Thus in addition to the polarization vector P_i, knowledge of the components P_{ij} of the polarization tensor, defined by $P_{ij} = \langle S_{ij}\rangle$, are needed for a complete description of a beam of spin-1 particles.

In order to expand ρ in terms of the S_i and S_{ij}, we make the ansatz:

$$\rho = aE + \sum_i b_iS_i + \sum_{ij} c_{ij}S_{ij}.$$

Although the S_{ij} are symmetric, we sum over all i, j. The overcompleteness of the basis matrices

may be taken into account by requiring $\sum_i C_{ii} = 0$. This constraint enables us to proceed as though we were dealing with an orthogonal set. Using the normalization condition (3) and Eq. (7), we calculate the expectation values and relate the coefficients a, b_i, and c_{ij} to P_i and P_{ij}. We get

$$\rho = \frac{1}{3}\left(E + \frac{3}{2}\sum_i P_i S_i + \frac{1}{3}\sum_{ij} P_{ij} S_{ij}\right). \tag{8}$$

Because from (6) it follows that $\sum_i P_{ii} = 0$ and remembering the normalization condition, the spin state of a spin-1 particle in general is characterized by eight real parameters, three components of the polarization vector, and the five independent components of the symmetric polarization tensor.

As an illustration we discuss the special case where the beam is in one of the pure spin states

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

which we denote in the following as $|+1\rangle$, $|0\rangle$, $|-1\rangle$. In the states $|+1\rangle$ and $|-1\rangle$ the polarization vector points along the positive or negative direction of the quantization axis (z axis). In the state $|0\rangle$ one can think of the spin vector as perpendicular to the quantization axis but precessing around it. This requires, that we consider quantities more complicated than the polarization vector.

In the simple case where the beam is an incoherent superposition of the three pure states $|m\rangle$ ($m = \pm 1, 0$), three numbers are sufficient to describe the ensemble, for example N_+, N_0, N_- which are the numbers of particles in the three states. An equivalent description is obtained by giving the values of the intensity I and of the components of the polarization vector and tensor:

$$I = N_+ + N_0 + N_-,$$

$$P_z = \langle S_z \rangle = \sum_m p_m \langle m|S_z|m\rangle = (N_+ - N_-)/I,$$

$$P_x = P_y = 0$$

($P_m = Nm/I$ is the probability of finding a particle in the state $|m\rangle$),

$$P_{zz} = \langle S_{zz} \rangle = \sum_m \langle m|S_{zz}|m\rangle p_m = (N_+ - 2N_0 + N_-)/I,$$

$$P_{xx} = P_{yy} = -\tfrac{1}{2}P_{zz},$$

$$P_{ij} = 0 \quad \text{for } i \neq j.$$

In describing systems of electrons and spin-1 atoms, the joint state of the two particles is represented by a 6×6 density matrix, acting in the composite spin space of electrons and atoms. Any operator in this space can be written in the form $\Omega_A \times \Omega_e$ where $\Omega_A(e)$ acts on the atomic (electronic) spinor and $\times$ denotes the direct product. (This type of product is often called a tensor or Kronecker product.)

The density matrix may be written as a linear combination of the 6×6 identity matrix 1 and 35 Hermitian traceless matrices. We choose the following set:

$$1, \quad \beta_i = E \times \sigma_i, \quad \alpha_i = S_i \times \epsilon,$$
$$\alpha_{ij} = S_{ij} \times \epsilon, \quad \alpha_i \beta_j, \quad \alpha_{ij}\beta_k \tag{9}$$
$$(i, j, k = x, y, z)$$

As in the case of spin-1 particles, this set is overcomplete and from (6) it follows that

$$\sum_i \alpha_{ii} = 0. \tag{10}$$

In terms of these operators the density matrix of the ingoing beam is given by the expression:

$$\rho_{\text{in}} = \frac{1}{6}\left(1 + \sum_i P_i^{(e)}\beta_i + \frac{3}{2}\sum_i P_i^{(A)}\alpha_i + \frac{1}{3}\sum_{ij} P_{ij}\alpha_{ij} + \frac{3}{2}\sum_{ij} Q_{ij}\alpha_i\beta_j + \frac{1}{3}\sum_{ijk} R_{ijk}\alpha_{ij}\beta_k\right), \tag{11}$$

which may be derived in a similar way to Eq. (8). $P_i^{(e)}$ and $P_i^{(A)}$ are the i components of the electronic and atomic polarization vector

$$P_i^{(e)} = \text{tr}\rho_{\text{in}}\beta_i, \quad P_i^{(A)} = \text{tr}\rho_{\text{in}}\alpha_i; \tag{12a}$$

P_{ij} is a component of the atomic polarization tensor

$$P_{ij} = \text{tr}\rho_{\text{in}}\alpha_{ij}; \tag{12b}$$

and Q_{ij} and R_{ijk} are correlation terms

$$Q_{ij} = \text{tr}\rho_{\text{in}}\alpha_i\beta_j, \quad R_{ijk} = \text{tr}\rho_{\text{in}}\alpha_{ij}\beta_k. \tag{12c}$$

From (10) follows:

$$\sum_i P_{ii} = 0, \quad \sum_i R_{iij} = 0. \tag{13}$$

We note that the correlation tensors are zero if at least one of the particle beams (electrons or atoms) is unpolarized. If both are completely polarized, these terms are given by

$$Q_{ij} = P_i^{(A)}P_i^{(e)}, \quad R_{ijk} = P_{ij}P_k^{(e)}.$$

These relations may be proved with methods similar to those in Ref. 1.

III. EXPANSION OF THE M MATRIX

After having characterized the initial beam, we start now with the second part of our program,

that is, an analysis of the scattering process between electrons and spin-1 atoms. We consider elastic scattering. If the ingoing electrons are in one of the pure states $|m^{(e)}\rangle$ ($m^{(e)}=\pm\frac{1}{2}$) and the atoms in one of the pure states $|m^{(A)}\rangle$ ($m^{(A)}=\pm1,0$), there will be ten amplitudes for each energy and scattering angle, because there are ten possible processes:

$$\begin{aligned}
&|+1\rangle|+\tfrac{1}{2}\rangle \to |+1\rangle|+\tfrac{1}{2}\rangle,\\
&|0\rangle|+\tfrac{1}{2}\rangle \to \begin{cases}|0\rangle|+\frac{1}{2}\rangle\\ |+1\rangle|-\frac{1}{2}\rangle\end{cases},\\
&|-1\rangle|+\tfrac{1}{2}\rangle \to \begin{cases}|-1\rangle|+\frac{1}{2}\rangle\\ |0\rangle|-\frac{1}{2}\rangle\end{cases},
\end{aligned} \tag{14}$$

and five further reactions with $|-\frac{1}{2}\rangle$ instead of $|+\frac{1}{2}\rangle$. These processes are the analogs to the six "basic reactions" listed in I.

In order to catalog the reactions (14), we define an operator M in spin space so that its matrix elements $\langle m^{(A)\prime}m^{(e)\prime}|M|m^{(A)}m^{(e)}\rangle$ are the amplitudes for a transition from an initial state $|m^{(A)}\rangle|m^{(e)}\rangle$ to the final state $|m^{(A)\prime}\rangle|m^{(e)\prime}\rangle$ for a given energy and scattering angle and that

$$\sigma(m^{(A)}m^{(e)}\to m^{(A)\prime}m^{(e)\prime}) = |\langle m^{(A)\prime}m^{(e)\prime}|M|m^{(A)}m^{(e)}\rangle|^2 \tag{15a}$$

is the differential cross section for this elastic transition. (We suppress here the dependence of M on energy and angle.)

If the wave function of the incident beam is given by $e^{i\vec{k}\cdot\vec{r}}\phi|m^{(A)}m^{(e)}\rangle$, the asymptotic wave function after scattering is

$$\psi_{as} \to [e^{i\vec{k}\cdot\vec{r}} + (1/r)e^{ikr}M]\,\phi|m^{(A)}\rangle|m^{(e)}\rangle, \tag{15b}$$

where ϕ is the wave function of the atom except the spin part. $M|m^{(A)}\rangle|m^{(e)}\rangle$ can be regarded as the spin state of the particles after the reaction. Thus the operator M transforms the initial spinor into the final one.

Any scattering process for any given ingoing beam in any arbitrary polarization state may be analyzed in terms of the M-matrix elements of the simple reactions (14). This has been discussed in I and Ref. 6 for the case of the one-electron atom.

Using Clebsch-Gordan techniques, we expand the element $\langle m^{(A)\prime}m^{(e)\prime}|M|m^{(A)}m^{(e)}\rangle$ in terms of the amplitudes $f^{(3/2)}$ and $f^{(1/2)}$ where $f^{(S_t)}$ denotes the scattering amplitude in the channel with total spin S_t: $f^{(S_t)}=\langle S_t, S_{tz}|M|S_t, S_{tz}\rangle$ (independent of S_{tz}). We get

$$M = \begin{bmatrix}
f^{(3/2)} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{3}(f^{(3/2)}+2f^{(1/2)}) & (\frac{2}{3})^{1/2}(f^{(3/2)}-f^{(1/2)}) & 0 & 0 & 0\\
0 & (\frac{2}{3})^{1/2}(f^{(3/2)}-f^{(1/2)}) & \frac{1}{3}(2f^{(3/2)}+f^{(1/2)}) & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{3}(2f^{(3/2)}+f^{(1/2)}) & (\frac{2}{3})^{1/2}(f^{(3/2)}-f^{(1/2)}) & 0\\
0 & 0 & 0 & (\frac{2}{3})^{1/2}(f^{(3/2)}-f^{(1/2)}) & \frac{1}{3}(f^{(3/2)}+2f^{(1/2)}) & 0\\
0 & 0 & 0 & 0 & 0 & f^{(3/2)}
\end{bmatrix}. \tag{15c}$$

Instead of using the amplitudes $f^{(S_t)}$, we may express the elements of M in terms of the direct (f) and exchange (g) amplitudes, defined by the expressions:

$$\begin{aligned}
&\langle +\tfrac{1}{2}, -1|M|+\tfrac{1}{2}, -1\rangle = f,\\
&\langle -\tfrac{1}{2}, +1|M|+\tfrac{1}{2}, 0\rangle = -\sqrt{2}\,g.
\end{aligned} \tag{16}$$

Physically f and g refer to scattering events in which the ingoing electron can definitely be stated to have, or not to have, suffered an exchange collision with one of the atomic electrons.

In elastic scattering processes between spin-$\frac{1}{2}$ and spin-1 particles involving explicit spin-dependent interactions, the expression for the M matrix is more complicated. However, by observing electrons scattered in forward direction, we have the same structure (15) for M. (That follows from the fact that there is no orbital-angular-momentum component in the direction of motion, which may be chosen as the z axis.)

One may go further in the analysis (without going into the details of the dynamics) by expressing M in terms of the set (9). Because all explicit spin-dependent forces are neglected M must have the simple form:

$$M = f^{(3/2)}\eta_{3/2} + f^{(1/2)}\eta_{1/2}.$$

η_{S_t} are the projection operators upon spin functions with total spin S_t:

$$\eta_{3/2} = \frac{1}{3}\left(2 + \sum_i \alpha_i\beta_i\right),$$

$$\eta_{1/2}=\frac{1}{3}\left(1-\sum_i \alpha_i\beta_i\right).$$

Reformulating the terms we get

$$M=\tfrac{1}{3}(2f^{(3/2)}+f^{(1/2)})1 + \tfrac{1}{3}(f^{(3/2)}-f^{(1/2)})\sum_i \alpha_i\beta_i \quad (17a)$$

$$=(f-g)1-g\sum_i \alpha_i\beta_i . \quad (17b)$$

Instead of using the projection operators, Eqs. (17) may be derived from invariance principles. The operator M is required to be invariant under rotation. Thus M must be a combination of all scalars which can be built from the set (9). Apart from the identity matrix, there is only one scalar combination $\vec{\alpha}\cdot\vec{\beta}=\sum_i \alpha_i\beta_i$. Thus the most general form of M in terms of the set (9) is

$$M=a1+b\sum_i \alpha_i\cdot\beta_i .$$

The coefficients a and b follow from

$$\mathrm{tr}M=6a,$$

$$\mathrm{tr}M\alpha_z\beta_z=b\sum_i \mathrm{tr}(S_i\cdot S_z)\,\mathrm{tr}(\sigma_i\cdot\sigma_z)=4b.$$

Using Eq. (15) and the explicit forms of S_z and σ_z, we get

$$a=\tfrac{1}{3}(2f^{(3/2)}+f^{(1/2)}),$$

$$b=\tfrac{1}{3}(f^{(3/2)}-f^{(1/2)}),$$

which gives us Eq. (17).

IV. POLARIZATION STATE OF THE FINAL PARTICLES

The main results of the previous discussion are both Eq. (11), which specifies the spin state of the initial beam, and Eq. (17), which extracts from the scattering process the properties independent of the dynamics. From both we obtain expressions for the quantities characterizing the final beam of spin-$\frac{1}{2}$ and spin-1 particles.

Because M transforms incident spinors into final spinors, the density matrix ρ_{out} of the outgoing beam is given by the relation:

$$\rho_{\mathrm{out}}=M\rho_{\mathrm{in}}M^{\dagger}. \quad (18)$$

We define polarization vectors and tensors for the final particles as

$$\begin{aligned}
&\sigma(\theta)=\mathrm{tr}\rho_{\mathrm{out}}, \quad \sigma(\theta)P'_{ij}=\mathrm{tr}\rho_{\mathrm{out}}\alpha_{ij},\\
&\sigma(\theta)P_i^{(A)\prime}=\mathrm{tr}\rho_{\mathrm{out}}\alpha_i,\\
&\sigma(\theta)Q'_{ij}=\mathrm{tr}\rho_{\mathrm{out}}\alpha_i\beta_j, \quad (19)\\
&\sigma(\theta)P_i^{(e)\prime}=\mathrm{tr}\rho_{\mathrm{out}}\beta_i,\\
&\sigma(\theta)R'_{ijk}=\mathrm{tr}\rho_{\mathrm{out}}\alpha_{ij}\beta_k .
\end{aligned}$$

Note that ρ_{out} is not normalized; thus, the differential cross section $\sigma(\theta)$ occurs in (19).

In order to relate the quantities (19) to the quantities (12), we put the Eqs. (11), (17), and (18) into (19). Using the relation

$$\mathrm{tr}[(A_1\times B_1)\cdot(A_2\times B_2)]=(\mathrm{tr}A_1\cdot A_2)\cdot(\mathrm{tr}B_1\cdot B_2),$$

we reduce the problem to the calculation of traces of Pauli- and spin-1 matrices. These traces may then be calculated with the help of Eqs. (4a) and (7).

We note that all the calculations are carried through without using the explicit forms of the matrices. All that is needed are the algebraic properties (4a) and (7). This illustrates the usefulness of the expansions (11) and (17).

In terms of f and g we get the following results for the quantities (19), where ϵ_{ijk} is the Kronecker tensor:

$$\sigma(\theta)=|f-g|^2+2|g|^2+(|g|^2-fg^*-f^*g)\sum_n Q_{nn}, \quad (20a)$$

$$\sigma(\theta)P_i^{(e)\prime}=(|f-g|^2-\tfrac{2}{3}|g|^2)P_i^{(e)}+(3|g|^2-fg^*-f^*g)P_i^{(A)}+i(fg^*-f^*g)\sum_{mn}Q_{mn}\epsilon_{imn}+\tfrac{2}{3}|g|^2\sum_j R_{ijj}, \quad (20b)$$

$$\sigma(\theta)P_i^{(A)\prime}=\tfrac{2}{3}(3|g|^2-fg^*-f^*g)P_i^{(e)}+(|f-g|^2+|g|^2)P_i^{(A)}+\tfrac{1}{2}i(f^*g-fg^*)\sum_{mn}Q_{mn}\epsilon_{imn}-\tfrac{1}{3}(fg^*+f^*g)\sum_j R_{ijj}, \quad (20c)$$

$$\sigma(\theta)P'_{ij}=(|f-g|^2-|g|^2)P_{ij}+\tfrac{1}{2}(4|g|^2-fg^*-f^*g)\left(-\delta_{ij}\sum_m Q_{mm}+\tfrac{3}{2}(Q_{ij}+Q_{ji})\right) + \tfrac{1}{2}i(f^*g-fg^*)\sum_{km}(R_{ikm}\epsilon_{jkm}+R_{jkm}\epsilon_{ikm}), \quad (20d)$$

$$\sigma(\theta)Q'_{ij}=\tfrac{2}{3}\delta_{ij}(|g|^2-fg^*-f^*g)+\tfrac{2}{3}i(f^*g-fg^*)\sum_k P_k^{(e)}\epsilon_{ijk}-\tfrac{1}{2}i(f^*g-fg^*)\sum_k P_k^{(A)}\epsilon_{ijk}+\tfrac{1}{3}(4|g|^2-f^*g-fg^*)P_{ij} + (|f-g|^2-\tfrac{1}{2}(fg^*+f^*g)+|g|^2)Q_{ij}+\tfrac{1}{2}(fg^*+f^*g)\sum_m Q_{mm}\delta_{ij}-\tfrac{1}{3}i(f^*g-fg^*)\sum_{kl}R_{ikl}\epsilon_{jkl}, \quad (20e)$$

$$\sigma(\theta)R'_{ijk}=\tfrac{1}{3}|g|^2(-2\delta_{ij}P_k^{(e)}+3\delta_{jk}P_i^{(e)}+3\delta_{ik}P_j^{(e)})+\tfrac{1}{4}(fg^*+f^*g)(2\delta_{ij}P_k^{(A)}-3\delta_{jk}P_i^{(A)}-3\delta_{ik}P_j^{(A)})$$
$$+\tfrac{1}{2}i(fg^*-f^*g)\cdot\sum_l(P_{il}\epsilon_{jkl}+P_{jl}\epsilon_{ikl})+\tfrac{1}{2}i(fg^*-f^*g)\sum_m(\tfrac{3}{2}Q_{im}\epsilon_{jkm}+\tfrac{3}{2}Q_{jm}\epsilon_{ikm}+\delta_{ij}\sum_l Q_{lm}\epsilon_{lmk})$$
$$+(|f-g|^2-\tfrac{1}{2}|g|^2)R_{ijk}-\tfrac{2}{3}|g|^2\delta_{ij}\sum_l R_{kll}+\tfrac{1}{2}(3|g|^2-fg^*-f^*g)(R_{ikj}+R_{jki})+\tfrac{1}{2}(-|g|^2+fg^*+f^*g)$$
$$\times\Big(\delta_{jk}\sum_l R_{ill}+\delta_{ik}\sum_l R_{jll}\Big). \tag{20f}$$

Equations (20) are our main results. They are the analogs of the expressions given by Burke and Schey[1] for one-electron atoms.

V. DISCUSSION AND APPLICATIONS

The main feature of Eqs. (20) is that the dynamical elements and the quantities which characterize the spin state of the ingoing beam are separated. Thus we easily obtain those properties of the scattering process under consideration which are independent of the special dynamics. We note the following.

(a) For an unpolarized ingoing beam, the polarization vectors and tensors of the outgoing particles vanish, and we get the well-known result that the polarization of the final beam is zero for unpolarized initial beams.

(b) In scattering processes between unpolarized electrons and polarized atoms, only the vector part of the polarization can be transferred; there will be no electron polarization if the atom has only tensor polarization. The direction of the atomic polarization vector remains unaltered, but the beam is depolarized. The polarization vector of the final electron is parallel to that of the target.

(c) If polarized electrons are scattered from unpolarized atoms the outgoing atoms can have no tensor polarization. The polarization vectors of both final particles are parallel to that of the incident electron.

A similar discussion for the case of spin-$\frac{1}{2}$ particles in terms of Stokes parameters has been given in Ref. 7.

Furthermore, from Eqs. (20) we can read off the information about the M-matrix elements, which can be extracted from experiments with polarized beams. Specifying the polarization state of the ingoing particles, and measuring polarization vectors and tensors of the outgoing ones, for each energy and scattering angle, the scattering amplitudes can be calculated from these experimental values with the help of Eq. (20), and compared with theoretical predictions. In addition the expressions (20) show how the results of special experiments can be combined in order to separate the contributions of direct and exchange scattering.

In order to determine f, g, and the relative phase between them, three independent measurements are required. As an example we mention the following set of experiments.

(a) Measurement of the unpolarized cross section: Because unpolarized particles are characterized by vanishing values of all the polarization vectors and tensors and correlation terms, we get from Eq. (20a)

$$\sigma_{un}=|f-g|^2+2|g|^2. \tag{21a}$$

(b) Scattering of unpolarized electrons from polarized atoms and spin analysis of the outgoing atom gives the depolarization ratio

$$d(\theta)=P_i^{(A)\prime}/P_i^{(A)}=1-|g|^2/\sigma(\theta), \tag{21b}$$

which follows from the expression (20c) with $P_i^{(e)\prime}=0$, $Q_{ij}=0$, $R_{ijk}=0$.

(c) Scattering of completely polarized electrons and atoms: The differential cross section for the process $|+\frac{1}{2}\rangle|+1\rangle\rightarrow|+\frac{1}{2}\rangle|+1\rangle$ follows from (20a)

$$\sigma(\theta)=|f-2g|^2. \tag{21c}$$

Although this experiment requires completely polarized particles no spin analysis of the final beam is necessary.

VI. INELASTIC SCATTERING

So far we discussed only elastic scattering events. However the formalism is more general. Thus in the case of an atomic triplet-triplet transition from $\Gamma=n_1l_1m_{l_1}$ to $\Gamma'=n_1'l_1'm_{l_1}'$, we only have to replace the elastic amplitudes in Eqs. (20) by the expression

$$f^{(S_t)}_{\Gamma'\Gamma}=\frac{i}{(k_nk_n')^{1/2}}\sum_{l_2l_2'L}i^{(l_2-l_2')}[4\pi(2l_2+1)]^{1/2}(l_1m_{l_1},l_20|LM_2)(l_1'm_{l_1}',l_2m_{l_2}'|LM_2)$$
$$\times(\delta_{n_1'k_n'l_1'l_2',nk_nl_1l_2}-S^{LS_t}_{n_1'k_n'l_1'l_2',nk_nl_1l_2})\gamma^{m_{l_2}'}_{l_2'}(\theta,\phi). \tag{22}$$

The functions γ_l^m are standard spherical harmonics, $(l_1 m_{l_1}, l_2 0 | LM_2)$ is a Clebsch-Gordan coefficient, and k_n' and k_n denote the wave numbers

Because all spin-dependent forces in the scattering matrix have been neglected, we get the same polarization pattern in both elastic and inelastic processes.

In singlet-triplet and triplet-singlet transitions, there is only one possible spin channel with total spin $S_t = \frac{1}{2}$. Thus measurements of the differential cross section at all energies and angles is sufficient in order to determine the amplitude $f_{\Gamma'\Gamma}^{(1/2)}$ (which is proportional to the exchange amplitude). We get the expressions:

$$\sigma_{\Gamma'\Gamma} = (k'/k) | f_{\Gamma'\Gamma}^{(1/2)} |^2$$

for singlet-triplet and

$$\sigma_{\Gamma'\Gamma} = \tfrac{1}{3}(k'/k) | f_{\Gamma'\Gamma}^{(1/2)} |^2$$

for triplet-singlet transitions.

In singlet-singlet transitions $f_{\Gamma'\Gamma}^{(1/2)}$ is a coherent superposition of the direct and exchange amplitude. The polarization state of the electron cannot change in collisions on spin-0 particles, as far as spin-dependent interactions are neglected. Thus in this case it is not possible to separate direct and exchange contributions by using polarized electrons.

VII. SUMMARY

We have given a description for polarization phenomena, arising from the exchange interaction, for collisions between electrons and two-electron atoms. In particular, scattering on spin-1 atoms has been discussed. For this case we derived general equations in which polarization vectors and tensors after scattering are related to their counterparts before scattering. In these equations the dynamical elements are separated from the quantities characterizing the polarization state of the incident beam. This feature allows us to extract information about the scattering amplitudes from polarization measurements. Furthermore, some of the expressions obtained may be useful for a discussion of reactions involving explicit spin-dependent forces.

ACKNOWLEDGMENTS

We would like to thank Professor P. G. Burke and Professor I. C. Percival for some stimulating discussions. One of us (K.B.) gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft and the hospitality of Stirling University.

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