Instabilities in continuous-wave light propagation in absorbing media

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The stability of a monochromatic constant-intensity laser beam propagating through a resonant medium is considered. It is found that many absorbers should exhibit a region of negative conductivity and, consequently, amplify perturbations. The associated gain is dependent on a number of variable parameters, such as laser-beam intensity, perturbation frequency, relaxation times, degeneracy, transverse mode, inhomogeneous broadening, etc. Putting absorbers inside Fabry-Perot interferometers allows the construction of plane-wave devices with a bistable output.

INTRODUCTION

The development of the laser has stimulated the study of nonlinear resonant light propagation in absorbing media. In addition to the 2π hyperbolic-secant pulse,[1] periodic frequency- and amplitude-modulated electric field pendulum-type solutions[2] satisfy the idealized plane-wave undamped Bloch-Maxwell equations.

Considered here is the stability of the steady-state solution to the Bloch-Maxwell equations with damping. The steady-state solution is found to be unstable and to exhibit gain when the light intensity is high. The unstable perturbations modulate the amplitude, but not the frequency.

The pendulum-type solutions might be unstable and evolve to some other solution, but the other solution cannot be a high-intensity steady-state solution, because it is itself unstable.

The instability of the steady-state solution implies that a resonant medium with a light beam of sufficient intensity passing through it can be used as a light amplifier. It is mainly for this reason that the regime and degree of instability is here examined in some detail.

Schwartz and Tan[3] considered the problem of a strong light beam and weak probe light beam passing through a resonant medium. They found that the weak probe beam underwent reduced absorption, particularly when the difference in frequency of the strong and weak beam was less than the inverse population lifetime. Mollow[4] pointed out that a homogeneously broadened two-level system in the presence of a strong light beam is unstable and amplifies fluctuations. Since this work was discovered during final manuscript preparation, no attempt has been made to reconcile notation, etc., and the reader is encouraged to read his work. Previous work is otherwise well referenced in the paper by Bloembergen and Shen.[5]

The gain or instability of the steady-state solution with respect to amplitude modulations of small frequency may be understood through general arguments. Let a resonant medium of finite length absorb an energy loss L from the energy of an incident light beam. As the incident light intensity is increased towards very large values, suppose that L approaches a given constant loss L_0. For example, if the only loss mechanism is that of spontaneous emission at the vacuum rate of quanta from the partially excited atoms, then L_0 equals the product of the two-level energy separation, half the number of atoms, and the inverse spontaneous-emission lifetime.

The loss L may be set equal to a product of the light's electric field ϵ and the total atomic current J. It then follows that $J = L_0/\epsilon$ in the region of large intensity (proportional to ϵ^2). The differential conductivity $dJ/d\epsilon$ is therefore negative in this large-field region. Slow amplitude-modulating perturbations of the electric field ϵ are therefore amplified.

A careful distinction between field and intensity should be made. Suppose the unperturbed field is ϵ_0, and the perturbing field is $\Delta\epsilon$. According to the previous paragraph, one may assume an apparatus where $\Delta\epsilon$ is amplified. The light intensity is proportional to $(\epsilon_0 + \Delta\epsilon)^2$. The intensity-modulation part of this is $2\epsilon_0\Delta\epsilon + \Delta\epsilon^2$. It may be that $\Delta\epsilon$ is amplified, but ϵ_0 is, at the same time, attenuated in such a way that the intensity modulation is decreased.

In such a situation observation of the total intensity with, for example, a phototube will reveal a decrease in the intensity modulation. On the other hand, if a frequency-sensitive filter (e.g., Fabry-Perot interferometer) is used to subtract the field ϵ_0 from the exit beam, the intensity of the modulating field, $\Delta\epsilon^2$, will be observed to be increased.

The absorber will be described through Bloch's equations with relaxation times T_1, T_2, T_2^*, with

T_2^* referring to a Gaussian spectral profile. The comparison of observed gain with that calculated with this or improved models may serve to provide a means of distinguishing between various models of the dynamics of, for example, collisional relaxation in gases.

The simplest case of $T_2^* = \infty$ (the homogeneously-broadened-line case) is considered first. Extension is then made to the case of a mixture of homogeneous and inhomogeneous broadening. It is found that the averaging effects of a transverse mode or degeneracy typically only decrease somewhat the maximum gain or instability. Inclusion of an absorbing medium in a Fabry-Perot structure allows the construction of plane-wave devices that have arbitrarily large differential gain, or have a bistable input-output relationship.

I. HOMOGENEOUSLY BROADENED LINE

In this section, the homogeneous-line case is examined. At any point in the absorber, the plane-wave electric field E is given by[6]

$$E(z,t) = \epsilon(z,t)e^{-i(kz-\omega t)} + \text{c.c.}, \tag{1}$$

where z and t are space and time coordinates, ω is a central fixed frequency, $k = \eta\omega/c$ where η is a background refractive index, c is the speed of light in vacuum, and the electric field envelope ϵ is complex in the case of frequency modulation. The letters c.c. denote the complex-conjugate term.

Two-level atoms without level degeneracy interact with the electric field according to the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\hbar w_0 \sigma_z - pE\sigma_x , \tag{2}$$

where σ_x, σ_y, σ_z are the usual Pauli spin matrices, p is a dipole-moment matrix element, and frequency $\omega_0 = \omega$. The electric field is tuned to the absorber line center.

In the slowly-varying-electric-field-envelope approximation,[6] and in the frame rotating according to a phase factor $e^{-i\omega t + ikz}$, the atom's behavior may be described through Bloch's equations:

$$\dot{v} = -w\kappa\epsilon_R - v/T_2' , \tag{3a}$$

$$\dot{u} = w\kappa\epsilon_I - u/T_2' , \tag{3b}$$

$$\dot{w} = v\kappa\epsilon_R - u\kappa\epsilon_I - (w+1)/T_1 , \tag{3c}$$

where Q $(= u + iv)$ and w are the expectation values of $\frac{1}{2}(\sigma_x - i\sigma_y)e^{+i\omega t - ikz}$ and σ_z, respectively, u and v are real, T_1 and T_2' damping terms have been added, and $\kappa = 2p/\hbar$, where $\hbar$ is Planck's constant divided by 2π, so that $\kappa\epsilon$ is the Rabi frequency. Here $\epsilon_R + i\epsilon_I = \epsilon$, with ϵ_R and ϵ_I real quantities.

Substitution into Maxwell's equations, using the slowly-varying-envelope approximation, leads to

$$\frac{\partial\epsilon}{\partial z} + \frac{\eta}{c}\frac{\partial\epsilon}{\partial t} = i\,\frac{2\pi\omega N p}{\eta c}\,Q . \tag{4}$$

Let u_0, v_0, w_0, ϵ_0 represent the steady-state solution, with $\dot{u}_0 = \dot{v}_0 = \dot{w}_0 = \dot{w}_0 = \dot{\epsilon}_0 = 0$. Then ϵ_0 may be chosen real, and the solutions to Bloch's equations are

$$u_0 = 0, \tag{5}$$

$$v_0 = T_2'\kappa\epsilon_0(1 + F^2)^{-1}, \tag{6}$$

$$w_0 = -(1 + F^2)^{-1}, \tag{7}$$

where

$$F = (T_1 T_2')^{1/2}\kappa\epsilon_0 \tag{8}$$

is a dimensionless electric field. Introducing $a = T_1/T_2'$, where $a \geq \frac{1}{2}$, one may write

$$v_0 = a^{-1/2}F(1 + F^2)^{-1}. \tag{9}$$

Figure 1 presents v_0 and w_0 as functions of F for various values of a. Notice that an increase in F causes a decrease in v_0 whenever $F > 1$. This implies that low-frequency modulation of ϵ_0 will experience gain, and the steady-state solution will be unstable.

Specifically, let

$$\epsilon = \epsilon_0 + \Delta\epsilon, \tag{10}$$

where ϵ_0 and $\Delta\epsilon$ are real, and let

$$\Delta\epsilon = \text{Re}\,\Delta\mathcal{E}\,e^{i\nu t} , \tag{11}$$

where, for the moment, $\nu \ll \kappa\epsilon, T_1^{-1}, T_2'^{-1}$, so that

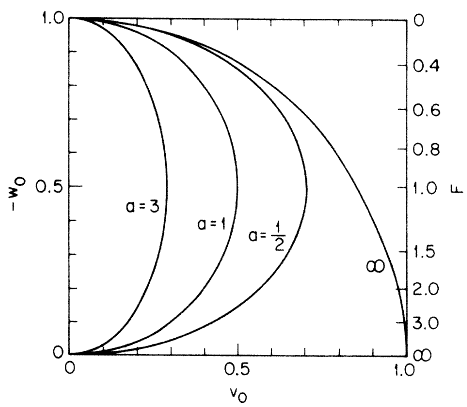

FIG. 1. $w < 0$, $v > 0$ quadrant of the Bloch circle $w^2 + v^2 = 1$. The circle represents the motion of an undamped Bloch vector subject to an electromagnetic field which is pulsed with temporal width $\ll T_2'$. The ellipses represent the steady-state solutions given by Eqs. (5)–(7). Note that a differential increase in F, for $F > 1$, leads to a decrease in v_0, and consequently to instability and differential gain.

the steady-state solution may be used. Then Eqs. (9) and (4) lead to

$$\frac{\partial(\epsilon_0+\Delta\epsilon)}{\partial z}+\frac{\eta}{c}\frac{\partial(\epsilon_0+\Delta\epsilon)}{\partial t} = -\tfrac{1}{2}\alpha(\epsilon_0+\Delta\epsilon)\{1+[F+(T_1T_2')^{1/2}\kappa\Delta\epsilon]^2\}^{-1}, \quad (12)$$

where $\alpha=8\pi T_2' Np^2/\eta\hbar c$; thus for small steady-state intensities in the region of linear absorption,

$$\epsilon^2(z)=e^{-\alpha z}\epsilon^2(0). \quad (13)$$

Subtracting terms describing the constant field ϵ_0 and neglecting terms higher than linear in $\Delta\epsilon$,

$$\frac{\partial\Delta\epsilon}{\partial z}+\frac{\eta}{c}\frac{\partial\Delta\epsilon}{\partial t}=-\frac{\alpha}{2}\frac{1-F^2}{(1+F^2)^2}\Delta\epsilon. \quad (14)$$

The saturated loss factor

$$\tfrac{1}{2}\alpha(1-F^2)(1+F^2)^{-2} \quad (15)$$

changes sign when F varies through 1, with $F>1$ representing gain or instability.

The gain

$$G=\frac{\alpha}{2}\frac{F^2-1}{(F^2+1)^2} \quad (16)$$

is also a function of a and frequency ν. If the restrictions previously placed on ν are lifted, the perturbative solutions to Bloch's equations are determined by

$$\Delta v=T_2'\,\mathrm{Re}\Delta\mathcal{E}e^{i\nu t}\frac{(1+iax-F^2)}{(1+iax)(1+ix)+F^2}, \quad (17)$$

where $x=T_2'\nu$.

The procedure used to determine Eq. (17) is given later in more detail. A complex gain may be defined

$$G(x,a,F)=\frac{\alpha}{2}\frac{F^2-(1+iax)}{F^2+(1+iax)(1+ix)}, \quad (18)$$

so that

$$\frac{\partial\Delta\mathcal{E}}{\partial z}+i\frac{\eta\nu}{c}\Delta\mathcal{E}=G\Delta\mathcal{E}. \quad (19)$$

A complete solution to Eq. (19) would involve the solution to Eq. (4) with $\Delta\epsilon=0$, using Eq. (6), to find F as a function of distance z. That result and Eq. (17) then allow a formal solution to Eq. (19). The information of interest is primarily contained in Eq. (18), however, so that Eq. (19) or later similar equations are not solved explicitly.

II. EXTENSION TO INCLUDE INHOMOGENEOUS BROADENING

Often optical-absorption lines are inhomogeneously broadened, perhaps owing to a range of Doppler velocity shifts or a variation in static local crystalline-field potentials. In such a case, one may not correctly speak of a single homogeneously broadened absorption line, but instead must include a parameter $\Delta\omega$, denoting the difference in frequency between the applied field and a given homogeneously broadened isochromat, in Bloch's equations. After solving Bloch's equations, the macroscopic polarization envelope is found by summing the contribution from each isochromat. Bloch's equations are now written[6]

$$\dot u=-\Delta\omega v+w\kappa\epsilon_I-u/T_2', \quad (20a)$$

$$\dot v=\Delta\omega u-w\kappa\epsilon_R-v/T_2', \quad (20b)$$

$$\dot w=v\kappa\epsilon_R-u\kappa\epsilon_I-(w+1)/T_1, \quad (20c)$$

and Maxwell's equation becomes

$$\frac{\partial\epsilon}{\partial z}+\frac{\eta}{c}\frac{\partial\epsilon}{\partial t}=i\frac{2\pi\omega Np}{\eta c}\int g(\Delta\omega)\,d\Delta\omega\,Q(\Delta\omega), \quad (21)$$

where $g(\Delta\omega)$ is the inhomogeneous distribution function, with

$$\int g(\Delta\omega)\,d\Delta\omega=1. \quad (22)$$

The steady-state solution is given by

$$u_0(\Delta\omega)=T_2'\epsilon_0 y(1+y^2+F^2)^{-1}, \quad (23a)$$

$$v_0(\Delta\omega)=T_2'\epsilon_0[1+y^2+F^2]^{-1}, \quad (23b)$$

$$w_0(\Delta\omega)=-(1+y^2)[1+y^2+F^2]^{-1}, \quad (23c)$$

where $y=T_2'\Delta\omega$, and ϵ_0 is chosen to be real and constant in time. If ϵ now is chosen to have a constant part ϵ_0 and a small part $\Delta\epsilon=\Delta\epsilon_R+i\Delta\epsilon_I$, Bloch's equations may be linearized with respect to $\Delta\epsilon$ to find

$$\frac{d}{dt}\Delta u=\Delta\omega\Delta v-T_2'^{-1}\Delta u+w_0\kappa\Delta\epsilon_I, \quad (24)$$

$$\frac{d}{dt}\Delta v=-\Delta\omega\Delta u-T_2'^{-1}\Delta v-\Delta w\kappa\epsilon_0-w_0\kappa\Delta\epsilon_R, \quad (25)$$

$$\frac{d}{dt}\Delta w=-T_1^{-1}\Delta w+\kappa\epsilon_0\Delta v+v_0\kappa\Delta\epsilon_R-u_0\kappa\Delta\epsilon_I. \quad (26)$$

Now u_0 is an odd function of $\Delta\omega$, and v_0 and w_0 are even functions of $\Delta\omega$. The Eqs. (24)–(26) therefore specify that the term in Δu proportional to $\Delta\epsilon_R$ is odd in $\Delta\omega$, and the term in Δv proportional to $\Delta\epsilon_I$ is also odd in $\Delta\omega$. Consequently, if $g(\Delta\omega)$ is symmetric in $\Delta\omega$, as will be the case here, Eq. (21) decouples into three equations, one involving only ϵ_0, one involving $\Delta\epsilon_R$ and Δv, and one involving $\Delta\epsilon_I$ and Δu. The amplitude-modulation case $\Delta\epsilon_I = 0$ will be considered first.

Let $\Delta\epsilon_I = 0$, and

$$\Delta\epsilon_R = \mathrm{Re}\Delta\mathcal{E}e^{i\nu t} . \tag{27}$$

Equations (25) and (26) then yield

$$\Delta v(\Delta\omega) = -T_2' \mathrm{Re}G(\Delta\omega, \nu)\Delta\mathcal{E}e^{i\nu t}, \tag{28}$$

where

$$G(\Delta\omega, \nu) = \frac{(1+ix)F^2 - (1+ix)(1+iax)(1+y^2)}{(1+y^2+F^2)[(1+ix)^2(1+iax)+(1+iax)y^2+(1+ix)F^2]} \tag{29}$$

and $y = T_2'\Delta\omega$ and $x = T_2'\nu$. The net gain for symmetrical $g(\Delta\omega)$ is therefore

$$G(\nu, F, a, g(\Delta\omega)) = \frac{\alpha}{2}\,\frac{\int g(\Delta\omega)G(\Delta\omega,\nu)\,d\Delta\omega}{-\int g(\Delta\omega)G(\Delta\omega,0)\,d\Delta\omega|_{F=0}}, \tag{30}$$

where α is the linear-absorption constant, so that weak fields ϵ_0 behave as $|\epsilon_0(z)|^2 = e^{-\alpha z}|\epsilon_0(0)|^2$, and

$$\left(\frac{\partial}{\partial z} + \frac{i\eta\nu}{c}\right)\Delta\mathcal{E} = G\Delta\mathcal{E}. \tag{31}$$

The choice $g(\Delta\omega) \propto e^{-T^2y^2}$ invites use of the complex function[7]

$$S(z) = e^{z^2}\,\mathrm{erfc}(z), \tag{32}$$

where $\mathrm{erfc}(z) = \int_z^{\infty+i0} e^{-r^2}dr$ is the complex complementary error function.

If $g(\Delta\omega) \propto e^{-T^2y^2}$, Eq. (30) reduces to

$$G(\nu, F, a, T) = \tfrac{1}{2}\alpha[A\gamma^{-1/2}S(\gamma^{1/2}T) + B\delta^{-1/2}\varepsilon^{-1/2} \times S(\delta^{1/2}\varepsilon^{-1/2}T)]S(T)^{-1}, \tag{33}$$

where

$$A = (\mu+\beta\gamma)(\delta-\gamma\varepsilon)^{-1}, \quad B = -\beta - \varepsilon A, \quad \varepsilon = 1 + iax,$$

$$\beta = (1+ix)\varepsilon, \quad \mu = F^2(1+ix)^2 - \beta, \quad \gamma = 1 + F^2,$$

$$\delta = (1+ix)(\beta + F^2).$$

In order to describe the amplitude-modulation gain function, some simplification is required. Only the real part of the function will be described, with the observation that the imaginary part of G (or an exponential of G) is determined through the knowledge that complex G is a causal function and obeys a Kramers-Kronig relation.

Figure 2 displays ReG as a function of x for various F, a, and T. Of the apparent general features, attention is brought to the decrease in gain (or increase in loss) when ν varies from zero to T_1^{-1}. This feature was discussed with respect to reduced absorption by Swartz and Tan.[3] It should be noted that, at least for values of a in the region of unity, ReG is approximately constant out to values $\nu \approx \kappa\epsilon_0$, where some peaking occurs. Furthermore, the case $T = 0$, which never has a zero-frequency instability, does develop regions near $\nu = \kappa\epsilon_0$ of gain or instability for relatively large values of F.

Only regions for which Re$G > 0$ will now be considered, and description will be given in terms of the value of ReG at $\nu = 0$, and in terms of $\nu_{\max}$, where $\nu_{\max}(F, a, T)$ is defined by $\mathrm{Re}G(\nu_{\max}, F, a, T) = 0$.

Equation (33) becomes, in the limit $\nu = 0$,

$$\mathrm{Re}G(0,F,a,T) = G(0,F,a,T) = \tfrac{1}{2}\alpha S\left\{\frac{2TF^2}{[\pi(1+F^2)]^{1/2}} - \left(2T^2F^2 + \frac{1}{1+F^2}\right) \times \frac{S(T(1+F^2)^{1/2})}{(1+F^2)^{1/2}}\right\} S(T)^{-1}, \tag{34}$$

which is independent of a. For large values of F,

$$G(0,F,T) \cong \frac{\frac{1}{2}\alpha}{T\sqrt{\pi}\,S(T)}F^{-2}. \tag{35}$$

One may introduce a nonlinear loss coefficient α_{NL} defined by

$$\alpha_{\mathrm{NL}} = -\frac{2}{\epsilon_0}\frac{d\epsilon_0}{dz}, \tag{36}$$

which, using Eqs. (21) and (23b), may be found to be given by

$$\alpha_{\mathrm{NL}} = \alpha\frac{S(T(1+F^2)^{1/2})}{(1+F^2)^{1/2}S(T)}, \tag{37}$$

which, for large F, becomes

$$\alpha_{\mathrm{NL}} \approx \frac{\alpha}{T\sqrt{\pi}\,S(T)}F^{-2}, \tag{38}$$

which is twice the expression in Eq. (35). The factor 2 is not restricted to the particular case discussed here, but follows from the assumption that the losses in erg/cm^3 sec at high fields are independent of field strength. The same assumption implies the F^{-2} behavior of Eqs. (38) and (35).

In Fig. 3, $\alpha_{\mathrm{NL}}/\alpha$ is given as a function of F for various T. $G(0,F,T)/\alpha$ is given in Fig. 4, and $G(0,F,T)/\alpha_{\mathrm{NL}}$ is given in Fig. 5.

The bandwidth function $\nu_{\max}(F, a, T)$ is given in Fig. 6. The bandwidth $\nu_{\max}$ approaches the Rabi

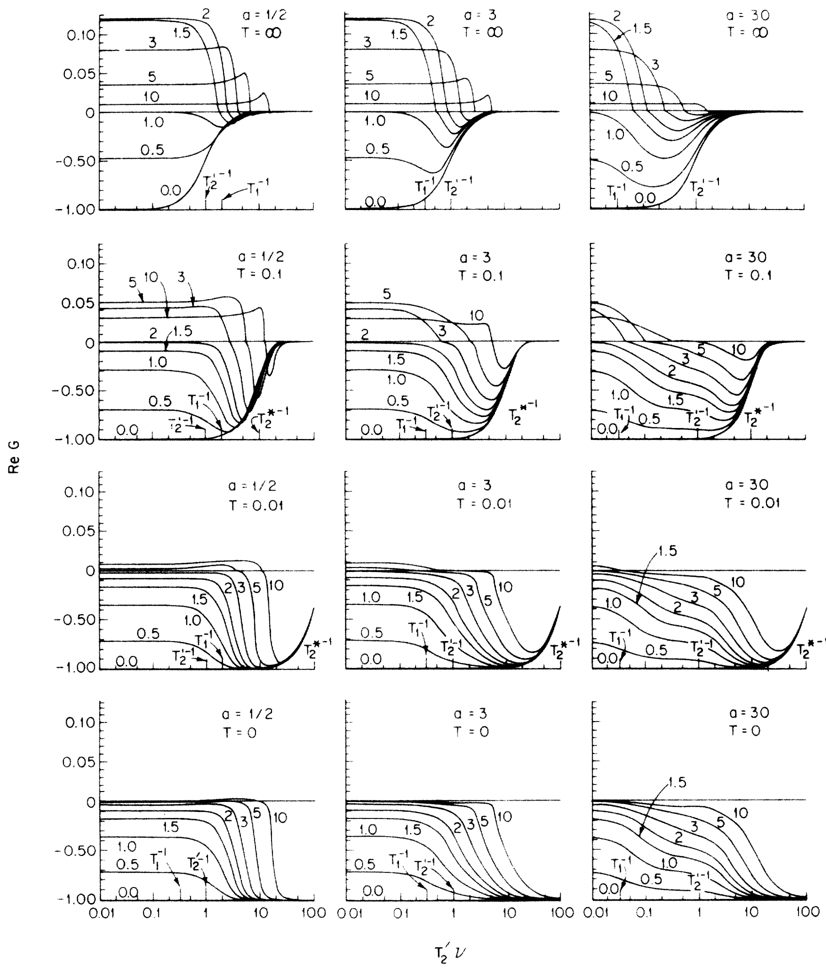

FIG. 2. ReG as a function of ν and F for various T_1/T_2' ratios a, and various inhomogeneous width parameters T. The numbers 0.0, 0.5, 1.0, ..., 10 represent the value of F for each curve. Note the scale change for Re$G > 0$. As is generally the case, absorber results may be carried over to the amplifying medium results through a change of sign. Thus, in the $a = \frac{1}{2}$, $T = \infty$ case the steady-state solution for $F = 1.5$ would admit of amplification of an additional field which amplitude modulates at frequency $1.5T_1^{-1}$, but would attenuate a similar field at frequency $0.5T_1^{-1}$. Frequencies ν are angular frequencies. The case $a = \frac{1}{2}$, $T = 0$ results in gain at frequencies near $\kappa\epsilon$, but never at zero frequency.

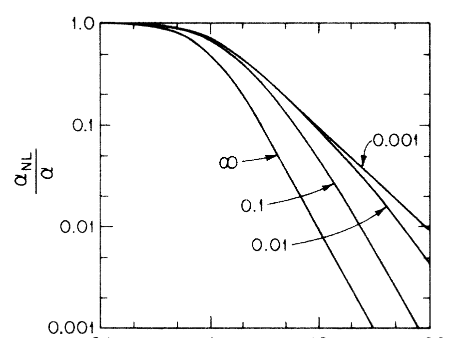

FIG. 3. Ratio α_{NL}/α of nonlinear absorption constant to weak-field absorption constant. The numbers labeling various curves refer to different values of T. For $T < 0.001$ and $F < 100$, $\alpha_{NL}/\alpha \approx \alpha/(1+F^2)^{1/2}$. The independence of α_{NL}/α in the case of small T is due to "hole burning," and the presence of spectral diffusion (e.g., velocity-changing collisions) would considerably change results in the small-T region.

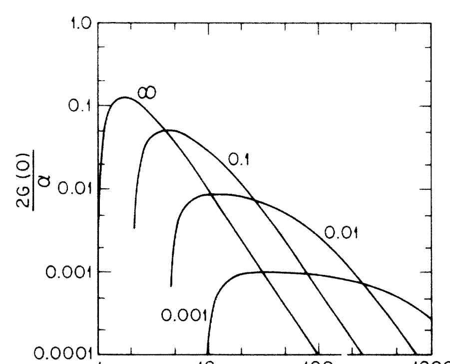

FIG. 4. ReG at zero frequency, $\nu = 0$. Plotted is $2G/\alpha$ for the zero-frequency case as a function of F. The figure is limited to the case Re$G > 0$. The numbers ∞, 0.1, etc., refer to values of T.

frequency $\kappa\epsilon_0$ as the field ϵ_0 becomes large.

The case of pure frequency modulation, $\Delta\epsilon_R = 0$, leads to

$$\Delta u = +T_2' \,\mathrm{Re}G_{\mathrm{FM}}(\Delta\omega, \nu)\Delta\epsilon' e^{i\nu t}, \tag{39}$$

where $\Delta\epsilon = i\,\mathrm{Re}\Delta\epsilon' e^{i\nu t}$. Equation (21) then specifies that an instability can exist only if $\mathrm{Re}G_{\mathrm{FM}}(\Delta\omega,\nu) > 0$ for some $\Delta\omega$, ν, and F. Use of Eqs. (24)–(26) allows G_{FM} to be expressed as

$$G_{\mathrm{FM}}(\Delta\omega, \nu) = -\frac{(1+y^2)(1+ix)(1+iax)+F^2}{(1+y^2+F^2)[(1+ix)^2(1+iax)+(1+iax)y^2+(1+ix)F^2]}, \tag{40}$$

which has a nonpositive real part, so that frequency-modulating perturbations are not amplified.

III. TRANSVERSE MODE AND DEGENERACY EFFECTS

There are two limits of transverse-mode behavior. When the Fresnel number of a light beam is so large that the intensity varies only slightly across a Fresnel zone, then individual zones may be treated in the plane-wave approximation. Each pencil may be treated independently, and the output pencil intensities may be individually summed.

At the opposite extreme, structure is presumed present which forces propagation in a single transverse mode, perhaps of Gaussian shape. Scattering into other transverse modes is heavily attenuated by the structure, but in such a way that the resultant losses of the favored mode may be neglected. Such a structure might consist of a sequence of stops. In this case, pencils cannot be individually treated, but the dispersive gain must be integrated over the beam cross section to obtain a net dispersion, and the result used to find the characteristics of the evolving perturbations. This second case will be discussed, with the reader referred to earlier work for background development.

Results of use[8] here may be outlined as follows. At any transverse position, r, θ, z, the given mode determines a variation of electric field

$$E(r, \theta, z, t) = \mathrm{Re}\,\xi(r, \theta, z)\epsilon(z, t)e^{i(\omega t - kz)}, \tag{41}$$

where $\xi(r, \theta, z)$ is the transverse-mode function. The slowly-varying-envelope approximation allows electric-field-envelope propagation to be expressed by

$$\frac{\partial\epsilon}{\partial z} + \frac{\eta\omega}{c}\frac{\partial\epsilon}{\partial t} = i\,\frac{2\pi\omega}{\eta c}\int r\,dr\,d\theta\,\mathcal{P}(r, \theta, z, t)\xi^*(r, \theta, z),$$

where $\mathcal{P}$ is the polarization envelope. Omitted from this equation is a term to take into account the change in ϵ due to focusing or defocusing, and also a normalization factor. The emphasis is that the contribution from a point r, θ, z is weighted by the mode function ξ. As in Ref. 6, the phase variation in ξ is canceled in the final result.

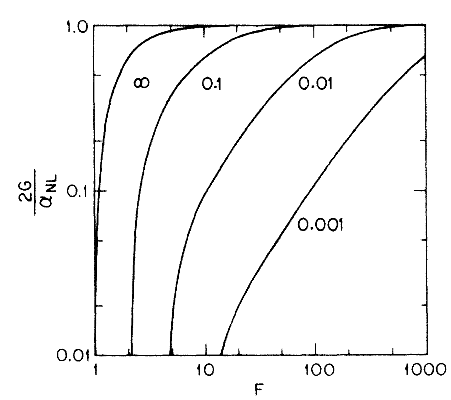

FIG. 5. $\mathrm{Re}G/\alpha_{\mathrm{NL}}$ at zero frequency, $\nu = 0$. Plotted is $2G/\alpha_{NL}$ at zero frequency as a function of F. The asymptotic approach to the value 1 depends only on the assumption of a bounded loss. The figure is limited to the region $\mathrm{Re}G > 0$.

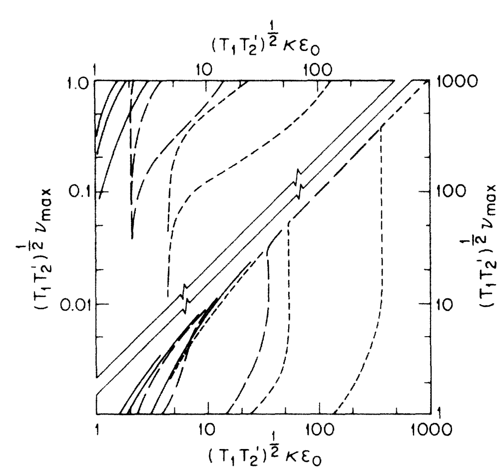

FIG. 6. Bandwidth $\nu_{\max}$ as a function of F. Only the case of zero-frequency $G > 0$ is considered. $\nu_{\max}$ rises abruptly, in the case of small T, because of the peaking of gain near frequency $\kappa\epsilon_0$. The nine curves are associated with values $T = \infty$, 0.1, 0.01, and $\alpha = 0.5, 3, 30$. The solid curves are for $T = \infty$; long-dashed, $T = 0.1$; short-dashed, $T = 0.01$.

In the case here, the additional polarization Δv at point r, θ, z is determined by a product:

$$\Delta v = \mathrm{Re}G(x, F, a, T)\Delta\epsilon, \tag{42}$$

except that now $\Delta\epsilon$ and F vary across the beam profile so that

$$\Delta v = \mathrm{Re}G(x, F|\xi|, a, T)\xi\Delta\epsilon, \tag{43}$$

where $F = (T_1 T_2')^{1/2}\kappa\epsilon$, G is that given by previous formulas, and $\Delta\epsilon$ refers to the variation of ϵ in Eq. (42). Consequently, the gain function appropriate for a transverse-mode function ξ is given by

$$G_\xi(x, F, a, T) = \frac{\alpha}{2}\,\frac{\int r\,dr\,d\theta\, G(x, F|\xi|, a, T)|\xi|^2}{-\int r\,dr\,d\theta\, G(0, 0, a, T)|\xi|^2}, \tag{44}$$

the denominator being introduced for normalization. If degeneracy is present, with a transition with Rabi coefficient κ introduced to define F, etc., and if T_1, T_2', etc., are independent of transition, then

$$G_{\xi,\kappa,\kappa_1,\kappa_2\cdots}(x, F, a, T) = \frac{\alpha}{2}\,\frac{\sum_{\kappa_i}\int r\,dr\,d\theta\, G(x, F|\xi|\kappa_i/\kappa, a_1 T)|\xi|^2(\kappa_i/\kappa)^2}{-\sum_{\kappa_i}\int r\,dr\,d\theta\, G(0, 0, a, T)(\kappa_i/\kappa)^2|\xi|^2}, \tag{45}$$

where the Rabi coefficients κ_i refer to the various transitions.

Among the various cases, the nondegenerate Gaussian-mode case with $T_2^* = \infty$ is singled out for discussion. In that case, one may take

$$|\xi(r, \theta)| = e^{-r^2/r_0^2}. \tag{46}$$

The gain function G at zero frequency then becomes

$$G_\xi(0, F) = \frac{\alpha}{2}\,\frac{\int r\,dr\,|\xi|^2[(F^2\xi^2 - 1)/(F^2\xi^2 + 1)^2]}{-\int r\,dr\,|\xi|^2}, \tag{47}$$

according to Eqs. (16) and (45). A change of variables of integration allows

$$G_\xi(0, F) = \frac{\alpha}{2}\int_0^1 ds\,\frac{sF^2 - 1}{(sF^2 + 1)^2} = \tfrac{1}{2}\alpha[F^{-2}\ln(1 + F^2) - 2(1 + F^2)^{-1}]. \tag{48}$$

This result is presented in Fig. 7. A maximum value 0.0602 occurs at $F \cong 3.65$.

The behavior of $G(\nu = 0)$ may be determined through the energy-loss dependence on F. Even in the Gaussian-mode case, the losses increase without bound as F increases, so that previous formulas for large F do not necessarily apply. In the Gaussian-mode case above, the nonlinear loss coefficient α_{NL} may be shown to be given by

$$\alpha_{\mathrm{NL}} = -\frac{2}{\epsilon_0}\frac{\partial\epsilon_0}{\partial z} = \alpha\frac{\ln(1 + F^2)}{F^2} \tag{49}$$

so that $G(0, F)/\alpha_{\mathrm{NL}}$ does approach the value $\frac{1}{2}$ for F large.

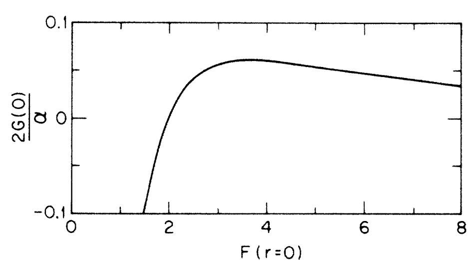

FIG. 7. Zero-frequency gain function for a Gaussian profile and no degeneracy. The curve is $2G/\alpha$, where G and α are given by Eqs. (48) and (49). The value of a for a given curve can then be determined through the inequality $\nu_{\max}(T, a_1) > \nu_{\max}(T, a_2)$ if $a_1 < a_2$.

IV. BISTABLE MIRROR

Equation (9) specifies an atomic polarization v as a function of steady-state field ϵ_0. For a fixed value of v, for example, $0.3a^{-1/2}$, there are two values of ϵ_0 which, with the given value of v, allow Eq. (9) to be satisfied; i.e., ϵ_0 is a double-valued function of v. If the optical analog of a current source could be made, one could construct, with it and some atoms, a bistable optical device.

Although a close analog of a current source may not exist, nevertheless it is possible to imagine ways in which polarizations and fields can be coupled so that an output field is a multivalued function of an input field. An example follows.

A Fabry-Perot interferometer with 95% reflectivity plates is adjusted for 100% transmission of light at frequency ω. A homogeneously broadened absorber, comprised of stationary atoms, that absorbs light of weak intensity at frequency ω is placed between the plates. Light of frequency ω is incident from the right. The electric field envelopes are E_I, incident from the right; E_R, reflected to the right; E_F, inside the cavity, moving to the left; E_B, inside the cavity, moving to the right; E_T, left of the cavity, moving to the left.

The electric field inside the cavity may then be expressed as

$$E(z, t) = E_F e^{-i(\omega t + kz)} + \mathrm{c.c.} + E_B e^{-i(\omega t - kz)} + \mathrm{c.c.}, \tag{50}$$

where E_F and E_B are taken to be real, and independent of distance and time. The out-of-phase susceptibility determined by Eq. (9) may vary rapidly in a light wavelength, so that the slowly-varying-envelope approximation must be used with care. Evaluation of Eq. (9) requires the quantity

$$\langle E^2\rangle_{\text{time}} = 2E_F^2 + 2E_B^2 + 2E_F E_B e^{-2ikz} + 2E_F E_B e^{2ikz}. \tag{51}$$

Then v_0, the polarization out of the phase with the electric field, may be written

$$v_0 = T_2' \kappa (E_F e^{-ikz} + E_B e^{ikz})(1 + T_1 T_2' \kappa^2 \langle E^2\rangle_{\text{time}})^{-1}, \tag{52}$$

which may be written in a Fourier sum,

$$v_0 = T_2' \kappa \{E_F e^{-ikz} + E_B e^{ikz}\}\{\sigma(0) + \sigma(2k) e^{-2ikz} + \sigma(-2k) e^{+2ikz} + \cdots\}, \tag{53}$$

where $\sigma(0)$ is a spatial average of the last factor in Eq. (53) averaged over a short distance, $\sigma(2k)$ is a similar average of the product of the last factor and a factor e^{2ikz}, etc. Thus

$$\sigma(0) = A^{-1}(1 - b^2)^{-1/2} \tag{54}$$

and

$$\sigma(2k) = \sigma(-2k)^* = A^{-1} b^{-1}[1 - (1 - b^2)^{-1/2}], \tag{55}$$

where

$$A = 1 + T_1 T_2' \kappa^2 \{E_F^2 + E_B^2\}$$

and

$$b = 2T_1 T_2' \kappa^2 E_F E_B A^{-1}.$$

Equations (50) and (53) may be substituted into Maxwell's electric field equation. The slowly-varying-envelope approximation is made in the usual way[6]; however, in this case terms of the form $E_F \sigma(-2k)$ will have the same sinusodial behavior as E_B, thus coupling the two waves. Consequently, the two resultant equations are

$$\frac{\partial E_F}{\partial z} = \frac{\alpha}{2}\sigma(0) E_F + \frac{\alpha}{2}\sigma(2k) E_B, \tag{56}$$

$$\frac{\partial E_B}{\partial z} = -\frac{\alpha}{2}\sigma(0) E_B - \frac{\alpha}{2}\sigma(2k) E_F. \tag{57}$$

It should be noted that contributions from waves varying as $e^{-3i\omega t}$ or as e^{-3ikz} are neglected, and that no motion of the absorbing atoms is allowed.

Let $z = 0$ coincide with the left-hand mirror, and $z = L$ coincide with the right-hand mirror. The boundary conditions may be chosen to be $E_T = \sqrt{T} E_F$, $E_B = \sqrt{R} E_F$, at $z = 0$, where R is the mirror reflectivity and $T = 1 - R$. At $z = L$, $E_F - \sqrt{R} E_B = \sqrt{T} E_I$

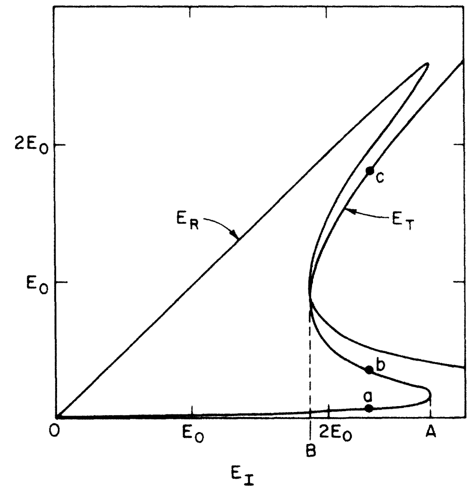

FIG. 8. Output field E_T and reflected field E_R as a function of input field E_I, as determined in Sec. IV. The mirror reflectivities are 95%, $\alpha L = 2$ is the amount of homogeneous absorber between the plates, and $E_0 = \kappa^{-1}(T_1 T_2')^{-1/2}$. Possibly a and c represent stable points, while b represents an unstable point.

and $E_B - \sqrt{R} E_F = \sqrt{T} E_R$. The equations may be solved with E_T specified to yield E_T and E_R as functions of E_I. Figure 8 displays these results for the case $R = 0.95$, $\alpha L = 2$.

The result that E_T is a multivalued function of E_I means that if E_I were varied starting from zero to a point between A and B, the output E_T would be relatively low. On the other hand, if the input were to exceed A, then the output would be relatively high, and would only decrease somewhat if E_I were then decreased to a value between A and B. A reasonable guess is that the portion of the curve with negative slope would be unstable, and the device would exhibit hysteresis between A and B.

If the reflectivity, or the amount of absorber, is changed enough, the multivalued property must disappear, and at some point be replaced by a large differential gain.

CONCLUSIONS

The T_1, T_2', T_2^* model used here is not the most general model. For example, such equations do not admit the description of spectral diffusion caused by velocity-changing collisions in a gas.

At large Rabi frequencies, the bandwidth of the gain or instability region extends to approximately the Rabi frequency, for example, 30 GHz in the case of a 1-W optical laser of 10-μ beam diameter in an absorber of large oscillator strength characteristic of alkali s-p transitions. If only the statistics of the absorbed photons limits the noise, as might be the case if a device considered in Sec. IV were included in an ideal Michaelson interferometer, then in principle, the noise figure could be quite low, and comparable to an ideal laser amplifier.

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