Condensation of photons in a hot plasma*

G. Chapline, G. Cooper, and S. Slutz†
Lawrence Livermore Laboratory, University of California, Livermore, California 94550
(Received 12 November 1973)

The Compton cooling of a photon gas is studied using the Compton-Fokker-Planck equation. Numerical solutions of the Fokker-Planck equation are presented which illustrate the possibility of forming a temporary excess of low-energy photons. The conditions under which it would be possible to observe this excess of low-energy photons are derived. The possibility of using stimulated Compton scattering to generate coherent x rays is also discussed.

I. INTRODUCTION

One of the most interesting consequences of Bose statistics is the condensation into the zero momentum state that is predicted[1] to occur when an ideal Bose gas is cooled below a certain temperature. Although this predicted condensation is a standard topic in statistical-mechanics textbooks, no simple physical illustration of this phenomenon is known. The most obvious example of an ideal Bose gas is a photon gas. Unfortunately, photons can be emitted and absorbed by all materials; whereas the existence of a condensation requires that the number of particles be conserved. It is possible, however, to conceive of circumstances in which the number of photons would be approximately conserved as thermodynamic equilibrium is approached. An example of a system in which it is possible to approach an equilibrium, where the number of photons is approximately conserved, is a very hot (multi-keV temperature) hydrogen plasma[2,3] where the energy exchange between radiation and matter takes place mainly via Compton scattering which conserves photons (radiative Compton scattering is small at keV temperatures). Thus one will expect that if the radiation and plasma are not in equilibrium, the photons will approach a distribution of the form

$$n(k) = 1/(Ce^{k/\Theta} - 1)\,, \qquad (1)$$

where Θ is the temperature and $C \geqslant 1$ is a constant determined by the number of photons present.

Detailed investigation shows,[4,5] in fact, that when the Compton-scattering optical depth of a hot plasma cloud becomes greater than $(mc^2/\Theta_e)^{1/2}$, where m is the electron mass and Θ_e is the electron temperature, the high-energy portion of the x-ray spectrum emitted by the plasma cloud does indeed correspond to a distribution of the form (1) (absorption effects become important at low energies). In general, Compton scattering will distort a bremsstrahlung-type spectrum towards a Bose-type spectrum, Eq. (1). This effect has been observed in the spectrum of the astronomical x-ray source SCOX-1.[6] Such an effect should also be present in x-ray spectra emitted by neutron stars.[7]

The interaction of a photon gas with matter via Compton scattering can lead either to an increase or decrease of the average photon energy depending on whether the radiation is "hotter" or "colder" than the matter. For example, if radiation described by a Bose distribution with a temperature Θ_r is brought into contact via Compton scattering with colder matter at temperature Θ_m, then the radiation will approach an equilibrium distribution (1) characterized by a temperature Θ_f such that $\Theta_r < \Theta_f < \Theta_m$. Conversely, if initially the radiation temperature Θ_r is greater than the matter temperature Θ_m then the equilibrium temperature Θ_f will satisfy $\Theta_m < \Theta_f < \Theta_r$; i.e., the radiation will be cooled.

The possibility of cooling radiation via Compton scattering leads, however, to a paradox. If the initial Bose distribution was a Planck distribution (i.e., $C = 1$), then the equilibrium distribution cannot be of the form given in Eq. (1) because the number of photons for a Planck distribution with temperature Θ_r exceeds the number of photons in any distribution of the form (1) with temperature $\Theta_f < \Theta_r$. This apparent contradiction is resolved by the fact that the excess photons will be scattered downward in energy and in the absence of absorption will condense into the zero momentum state.[8] Of course, a condensation of photons into the zero momentum state cannot actually happen because the low-energy photons will be strongly absorbed by the inverse bremsstrahlung process. However, one expects that Compton cooling of a radiation spectrum can lead to a temporary excess of low-energy photons. In fact, it has been suggested[9] that an excess of low-energy photons due to Compton scattering occurs in some astrophysical contexts. In this paper we shall be primarily concerned with the conditions needed to produce such an excess of low-energy photons in the laboratory. Since Bose particles approaching a condensation become partially coherent[10] one might hope that this process could be used to produce coherent x rays.

In Sec. II we discuss the Compton cooling of a radiation spectrum in the absence of photon absorption. The Bose condensation of Compton-scattered photons is illustrated by numerically solving the Compton-Fokker-Planck equation for the cooling of a Planck spectrum. It is shown that in the absence of absorption the excess photons will scatter downward in energy and after a certain time will start to condense into the zero momentum state.

The effect of absorption on the formation of a condensation is discussed in Sec. III. The conditions needed to produce a temporary excess of low-energy photons are derived.

II. CONDENSATION OF COMPTON-SCATTERED PHOTONS

In this section we are concerned with the time evolution of radiation spectra due to Compton scattering in an infinite medium. We use a distribution function $n(k,t)$ normalized so that

$$\frac{8\pi}{(hc)^3}\int_0^\infty n(k,t)k^2\,dk = N, \tag{2}$$

where N is the number of photons/cm^3. If the mean photon energy is not very different from the electron temperature, then the changes in $n(k,t)$ due to Compton scattering can be described using the Compton-Fokker-Planck equation[2,3,11]

$$k^2\frac{\partial n(k,t)}{\partial t} = \frac{\partial}{\partial k}\left[\alpha(k,\Theta_e)\left(n(k,t)[1+n(k,t)] + \Theta_e\frac{\partial n(k,t)}{\partial k}\right)\right], \tag{3}$$

where Θ_e is the electron temperature. The coefficient $\alpha(k,\Theta_e)$ is related to the mean-square energy change in a Compton collision of photons with energy k:

$$\alpha(k,\Theta_e) = \frac{\langle (k'-k)^2\rangle}{2\Theta_e}. \tag{4}$$

It should be noted that Eq. (3) guarantees two important properties, namely, conservation of photons and a Bose-Einstein equilibrium.

To evaluate the quantity $\alpha(k,\Theta)$ one must perform an average involving the Klein-Nishina cross section and a relativistic Maxwellian electron distribution. The multiple integrations involved are sufficiently complicated so that no general analytic expression is available for $\alpha(k,\Theta)$. However, for electron temperatures less than 20 keV and photon energies less than 10 keV one can use the nonrelativistic approximation[2,3]

$$\alpha_{\rm NR}(k) = n_e\sigma_T k^4/mc,$$

where n_e is the electron density and σ_T is the Thompson cross section. For photon energies above 10 keV it is necessary to take into account relativistic corrections.[11] Given an expression for $\alpha(k,\theta)$ it is possible to follow the time evolution of a radiation spectrum by numerical integration of Eq. (3). For example, one may study the formation of a condensation by obtaining the numerical solution of Eq. (3) when $n(k,0)$ is a Planck spectrum whose blackbody temperature Θ_r is greater than the initial electron temperature Θ_e.

Figure 1 shows an example of such a numerical calculation, using the equation of Ref. 11 and the finite-difference scheme of Chang and Cooper.[12] The initial $n(k,0)$ was a Planck distribution with $\Theta_r = 12$ keV. The initial matter temperature was $\Theta_m = 10$ keV. The matter density was chosen so that the matter specific heat is comparable to the radiation specific heat, thus giving rise to significant radiation cooling. The calculation shows clearly that after a certain time photons begin to pile up in the zero momentum state.

The fact that the condensation into the zero momentum state begins after a finite time may seem a little surprising but can be made plausible analytically.[8] For small k we have $n \gg 1$ so that Eq. (3) becomes

$$k^2\frac{\partial n}{\partial t} = \frac{\partial}{\partial k}\left[\alpha(k,\Theta_e)\left(n^2 + \Theta_e\frac{\partial n}{\partial k}\right)\right]. \tag{5}$$

Let us assume that we also have

$$n^2 \gg \Theta_e\left|\frac{\partial n}{\partial k}\right|. \tag{6}$$

If we assume further that the electron temperature and photon energy are low enough so that the

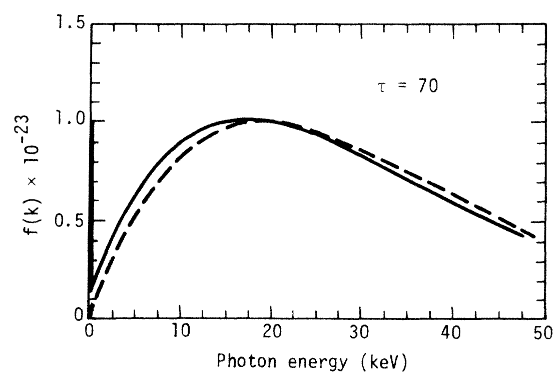

FIG. 1. Numerical calculation of $f(k) \equiv 2(hc)^{-3}k^2 n(k,t)$ at time $\tau = 7\times10^{-11}$ sec ($\tau = 70$). The initial photon distribution (dashed line) corresponds to a 12-keV blackbody spectrum. The initial electron temperature was 10 keV. The electron density is 5×10^{25} cm^{-3}. The critical time $(n_e\sigma_T C)^{-1}(mc^2/2\Theta_r)$ is 2.1×10^{-11} sec ($\tau = 21$).

nonrelativistic expression for $\alpha(k,\Theta)$ is valid, then Eq. (5) can be written in the simple form

$$x^2\frac{\partial n}{\partial t}=\frac{\partial}{\partial x}\{x^4n^2\}, \quad (7)$$

where $x=k/mc^2$ and $\tau=(n_e\sigma_T c)t$. Introducing the function $f\equiv x^2 n$ one sees immediately that the number of photons N is constant and that the solution to Eq. (7) is

$$f(x,\tau)=F(x+2f\tau), \quad (8)$$

where the function F is determined by the initial conditions. As an example, suppose that the initial distribution is a Planck distribution:

$$n(k,0)=1/(e^{k/\Theta_r}-1). \quad (9)$$

This corresponds to

$$F(x)=x^2/(e^{x/x_0}-1),$$

where $x_0=\Theta_r/mc^2$. Substituting into Eq. (8) and setting $x=0$ gives an equation for the time τ_c to reach $x=0$:

$$4f\tau_c^2=e^{2f\tau_c/x_0}-1. \quad (10)$$

From Eq. (10) one finds that the low-energy photons ($k\ll\Theta_r$) start to pile up at $k=0$ after a time $\tau_c=mc^2/2\Theta_r$. For the example shown in Fig. 1 this time is 21 psec. In the numerical calculation we find that the buildup of photons in the $k=0$ state becomes noticeable after about 50 psec.

III. EFFECT OF INVERSE BREMSSTRAHLUNG

The Bose condensation into the zero momentum state that we have just discussed will in fact never happen because the low-energy photons will be absorbed. Nevertheless, under certain conditions Compton cooling of a radiation spectrum can lead to a temporary excess of low-energy photons. This is illustrated in Fig. 2, which shows the cooling of a Planck spectrum taking into account bremsstrahlung absorption. It can be seen that Compton scattering creates a "wave" of excess photons that is eaten away by photon absorption as the wave moves downward in energy. Eventually, one is left with a Planck distribution corresponding to the equilibrium temperature.

If the excess of photons occurs in a region where $n\gg1$ then the excess photon "wave" will be strongly influenced by stimulated Compton scattering [Eq. (6)]. Under these circumstances it is reasonable to expect that the excess photons will develop some degree of coherence. Indeed, as was first pointed out by Zel'dovich and Levich,[8] stimulated Compton scattering can lead to the formation of a "shock wave" in the photon spectrum in which the number of photons per quantum state rises very sharply at particular energy. To see how this comes about we differentiate Eq. (8)

$$\frac{\partial f}{\partial x}=\frac{F'}{1-2F'\tau}. \quad (11)$$

We see that $\partial f/\partial x\to\infty$ when $\tau\to(2F')^{-1}$, if $F'>0$. Thus the photon spectrum can develop a very steep gradient if the approximations leading to Eq. (7) are satisfied and if the initial photon spectrum is sufficiently steep.

In order to delineate the conditions where stimulated Compton scattering can produce coherence in a pulse of photons, we must first compare the time it takes Compton scattering to change the spectral shape of the pulse to the bremsstrahlung absorption time. The characteristic time for changes in the spectral shape of the pulse due to Compton scattering is $(n_e\sigma_T c)^{-1}(1/2F')$, where F' is a characteristic value for the slope of the photon spectral-distribution function. As an order-of-magnitude estimate we can set $F'=\Theta_B/mc^2$, where Θ_B is the peak luminosity temperature of the pulse. The characteristic time for absorption of the pulse is $(\kappa c)^{-1}$, where κ is the inverse bremsstrahlung opacity. Thus Compton scattering will dominate over absorption if

$$(n_e\sigma_T c)(2\Theta_B/mc^2)>\kappa c. \quad (12)$$

Using the Kramers approximation for the inverse bremsstrahlung opacity, inequality (12) may be written in the form

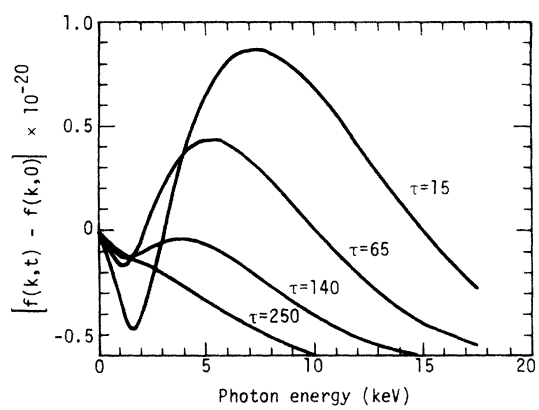

FIG. 2. Numerical calculation of the "excess photon" distribution, $f(k,t)-f(k,0)$, as a function of time. The time is given in units of $(n_e\sigma_T c)^{-1}=2\times10^{-10}$ sec. The initial electron temperature was 4 keV. The initial radiation temperature was 5 keV. The density is $n_e=2.5\times10^{23}$ cm^{-3}. As a result of photon absorption and emission, $f(k,t)$ will approach a Planck distribution corresponding to the equilibrium temperature 4.99 keV. The equilibrium distribution is established out to $k=5$ keV in a time $\tau=6\times10^{-8}$ sec ($\tau=300$).

$$k^3 > \frac{0.6}{\sqrt{\Theta_e}}\,\eta(1-e^{-k/\Theta_e})\left(\frac{mc^2}{2\Theta_B}\right), \qquad (13)$$

where k and Θ_e are measured in keV and η is the plasma density in units of 5×10^{22} cm^{-3}. The reason for considering such high plasma densities is that the matter specific heat must be comparable to the radiation specific heat in order to achieve significant radiation cooling. This means that the density of hydrogen atoms must be comparable to the density of photons. The density of photons in blackbody radiation of temperature Θ_r is

$$N = 2.6\times10^{22}\Theta_r^3 \text{ cm}^{-3},$$

where Θ_r is measured in keV. Thus we see that if Θ_r is in the multi-keV range the matter density must be at least on the order of liquid-hydrogen density ($n_e = 5\times10^{22}$ cm^{-3}). In general, we require

$$\eta \approx \Theta_r^3. \qquad (14)$$

Combining (13) and (14) and assuming $k\approx\Theta_e$ leads to the following condition on Θ_B:

$$\frac{\Theta_B}{mc^2} \gtrsim \frac{0.2}{\sqrt{\Theta_e}}\left(\frac{\Theta_r}{\Theta_e}\right)^3. \qquad (15)$$

This shows that in order to achieve significant Compton cooling Θ_B/mc^2 must be on the order of 0.2 or that Θ_B must be on the order of 100 keV. We have checked the estimate by calculating the "gain"[13] $(1/n)(dn/dx)$ in a hot plasma due to stimulated Compton scattering:

$$\frac{1}{n(k)}\frac{dn(k)}{dx} = \int_0^\infty \left(\frac{k'}{k}\right)^2 T(k',k)n(k')dk' - \int_0^\infty T(k,k')[1+n(k')]\,dk', \qquad (16)$$

where $T(k,k')$ is the probability for scattering $k \to k'$. No general analytic formula for $T(k,k')$ is known, but in the nonrelativistic regime one may note from the numerical work of Matteson *et al.*[14] that T is nearly exponential in k' with different decay parameters for $k'<k$ and $k'>k$. Using this observation and the requirement that the first three moments of $k'-k$ be correct, one finds

$$T(k,k') \simeq (n_e\sigma_T)\frac{mc^2}{k[4\Theta - k^2 + 4\Theta mc^2]^{1/2}} \times \exp\left[\left(\frac{2}{k}-\frac{1}{2\Theta}\right)(k'-k) - \frac{[(4\Theta-k)^2+4\Theta mc^2]^{1/2}}{2k\Theta}\,|k'-k|\right]. \qquad (17)$$

Using expressions (17) and (16) the stimulated Compton gain can be evaluated for any given $n(k)$. We have numerically evaluated the gain for distributions of the form $Ae^{-(k-k_0)^2/\Delta k^2}$ superimposed on a Planck distribution in equilibrium with the plasma (i.e., at temperature Θ_e). The gain will depend somewhat on the shape and position of the Gaussian, but we find that the gain calculated from Eq. (16) will not exceed the inverse bremsstrahlung loss unless the peak brightness temperature of the Gaussian is large compared to the electron temperature. For example, for $\Theta_e = 5$ keV, $\eta = 50$, we find the stimulated Compton gain exceeds the bremsstrahlung loss only if $k \simeq 5$ keV and the peak luminosity temperature of the Gaussian exceeds 120 keV.

IV. CONCLUSION

The high densities needed to produce a significant excess of low-energy x rays are beyond the densities that have been produced in the laboratory. Further, luminosity temperatures on the order of 100 keV in the region $k \gtrsim 1$ keV are much higher than can be produced at present. Nevertheless, high-power lasers capable of producing compressions in hydrogen of $10^4\times$liquid density will soon be available.[15] Also, by using exotic nuclear fuels[16] it may be possible to produce radiation temperatures near 100 keV. Therefore, it is not unthinkable that at some time in the future it will be possible to observe effects related to the Bose condensation of photons.

---

*Work performed under the auspices of the United States Atomic Energy Commission.

†Present address: Physics Department, California Institute of Technology, Pasadena, Calif.

[1]A. Einstein, Sitzungsber. Dtsch. Akad. Wiss. Berl. 3, (1924).

[2]A. S. Kompaneets, Zh. Eksp. Teor. Fiz. 31, 876 (1956) [Sov. Phys.—JETP 4, 730 (1957)].

[3]R. Weymann, Phys. Fluids 8, 2112 (1965).

[4]A. F. Illarionov and R. A. Sunyaev, Sov. Astron.–AJ 16, 45 (1972).

[5]G. Chapline and J. Stevens, Astrophysical J. 184, 1041 (1973).

[6]E. D. Loh and G. P. Garmire, Astrophysical J. 166, 301 (1971).

[7]Y. B. Zel'dovich and N. I. Shakura, Sov. Astron.–AJ 13, 175 (1969).

[8]Y. B. Zel'dovich and E. V. Levich, Zh. Eksp. Theor. Fiz. 55, 2423 (1969) [Sov. Phys.—JETP 28, 1287 (1969)].

[9]R. A. Sunyaev, Sov. Astron.–AJ 15, 190 (1971).

[10]G. Chapline, Phys. Rev. A 3, 1671 (1971).

[11]G. Cooper, Phys. Rev. D 3, 2312 (1970).
[12]J. S. Chang and G. Ccoper, J. Comput. Phys. 6, 1 (1970).
[13]Positive gain does not mean that a beam would be amplified but only that the photons would tend to pile up into the same quantum state.
[14]L. Matteson, G. C. Pomraning, and H. L. Wilson, Gulf General Atomic Report No. GA-9694 (1969) (unpublished).
[15]J. Nuckolls, L. Wood, and J. Emmett, Phys. Today 26, 46 (1973).
[16]T. A. Weaver and L. Wood, Bull. Am. Phys. Soc. 18, 1300 (1973).