Effects of retardation on electromagnetic self-energy of atomic states*

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The significance of retardation effects in photon emissions and absorptions is emphasized in the calculation of self-energy. It is explicitly demonstrated that inclusion of such effects leads to a finite answer for the shifts of atomic energy levels in a nonrelativistic theory without cutoff. Ambiguities that exist in the mass renormalization in the nonrelativistic approach are pointed out. Such ambiguities vanish from the relativistic theory. Explicit calculation is carried out in the case of hydrogen by utilizing the Coulomb Green's function. The advantage of the present approach in calculating self-energy shift in high-Z hydrogenic ions is suggested.

I. INTRODUCTION

It is well known that Bethe's original calculation of the Lamb shift,[1] which introduces a photon energy cutoff at the electron mass, gives surprisingly close agreement with the experimental value. Without the cutoff, Bethe's result would be logarithmically divergent. In addition to being nonrelativistic, Bethe's calculation neglects retardation in the photon emission and absorption, an approximation which has been argued to be accurate.[2] While a correct treatment of electromagnetic radiative correction must be based on a relativistic formalism, it is interesting to note that finite value for the Lamb shift can be obtained from a nonrelativistic calculation without an ultraviolet photon cutoff, but including all multipole interactions.[3] In such a calculation, there exist ambiguities in the correct mass renormalization counterterm, which can only be properly treated by considering the nonrelativistic model as a suitable limit of a relativistic theory, where such ambiguities are absent. However, the ambiguities do not affect the convergence of the result.

In this paper, we carry out a nonrelativistic calculation including retardation, and show how and why it leads to a finite Lamb shift. Our technique involves the relation between energy shift and a part of the forward Compton scattering amplitude, a relation known in particle physics.[4] In the present case, the crossed forward scattering amplitude can be calculated exactly, and the necessary integral over photon momentum can easily be carried out numerically. In Sec. II we present and derive the relevant formula for energy shift and the Compton amplitude. In Sec. III we discuss the reason for the convergence of the Lamb shift, the renormalization prescription, and the ambiguities thereof. In Sec. IV we give some numerical results for hydrogen. These results are not especially useful for hydrogen, where very accurate results from a relativistic theory including retardation are available. However, our technique may be more useful for hydrogenic ions of high Z, where retardation corrections may be substantial, and there are not equally accurate calculations available.

II. FORMULATION

In this section we derive the relation between the self-energy shift and the *on-shell* Compton amplitude. The unrenormalized energy shift is well known to be given by[4]

$$\Delta E^{\mathrm{unr}} = \mathrm{Re}\Delta M\,, \tag{2.1}$$

$$\Delta M = \frac{i^2}{(2\pi)^4}\frac{1}{2!}\int_{-\infty}^{\infty}\frac{T_{\mu\nu}(\vec{k},k_0)g_{\mu\nu}}{k^2-k_0^2-i\epsilon}d^4k\,, \tag{2.2}$$

where

$$T_{\mu\nu}(\vec{k},k_0) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\vec{k}\cdot(\vec{y}-\vec{x})}d^3x\,d^3y\int_0^{\infty} e^{ik_0x_0}dx_0\langle n|j_\mu(\vec{x},x_0)\sum_m|m\rangle\langle m|j_\nu(\vec{y},0)|n\rangle$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty} e^{i\vec{k}\cdot(\vec{y}-\vec{x})}d^3x\,d^3y\int_{-\infty}^{0} e^{ik_0x_0}dx_0\langle n|j_\nu(\vec{y},0)\sum_m|m\rangle\langle|j_\mu(\vec{x},x_0)|n\rangle+(\mu\leftrightarrow\nu)\,, \tag{2.3}$$

and $\epsilon_\mu \epsilon_\nu T_{\mu\nu}$ is the forward Compton scattering amplitude for the scattering of a virtual photon with three-momentum k and energy k_0, and polarization vector ϵ_μ from an atom in state n. Using $j_\mu(t) = e^{iH_a t} j_\mu(0) e^{-iH_a t}$, where H_a is the atomic Hamiltonian

$$\int_0^\infty dt = \int_{-\infty}^{+\infty} dt\, \theta(t),$$

$$\int_{-\infty}^0 dt = \int_{-\infty}^{\infty} dt\, \theta(-t),$$

and

$$\theta(t) = \frac{1}{2\pi i} \int \frac{e^{i\omega t}}{\omega - i\epsilon},$$

Eq. (2.3) can be reduced to

$$T_{\mu\nu}(\vec{k}, k_0) = \frac{-i}{2} \sum_m \int_{-\infty}^{+\infty} e^{i\vec{k}\cdot(\vec{y}-\vec{x})} d^3x\, d^3y \langle n|j_\mu(\vec{x})|m\rangle \langle m|j_\nu(\vec{y})|n\rangle \left(\frac{1}{E_m - E_n - k_0 - i\epsilon} + \frac{1}{E_m - E_n + k_0 - i\epsilon} \right) + (\mu \rightleftharpoons \nu) \,. \tag{2.4}$$

Let us now examine the poles in k_0 in the integrand in Eq. (2.2). The poles coming from the photon propagator are determined by the Feynman prescription (Fig. 1). The poles in the first term in (2.4) are all below the real axis and those in the second term are all above. Hence in each case, we close the k_0 contour accordingly, as in Figs. 2 and 3, and pick up only the contribution from the photon pole. Therefore, only Compton amplitude of on mass-shell photons enters. These two terms give equal contribution, and correspond to the crossed Born term in the on-shell forward Compton amplitude. The $(\mu \rightleftharpoons \nu)$ term also gives a factor of 2. Hence, after doing the k_0 integration, we obtain

$$\Delta M = \frac{-1}{(2\pi)^3} \int d^3k \frac{g_{\mu\nu}}{2|k|} \sum_m \int d^3x\, d^3y\, e^{i\vec{k}\cdot(\vec{y}-\vec{x})} \times \frac{\langle n|j_\mu(\vec{x})|m\rangle \langle m|j_\nu(\vec{y})|n\rangle}{E_m - E_n + |k| - i\epsilon}, \tag{2.5}$$

which is exactly what one gets from second-order perturbation theory, according to the interaction $\vec{j}\cdot\vec{A}$.

However, in the present work, we only evaluate the scattering amplitude in (2.5) nonrelativistically where a closed form exists and show how it leads to a finite Lamb shift with mass renormalization. Ambiguities that exist in the mass renormalization counter term in nonrelativistic calculations are discussed in Sec. III.

In the nonrelativistic limit, $g_{\mu\nu} j_\mu j_\nu$ in Eq. (2.5) is replaced by $\sum_\lambda \epsilon_i^\lambda \epsilon_j^\lambda (e/m)^2 p_i p_j$, which just comes from the interaction $-(e/m)\vec{p}\cdot\vec{A}$ taken to second order. However, it is also well known that in the nonrelativistic interaction Hamiltonian, there is also the $e^2A^2/2m$ term. This term gives rise to the process in Fig. 4, whose matrix element $\langle n|(e^2A^2/2m)|n\rangle$ is independent of the wave function of the particle. This graph is present in both cases of bound and free electrons. The net result is that it does not contribute to the energy shift after mass renormalization. Thus in subsequent discussions, we shall ignore this term. Also, to bring out the dependence on the state in question, we shall write W_n in place of ΔM in Eq. (2.5), and we have

$$W_n = \frac{1}{4\pi^2} \int k dk \sum_\lambda d\Omega_{\vec{k}}\, \mathfrak{M}_n(\lambda, \vec{k}, \omega), \tag{2.6}$$

where λ is the polarization mode, $\omega = |k|$ is the on-shell photon energy, and $\mathfrak{M}_n$ is the crossed forward-Compton-scattering amplitude in the given state n. $\mathfrak{M}_n$ can be written

$$\mathfrak{M}_n(\lambda, \vec{k}, \omega) = (e^2/m^2) \langle n|\vec{\epsilon}^{\,\lambda}\cdot\vec{p}\, e^{ikx} G \vec{\epsilon}^{\,\lambda}\cdot\vec{p} e^{-ikx}|n\rangle, \tag{2.7}$$

where the dipole approximation is not used. In Eq.

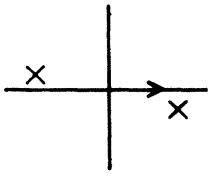

FIG. 1. Equivalent pole distribution in the photon propagator according to the Feynman contour.

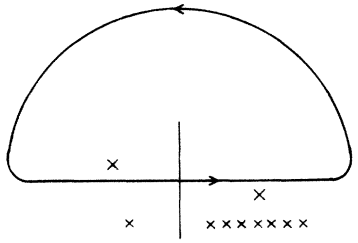

FIG. 2. k_0 contour for first integrand in Eq. (2.4).

(2.7), $G=(E-H+i\epsilon)^{-1}$ is the Coulomb Green's function; it can be written, in the momentum representation according to Schwinger,[5]

$$G(\vec{p}_2,\vec{p}_1,\Omega_n)=-\frac{m}{2\pi^3}X^2\left(\frac{i\epsilon^{i\pi\tau}}{2\sin\pi\tau}\right)\int_1^{0+}\rho^{-\tau}\frac{d}{d\rho}\left\{[(1-\rho^2)/\rho][X^2(\vec{p}_1-\vec{p}_2)^2+(p_1^2+X^2)(p_2^2+X^2)(1-\rho)^2/4\rho]^{-2}\right\}d\rho\ , \tag{2.8}$$

where

$$\tau=\lambda/X\ , \tag{2.9a}$$

$$X^2=-2m\Omega_n\ , \tag{2.9b}$$

$$\lambda=\alpha Zm\ , \tag{2.9c}$$

$$\alpha=e^2/\hbar c=e^2=\tfrac{1}{137}\ ,\quad \hbar=c=1\ , \tag{2.9d}$$

$$\Omega_n=E_n+\omega+i\epsilon \text{ for the direct Born graph,} \tag{2.9e}$$

$$\Omega_n=E_n-\omega+i\epsilon \text{ for the crossed Born graph.} \tag{2.9f}$$

Note that the $\Omega_{\vec{k}}$ in (2.6) is the solid angle. The $i\epsilon$ is inserted to prevent divergence at resonances. A similar calculation of the Lamb shift using similar techniques but without the inclusion of retardation has been carried out by Lieber.[6]

Since the photon momentum vector is to be averaged over angles in the calculation of the Lamb shift, we are allowed to replace tensor terms such as $\epsilon_i\epsilon_j$ and k_ik_j in the scattering amplitude by $\frac{1}{3}\vec{\epsilon}\cdot\vec{\epsilon}\,\delta_{ij}$ and $\frac{1}{3}k^2\delta_{ij}$ and for forward scattering $\vec{\epsilon}\cdot\vec{\epsilon}=1$. Thus, the summation over polarization modes reduces to a factor of 2 after the above modifications are made; W_n in Eq. (2.6) then becomes

$$W_n=\frac{2}{\pi}\int kdk\,\tilde{\mathfrak{M}}_n(k)\ , \tag{2.10}$$

where $\tilde{\mathfrak{M}}_n$ is the scattering amplitude $\mathfrak{M}_n$ with the above modifications. The problem thus reduces in part to the calculation of $\mathfrak{M}_n$. Calculations utilizing Coulomb Green's functions for Compton scattering amplitudes have been carried out by Gavrila[7] in the dipole approximation for the $1s$ hydrogenic state, by Gavrila and Costescu,[8] with retardation, on the same state, and also by Klarsfeld[9] in the dipole approximation for general s states in hydrogenic atoms. We have used techniques similar to those used by Gavrila and Costescu to obtain $\tilde{\mathfrak{M}}_{2s}$ and $\tilde{\mathfrak{M}}_{2p}$. We find

$$\tilde{\mathfrak{M}}_{2s}=\frac{-\lambda^2\alpha^2a}{(X+\beta)^2+k^2}\left(\frac{4[(X^2-\beta^2)+k^2]^2}{[(X+\beta)^2+k^2]^2}\,\frac{F(3;-1-\tau,5-\tau;u)}{(4-\tau)(3-\tau)(2-\tau)}+\frac{1}{2}\,\frac{F(1;-2-\tau,4-\tau;u)}{3-\tau}\right) \tag{2.11}$$

and

$$\tilde{\mathfrak{M}}_{2p}=\frac{-\lambda^2\alpha^2a}{(X+\beta)^2+k^2}\left\{\frac{u}{3}\,\frac{F(1;-1-\tau,3-\tau;u)}{(2-\tau)}-\frac{2}{3}\,\frac{[\beta^2+X^2+k^2]}{[(X+\beta)^2+k^2]}\,\frac{F(1;-1-\tau,4-\tau;u)}{(3-\tau)}+\frac{5}{6}\,\frac{F(1;-2-\tau,4-\tau;u)}{(3-\tau)}\right.$$

$$\left.+\frac{16k^2\beta^2}{3[(X+\beta)^2+k^2]^2}\,\frac{F(3;-1-\tau,5-\tau;u)}{(4-\tau)(3-\tau)(2-\tau)}+\frac{1}{12}\,\frac{F(1;-\tau,4-\tau;u)}{(3-\tau)}+\frac{1}{12}\,\frac{F(1;-2-\tau,2-\tau;u)}{(1-\tau)}\right\}, \tag{2.12}$$

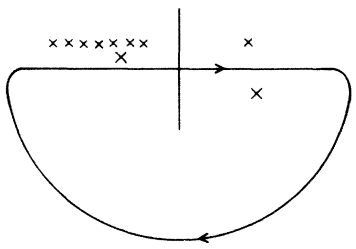

FIG. 3. k_0 contour for second integrand in Eq. (2.4).

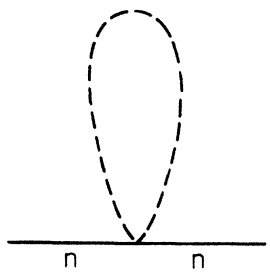

FIG. 4. Feynman diagram for process arising from the $e^2A^2/2m$ term in the nonrelativistic interaction Hamiltonian. The dashed line represents the photon.

where

$$a = \text{Bohr radius}, \tag{2.13}$$

$$\beta \doteq \tfrac{1}{2}\lambda, \tag{2.14}$$

$$u = [(X-\beta)^2 + k^2]/[(X+\beta)^2 + k^2], \tag{2.15}$$

and $F(\alpha;\beta,\gamma;z)$ is the Gauss type of hypergeometric function.[10]

The energy variable is contained exclusively in the factors τ and X. The k^2 terms come strictly from the higher multipoles. We therefore recover the dipole form of the scattering amplitude by putting k^2 equal to zero. However, by retaining the k^2 term, we see that the integrand in (2.6) is now less divergent, in fact, only logarithmically divergent. More interesting is that the $2s$ and $2p$ state have the same logarithmic divergence. We shall see in the next section that by including retardation, the leading divergence in the mass renormalization counter-term has identical logarithmic behavior.

III. CONVERGENCE AND RENORMALIZATION

It is both interesting and instructive to see why the nonrelativistic Lamb shift including retardation is convergent. While this fact follows from the asymptotic behavior of the integrands $\tilde{\mathfrak{M}}_{2s}$ and $\tilde{\mathfrak{M}}_{2p}$, it can be seen more readily from Eq. (2.7). We can write the Coulomb Green's function as an operator

$$G = (E - p^2/2m - V - k)^{-1}. \tag{3.1}$$

By a well-known identity

$$e^{i\vec{k}\cdot\vec{x}} G e^{-i\vec{k}\cdot\vec{x}} = [E - V - k - (\vec{p}-\vec{k})^2/2m]^{-1}. \tag{3.2}$$

The physical content of this is just that the inclusion of retardation in the photon emission and absorption imposes momentum conservation between the photon and the electron, changing the intermediate electron momentum to $\vec{p}-\vec{k}$. From the identity, we can see that for large k, $\tilde{\mathfrak{M}}_n$ behaves as $2m/k^2$, compared to the $1/k$ behavior of the nonretarded integral. This is sufficient to make the unrenormalized energy shift of each state, such as $2s$, only logarithmically divergent. Furthermore, for degenerate states such as $2s$ and $2p$, the coefficient of the logarithmic divergence are equal, as shown below, so that the Lamb shift is convergent without renormalization. However, the energy shift of an individual bound state such as $1s$, $2s$, or $2p$, which is logarithmically divergent, does require mass renormalization to be finite. We consider how to carry out that renormalization below.

It is worth noting that the extra convergence does not appear to occur in a relativistic calculation in which

$$G \sim (E - \vec{\alpha}\cdot\vec{p} - \beta m - V - k)^{-1}$$

and

$$e^{i\vec{k}\cdot\vec{x}} G e^{-i\vec{k}\cdot\vec{x}} \sim (E - \vec{\alpha}\cdot\vec{p} - \beta m - V - \vec{\alpha}\cdot\vec{k} - k)^{-1},$$

which still goes as $1/k$ for large k. The finiteness of the relativistic Lamb shift occurs for other reasons which we do not consider here.

In the relativistic theory of bound state energy shift, mass renormalization is accomplished by introducing a counter term $\delta m\bar{\psi}\psi$ into the Hamiltonian, and evaluating this term, together with the usual second order shift, between the bound states in question. The effect of the mass counter term is to cause the propagator of a dressed particle near the mass shell to differ by only a multiplicative constant from that of a bare free particle.[11] This can be accomplished in the relativistic theory because the electron propagator depends only on the Lorentz invariant combination $\not{p}$, and therefore once the electron is on the mass shell ($\not{p} = m$), the propagator has no further dependence on the momentum.

On the other hand, in the model we have been using, in which the electron is treated nonrelativistically and the photon relativistically, the dressed propagator depends both on the energy E and three-momentum squared p^2 of the particle. Therefore, even when the particle is put on the mass shell ($E = p^2/2m$), the propagator retains a complicated dependence on p. As a result, it is impossible by a simple mass renormalization, to make the propagator of a dressed particle agree with that of a bare particle, for all momenta on the mass shell.

It is possible, by a simple mass renormalization as first carried out by Bethe,[1] to make the dressed and bare propagator agree at small p^2. When this is done in the model neglecting retardation, the dressed propagator still has a pole at $E = p^2/2m$, for all value of p^2. On the other hand when retardation is included, the pole of the dressed propagator occurs at $E = f(p^2)$ where $f(p^2)$ is a complicated function which reduces to $p^2/2m$ only for small p^2. In order to make the dressed propagator have a simple pole at $E = p^2/2m$, when retardation is included, it would be necessary to add a counter-term with complicated dependence on p^2. This is equivalent to requiring that the self-energy of a free particle should vanish identically, and is not obviously justified. A more convincing way to solve this problem would be to treat the nonrelativistic model as a suitable limit of the relativistic theory where the normalization prescription is unambiguous. This we have not yet done. What we do here is to adopt the simple

mass renormalization prescription of Bethe, with the mass renormalization parameter δm evaluated so as to make the propagator, including the effect of retardation, agree at small p^2. It is not difficult to see that the result of this is to give δm by

$$\delta m = \frac{4\alpha}{3\pi}\int_0^\infty \frac{k\,dk}{k + k^2/2m},$$

which is logarithmically rather than linearly divergent. The counter term that must be added to the Hamiltonian to give the proper mass renormalization is

$$(\delta m/m)p^2/2m$$

so that the correct energy shift is given by adding this to W_n given by Eq. (2.10). Now the asymptotic behavior of the integral $\tilde{\mathfrak{M}}_n(k)$ in Eq. (2.6) can be seen to be

$$\tilde{\mathfrak{M}}_{2s} \xrightarrow[k\to\infty]{} -\frac{Z^2\alpha^3 m}{6k^2}$$

and

$$\tilde{\mathfrak{M}}_{2p} \xrightarrow[k\to\infty]{} -\frac{Z^2\alpha^3 m}{6k^2}$$

from the properties of the Gaussian hypergeometric functions.[10] Thus,

$$W_n \sim \int \frac{-\alpha^3 m Z^2}{3\pi}\frac{dk}{k}$$

for $2s$ and $2p$, where only the asymptotic contribution is written, i.e., the divergent contributions are equal. On the other hand, the quantities $(\delta m/m)p^2/2m$ for these states are also equal by the virial theorem

$$\langle p^2/2m\rangle_n = -E_n = \tfrac{1}{8}Z^2\alpha^2 m$$

for each state, and $\delta m/m$ is a number. The renormalized energy shift of either state, given by

$$W_n^{\text{ren}} = \frac{2}{\pi}\int_0^\infty k\,dk\,\tilde{\mathfrak{M}}_n + \frac{\alpha^3}{6\pi}Z^2\int_0^\infty \frac{k\,dk}{k + k^2/2m}$$

is now clearly finite because the asymptotic form of the terms with $\tilde{\mathfrak{M}}$ is equal but with opposite sign to the δm term.

Another consequence of these considerations is that if we are interested only in the energy difference of the $2s$ and $2p$ states, it is unnecessary to perform any mass renormalization, because this will cancel between the two states anyway. This is also true in the absence of retardation, but then the energy difference is still logarithmically divergent, while here it is finite.

If we instead adopt the prescription of adding a momentum dependent contribution to the Hamiltonian in order to make $\Delta E = 0$ for a free particle, we still obtain a finite result for W_n^{ren}, but this result differs somewhat from the result with the sim-

TABLE I. Numerical integration for $W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}}$.[a]

	Retarded calculation		Nonretarded calculation	
k	Integrand	Lamb shift	Integrand	Lamb shift
0.1	−0.923 02(−2)	−0.582 86(0)	−0.923 02(−2)	−0.582 86(0)
0.5	0.972 61(−1)	0.121 49(1)	0.972 63(−1)	0.121 52(1)
1.0	0.303 52(−1)	0.406 99(2)	0.303 54(−1)	0.407 00(2)
5.0	0.814 63(−2)	0.130 92(3)	0.814 87(−2)	0.130 94(3)
1.0(1)	0.474 09(−2)	0.180 79(3)	0.474 40(−2)	0.180 83(3)
5.0(1)	0.122 93(−2)	0.324 70(3)	0.123 39(−2)	0.325 00(3)
1.0(2)	0.660 66(−3)	0.396 99(3)	0.665 68(−3)	0.397 68(3)
b 1.4(2)	0.484 92(−3)	0.433 82(3)	0.490 13(−3)	0.434 85(3)
2.0(2)	0.347 46(−3)	0.473 85(3)	0.352 86(−3)	0.475 39(3)
5.0(2)	0.143 53(−3)	0.579 95(3)	0.149 26(−3)	0.584 24(3)
1.0(3)	0.710 92(−4)	0.661 06(3)	0.769 07(−4)	0.670 08(3)
5.0(3)	0.109 79(−4)	0.832 38(3)	0.160 51(−4)	0.877 19(3)
1.0(4)	0.394 47(−5)	0.886 24(3)	0.811 16(−5)	0.968 57(3)
c 1.9(4)	0.122 41(−5)	0.919 10(3)	0.430 02(−5)	0.105 39(4)
5.0(4)	0.119 15(−6)	0.940 50(3)	0.164 63(−5)	0.118 36(4)
1.0(5)	0.139 20(−7)	0.944 19(3)	0.826 08(−6)	0.127 70(4)
2.0(5)	0.106 84(−8)	0.944 93(3)	0.414 11(−6)	0.137 07(4)
d ∞	...	0.9503(3)	...	...

[a] k is given in units of α/a. Integrand is $k(\tilde{M}_{2s} - \tilde{M}_{2p})$ in units such that $\alpha/a = 1$. Lamb shift is given in MHz. Numbers in parentheses indicate powers of 10 to be multiplied.

[b] Where $ka \sim 1$, i.e., where retardation starts to be significant.

[c] Where Bethe introduces his cutoff. Note that we recover the 1053.9 MHz in our nonretarded calculation.

[d] Estimated by assuming integrand scales as $1/k^2$.

ple mass renormalization. The details are given in the Appendix. Numerical results are given in Sec. IV.

IV. NUMERICAL RESULTS

In this section we present numerical results for $W_{2s}^{\rm ren}$ and $W_{2s}^{\rm ren}-W_{2p}^{\rm ren}$, where in the latter case, the mass renormalization terms cancel because of the virial theorem as shown in Sec. III. In order to illustrate the effect of the inclusion of retardation, we give in Table I the value of the integrand in $W_{2s}^{\rm ren}-W_{2p}^{\rm ren}$ (apart from constant factors) as a function of k, as well as the would be value of $W_{2s}^{\rm ren}-W_{2p}^{\rm ren}$ in MHz, had there been an ultraviolet photon cutoff at k, under the column "Lamb shift," for both cases with and without retardation. The numerical integration is done by trapezoidal rule up to a cutoff about ten times the electron mass. The contribution from frequencies higher than this is estimated by assuming that the integrand scales simply as $1/k^2$. The table illustrates where retardation becomes significant and where contribution to the energy shift is maximum, facts which help explain why Bethe's answer is so surprisingly close to the actual value, as will be discussed in the conclusion of this paper.

We note that with a cutoff at the electron mass we do obtain in our nonretarded calculation $W_{2s}^{\rm ren}=1056.7$ MHz, and $W_{2s}^{\rm ren}-W_{2p}^{\rm ren}=1053.9$ MHz, which is in close agreement with Bethe.[1] In the retarded calculation, we obtain

$$W_{2s}^{\rm ren}=931.1\ \text{MHz}\,,\quad W_{2s}^{\rm ren}-W_{2p}^{\rm ren}=950.3\ \text{MHz}\,.$$

In the Appendix, we discuss an alternative approach to treat the mass renormalization term. There we obtain

$$\tilde{W}_{2s}^{\rm ren}=1330\ \text{MHz}\,,\quad \tilde{W}_{2s}^{\rm ren}-\tilde{W}_{2p}^{\rm ren}=996.6\ \text{MHz}\,.$$

V. CONCLUSION

We have shown that the inclusion of higher multipole effects in the calculation of the nonrelativistic Lamb shift produces a finite answer for the energy shift of the hydrogen bound states, without the introduction of an ultraviolet cutoff as in the calculation of Bethe. The modification that is introduced in the integrand of the integral over photon energy by higher multipole effects become significant in the range $Z\alpha m<k<m$, whereas the dominant contribution of the nonrelativistic calculation to the Lamb shift (the $\ln Z\alpha$ term) comes from $k\sim(Z\alpha)^2 m$. Therefore, it may be argued that the surprisingly accurate agreement of Bethe's answer with experiment is a consequence of two facts. In the region of maximum contribution, the integrand with and without cutoff do not sensibly differ, and in the region where Bethe introduces a cutoff, the retardation effects produces an effective cutoff anyway.

Apart from whatever light our calculation may shed on this point, we wish to emphasize that retardation effects evidently play a sizable role in the numerical value of the Lamb shift even in hydrogen, and would contribute much more significantly in heavy hydrogenic ions, where $Z\alpha\sim(Z\alpha)^2\sim 1$. In view of experimental interest in such ions,[12] it would appear worthwhile to examine the contribution of retardation effects there in details. Of course, this must be done within a relativistic theory, since $(v/c)^2\sim(Z\alpha)^2$ also. It may, however, be feasible to use the technique of the Coulomb Green's function in that case also,[13] or at least to carry out the low-frequency part of the calculation in that way. We hope to return to that question elsewhere.

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APPENDIX: ALTERNATIVE APPROACH FOR MASS RENORMALIZATION

As discussed in Sec. III, a second alternative for the mass renormalization counter term is momentum dependent and makes $\Delta E=0$ for a free particle. In such cases, the counterterm would be given by

$$\Delta E'=\frac{2\alpha}{3\pi}\frac{p^2}{2m^2}\int\frac{k\,dk\,d\cos\theta}{k^2/2m+k-pk\cos\theta/m}\tag{A1}$$

as compared to $(p^2/2m)\delta m/m$, where

$$\delta m=\frac{4\alpha}{3\pi}\int_0^\infty\frac{k\,dk}{k+k^2/2m}$$

as given in Sec. III. The difference between these two forms of counter-terms is equal to $(p^2/2m^2)\times F(p)$, where

$$\begin{aligned}F(p)&=\frac{2\alpha}{3\pi}\int\frac{k\,dk\,d\cos\theta}{k^2/2m+k-pk\,d\cos\theta/m}-\frac{2\alpha}{3\pi}\int_0^\infty\frac{k\,dk}{k^2/2m+k}\\&=\frac{4\alpha m^2}{3\pi p}\left[\left(1-\frac{p}{m}\right)\ln\left(1-\frac{p}{m}\right)\right.\\&\qquad\left.-\left(1+\frac{p}{m}\right)\ln\left(1+\frac{p}{m}\right)+\frac{2p}{m}\right]\equiv\frac{4\alpha m^2}{3\pi p}G(p)\,.\end{aligned}\tag{A2}$$

This would give an additional contribution to the energy shift

$$\Delta E_n'=\langle n|[F(p)/m](p^2/2m)|n\rangle=\langle n|(2\alpha/3\pi)pG(p)|n\rangle\,.\tag{A3}$$

In particular, for the $2s$ and $2p$ states in hydrogen, we have

$$\Delta E'_{2s} = 399 \text{ MHz}, \tag{A4}$$

$$\Delta E'_{2p} = 352 \text{ MHz}. \tag{A5}$$

Thus,

$$\Delta E'_{2s} - \Delta E'_{2p} = 47 \text{ MHz}; \tag{A6}$$

so

$$\tilde{W}^{\text{ren}}_{2s} = W^{\text{ren}}_{2s} + \Delta E_{2s} = 1330 \text{ MHz} \tag{A7}$$

and

$$\tilde{W}^{\text{ren}}_{2s} - \tilde{W}^{\text{ren}}_{2p} = 996.6 \text{ MHz} \tag{A8}$$

in such an approach.

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