PHYSICAL REVIEW A **89**, 064301 (2014)

Entanglement swapping for X states demands threshold values

Luis Roa,[1] Ariana Muñoz,[1] and Gesa Grüning[1,2]

[1]*Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

[2]*Ruprecht Karls Universität Heidelberg, Heidelberg, Germany*

(Received 26 November 2013; revised manuscript received 6 March 2014; published 13 June 2014)

The basic entanglement-swapping scheme can be seen as a process which allows one to redistribute the Bell states properties between different pairs of a four-qubit system. We analyze a similar scheme by performing a Bell–von Neumann measurement over two local qubits, each one initially correlated through an X state with a spatially distant qubit. This process swaps the X feature without conditions, whereas the input entanglement is partially distributed in the four possible outcome states under certain conditions. Specifically, we obtain two threshold values for the entanglement of formation of the input X states in order for the outcome states to be nonseparable.

DOI: 10.1103/PhysRevA.89.064301 PACS number(s): 03.67.Bg, 03.65.Ud, 03.67.Ac

The quantum correlations are the key ingredients for implementing processes both assisted [1] and between distant particles [2]. In this context, quantum correlations have been well characterized until now by the so-called entanglement of formation (EOF) [3] and quantum discord (QD) [4]. The entanglement-swapping [5] procedure redistributes quantum correlation between different parts of two composed systems. The advantage of this process lies in the fact that two factorized parts can acquire entanglement even though they are away from each other. Here we analyze a generalization of the basic entanglement-swapping scheme by considering two pairs of qubits being initially correlated via X states instead of pure states. Specifically, we focus with special interest on knowing how the entanglement of formation is redistributed onto the outcome states. Motivations for studying X states stem from the fact that they are usually encountered in different areas [6]. Interesting approaches to this problem have been addressed in Refs. [7,8], where initial Bell states undergo non-Markovian decoherence mechanisms.

Let us begin by considering the simplest case of two pairs of qubits $A,C1$ and $B,C2$ prepared in the Bell states $|\phi^+_{A,C1}\rangle$ and $|\phi^+_{B,C2}\rangle$. The entanglement-swapping protocol can readily be read out of the identity

$$|\phi^+_{A,C1}\rangle|\phi^+_{B,C2}\rangle = \tfrac{1}{2}(|\phi^+_{A,B}\rangle|\phi^+_{C1,C2}\rangle + |\phi^-_{A,B}\rangle|\phi^-_{C1,C2}\rangle + |\psi^+_{A,B}\rangle|\psi^+_{C1,C2}\rangle + |\psi^-_{A,B}\rangle|\psi^-_{C1,C2}\rangle), \quad (1)$$

where $|\phi^\pm_{U,V}\rangle = (|0_U\rangle|0_V\rangle \pm |1_U\rangle|1_V\rangle)/\sqrt{2}$ and $|\psi^\pm_{U,V}\rangle = (|0_U\rangle|1_V\rangle \pm |1_U\rangle|0_V\rangle)/\sqrt{2}$ are the Bell states of the generic systems U and V, and $\{|0\rangle,|1\rangle\}$ are the eigenstates of σ_z. By implementing a measurement which projects the qubits $C1$ and $C2$ onto one of their Bell states, the pair A,B is also projected onto one of its Bell states. Each one of the four results has the same probability. Therefore the entanglement contained in the pairs $A,C1$ and $B,C2$ is redistributed to the pair A,B, even though there is no interaction between them.

If the pairs $A,C1$ and $B,C2$ are in partially entangled pure states, then (1) becomes

$$|\tilde{\phi}^+_{A,C1}\rangle|\tilde{\phi}^+_{B,C2}\rangle = \sqrt{p_\phi}(|\ddot{\phi}^+_{A,B}\rangle|\phi^+_{C1,C2}\rangle + |\ddot{\phi}^-_{A,B}\rangle|\phi^-_{C1,C2}\rangle) + ab(|\psi^+_{A,B}\rangle|\psi^+_{C1,C2}\rangle + |\psi^-_{A,B}\rangle|\psi^-_{C1,C2}\rangle), \quad (2)$$

where

$$|\tilde{\phi}^+_{U,V}\rangle = a|0_U\rangle|0_V\rangle + b|1_U\rangle|1_V\rangle,$$

$$|\ddot{\phi}^\pm_{A,B}\rangle = \frac{a^2|0_A\rangle|0_B\rangle \pm b^2|1_A\rangle|1_B\rangle}{\sqrt{|a|^4+|b|^4}},$$

with $p_\phi = (|a|^4+|b|^4)/2$ and $|a|^2+|b|^2 = 1$. Note that by projecting the qubits $C1$ and $C2$ onto one of the Bell states $|\psi^\pm_{C1,C2}\rangle$ the pair A,B is projected as well onto one of the Bell states $|\psi^\pm_{A,B}\rangle$ with probability $p_\psi = |a|^2|b|^2$. Otherwise when the pair $C1,C2$ is projected onto one of the Bell states $|\phi^\pm_{C1,C2}\rangle$ the composed system A,B is projected onto one of the partially entangled states $|\ddot{\phi}^\pm_{A,B}\rangle$ with probability $p_\phi \geqslant p_\psi$. Thus, even though the initial entanglement $E(|\tilde{\phi}^+_{U,V}\rangle)$ is not maximal, there are two possible outcome states maximally entangled. However, the other two resulting states have an amount of entanglement $E(|\ddot{\phi}^\pm_{A,B}\rangle)$ smaller than the initial, and in general, the average outcome entanglement value $\bar{E}$ is also smaller than the initial value $E(|\tilde{\phi}^+_{U,V}\rangle)$. We illustrate that behavior of the redistributed entanglement in Fig. 1 as a function of $|a|$. We note that the four outcome states are maximally entangled only for $|a| = 1/\sqrt{2}$, which is the case of identity (1). It is worth emphasizing three things: (i) the outcome entanglement is maximal with probability $2|a|^2|b|^2$, which vanishes only at the end values $|a| = 0,\ 1$, where there is no input entanglement; (ii) the probability of increasing the entanglement is always smaller than that of decreasing it, since $2|a|^2|b|^2 \leqslant |a|^4+|b|^4$, and (iii) for having entangled outcome states it is required only to have the initial entanglement different from zero. Finally, we want to emphasize that there is an asymmetry in the distribution of the entanglement over the four outcome states, which in identity (2) is in favor of the $|\psi^\pm_{A,B}\rangle$ outcome Bell states.

Now we succinctly review the properties of a general X state $\hat{\rho}_{a,b}$ of two qubits a and b. In the logic basis $\{|0_a\rangle|0_b\rangle, |0_a\rangle|1_b\rangle, |1_a\rangle|0_b\rangle, |1_a\rangle|1_b\rangle\}$ the state is represented by the matrix

$$\hat{\rho}_{a,b} \equiv \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \quad (3)$$

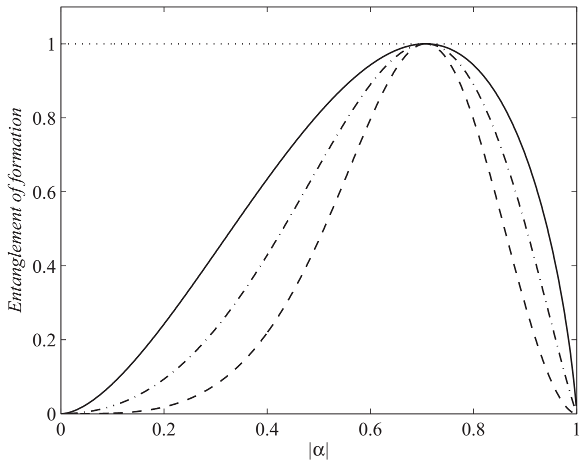

FIG. 1. EOF as functions of $|a|$ of the initial states (solid line) and of the outcome states $|\ddot{\phi}^{\pm}_{A,B}\rangle$ (dashed line) and $|\psi^{\pm}_{A,B}\rangle$ (dotted line). The dot-dashed line corresponds to the average EOF $\bar{E}$ of the four possible outcome states.

The $\hat{\rho}_{a,b}$ state must satisfy normalization and the positivity conditions,

$$|\rho_{14}| \leqslant \sqrt{\rho_{11}\rho_{44}} \quad \text{and} \quad |\rho_{23}| \leqslant \sqrt{\rho_{22}\rho_{33}}. \tag{4}$$

In what follows we consider *fixed diagonal elements* and we do the analysis in terms of the off-diagonal elements. The off-diagonal elements account for the coherence degree inside two orthogonal subspaces, say, $\mathcal{H}_{00,11}$ spanned by the basis $\{|0_a\rangle|0_b\rangle, |1_a\rangle|1_b\rangle\}$ and $\mathcal{H}_{01,10}$ spanned by $\{|0_a\rangle|1_b\rangle, |1_a\rangle|0_b\rangle\}$. For instance, when $\rho_{14} = 0$ there is absolute decoherence into $\mathcal{H}_{00,11}$, while for $|\rho_{14}| = \sqrt{\rho_{11}\rho_{44}}$ there is a pure state into $\mathcal{H}_{00,11}$. The ρ_{23} element has a similar meaning in the subspace $\mathcal{H}_{01,10}$. Accordingly, depending on the moduli $|\rho_{14}|$ and $|\rho_{23}|$, the X state (3) goes from an incoherent superposition of the four factorized logic states to an incoherent superposition of two partially entangled pure states. There are two special features of an X state: (i) the X form itself which arises because it populates the two subspaces $\mathcal{H}_{00,11}$ and $\mathcal{H}_{01,10}$ without having off-diagonal terms between the elements of those subspaces, and (ii) the entanglement between the two involved systems. The EOF can be evaluated by the concurrence, which for state (3) is given by [9]

$$\mathcal{C}_{\text{in}} = 2\max\{0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}. \tag{5}$$

Thus there is entanglement when one of the two inequalities is satisfied:

$$|\rho_{14}| > \sqrt{\rho_{22}\rho_{33}}, \quad |\rho_{23}| > \sqrt{\rho_{11}\rho_{44}}\,; \tag{6}$$

otherwise entanglement is absent. Complementing inequalities (6) with those for positivity (4) leads to the following two equations for having entanglement:

$$|\rho_{23}| \leqslant \sqrt{\rho_{22}\rho_{33}} < |\rho_{14}| \leqslant \sqrt{\rho_{11}\rho_{44}}, \tag{7a}$$

$$|\rho_{14}| \leqslant \sqrt{\rho_{11}\rho_{44}} < |\rho_{23}| \leqslant \sqrt{\rho_{22}\rho_{33}}, \tag{7b}$$

where clearly only one or none of them can be fulfilled. We extract in the following three sentences what the inequalities (7) are saying to us: (i) For $\mathcal{C}_{\text{in}} \neq 0$ it is necessary that $\rho_{11}\rho_{44} \neq \rho_{22}\rho_{33}$. (ii) For $\mathcal{C}_{\text{in}} \neq 0$ it is necessary and sufficient to have a certain nonzero coherence degree inside only one subspace $\mathcal{H}_{00,11}$ or $\mathcal{H}_{01,10}$. (iii) The coherence degree of one of the subspaces does not contribute to $\mathcal{C}_{\text{in}}$.

We stress that for fixed diagonal elements, the maximum amount of entanglement is reached with total coherence into only one subspace, $\mathcal{H}_{00,11}$ or $\mathcal{H}_{01,10}$.

Now, for the entanglement-swapping process, we assume that both pairs of qubits $A,C1$ and $B,C2$ are in the X states $\hat{\rho}_{A,C1}$ and $\hat{\rho}_{B,C2}$, which we call input states. In the respective logic bases they are represented by

$$\hat{\rho}_{A,C1} \equiv \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \qquad \hat{\rho}_{B,C2} \equiv \begin{pmatrix} \rho'_{11} & 0 & 0 & \rho'_{14} \\ 0 & \rho'_{22} & \rho'_{23} & 0 \\ 0 & \rho'_{32} & \rho'_{33} & 0 \\ \rho'_{41} & 0 & 0 & \rho'_{44} \end{pmatrix}. \tag{8}$$

Let us consider that the factorized state $\hat{\rho}_{A,C1} \otimes \hat{\rho}_{B,C2}$ undergoes a von Neumann measurement for projecting the pair $C1,C2$ onto one of the four Bell states $|\phi^{\pm}_{C1,C2}\rangle$, $|\psi^{\pm}_{C1,C2}\rangle$. Consequently, the pair A,B also is projected onto one of the states $\hat{\rho}^{\phi^{\pm}}_{AB}$ or $\hat{\rho}^{\psi^{\pm}}_{AB}$, which we call outcome states. In the logic basis the outcome states are

$$\hat{\rho}^{\phi^{\pm}}_{AB} \equiv \frac{1}{N_\phi}\begin{pmatrix} \rho_{11}\rho'_{11} + \rho_{22}\rho'_{22} & 0 & 0 & \pm(\rho_{14}\rho'_{14} + \rho_{23}\rho'_{23}) \\ 0 & \rho_{11}\rho'_{33} + \rho_{22}\rho'_{44} & \pm(\rho_{14}\rho'_{32} + \rho_{23}\rho'_{41}) & 0 \\ 0 & \pm(\rho_{41}\rho'_{23} + \rho_{32}\rho'_{14}) & \rho_{33}\rho'_{11} + \rho_{44}\rho'_{22} & 0 \\ \pm(\rho_{41}\rho'_{41} + \rho_{32}\rho'_{32}) & 0 & 0 & \rho_{33}\rho'_{33} + \rho_{44}\rho'_{44} \end{pmatrix}, \tag{9a}$$

$$\hat{\rho}^{\psi^{\pm}}_{AB} \equiv \frac{1}{N_\psi}\begin{pmatrix} \rho_{11}\rho'_{22} + \rho_{22}\rho'_{11} & 0 & 0 & \pm(\rho_{14}\rho'_{23} + \rho_{23}\rho'_{14}) \\ 0 & \rho_{11}\rho'_{44} + \rho_{22}\rho'_{33} & \pm(\rho_{14}\rho'_{41} + \rho_{23}\rho'_{32}) & 0 \\ 0 & \pm(\rho_{41}\rho'_{14} + \rho_{32}\rho'_{23}) & \rho_{33}\rho'_{22} + \rho_{44}\rho'_{11} & 0 \\ \pm(\rho_{41}\rho'_{32} + \rho_{32}\rho'_{41}) & 0 & 0 & \rho_{33}\rho'_{44} + \rho_{44}\rho'_{33} \end{pmatrix}, \tag{9b}$$

where the normalization constants $N_\phi = (\rho_{11} + \rho_{33})(\rho'_{11} + \rho'_{33}) + (\rho_{22} + \rho_{44})(\rho'_{22} + \rho'_{44})$, $N_\psi = (\rho_{11} + \rho_{33})(\rho'_{22} + \rho'_{44}) + (\rho_{22} + \rho_{44})(\rho'_{11} + \rho'_{33})$. The probabilities of obtaining each one of the four possible outcomes (9) are $P_{\phi^\pm} = N_\phi/2$ and $P_{\psi^\pm} = N_\psi/2$, which are generally different. We note that the four outcomes are X states as well. This means that the X feature of the input states is swapped to the state of the pair A,B. The diagonal (off-diagonal) elements of the outcomes (9) depend only on the diagonal (off-diagonal) elements of the input states. Thus both terms ρ_{14} and ρ_{23} affect and contribute to the coherence into both subspaces $\mathcal{H}_{00,11}$ and $\mathcal{H}_{01,10}$ of the pair A,B.

The states $\hat\rho_{AB}^{\phi^+}$ and $\hat\rho_{AB}^{\phi^-}$ are equivalent by means of local unitary operators, e.g., $I_A \otimes (|0_B\rangle\langle 0_B| - |1_B\rangle\langle 1_B|)$; this means that they have equal amounts of quantum correlation. Similarly, the states $\hat\rho_{AB}^{\psi^+}$ and $\hat\rho_{AB}^{\psi^-}$ are equivalent with the same local unitary operators. However, the two states $\hat\rho_{AB}^{\phi^\pm}$, in general, are not local-unitarily equivalent to the two states $\hat\rho_{AB}^{\psi^\pm}$. Therefore, in this process, the entanglement is distributed probabilistically with concurrence $\mathcal{C}_{AB}^{\phi}$ in the two outcomes $\hat\rho_{AB}^{\phi^\pm}$ and with $\mathcal{C}_{AB}^{\psi}$ in both states $\hat\rho_{AB}^{\psi^\pm}$, which, in general, are different. This asymmetric distribution is reminiscent of what happens with pure input states. From Eqs. (9) we realize that the matrix elements of $\hat\rho_{A,C1}$ are transferred to the outcome states of pair A,B when the state $\hat\rho_{B,C2} = |\phi^+_{B,C2}\rangle\langle\phi^+_{B,C2}|$. In this particular case, the states $\hat\rho_{AB}^{\phi^\pm}$ are local-unitarily equivalent with the $\hat\rho_{AB}^{\psi^\pm}$ states by means of the local unitary transformation, e.g., $I_A \otimes \sigma_x^B$ or $\sigma_x^A \otimes I_B$. This means that all the features of the X state $\hat\rho_{A,C1}$ are transferred to the state of the two remote qubits A and B when $\hat\rho_{B,C2}$ is a Bell state, thus obtaining the result of Ref. [7] as a special case of ours. In Ref. [7], $\hat\rho_{A,C1}$ is only inside $\mathcal{H}_{01,10}$ and undergoes an Ornstein-Uhlenbeck [9] decoherence process, and $\hat\rho_{B,C2}$ is a steady Bell state.

In order to simplify the analysis and to shed some light on the principal aspects and the scope of this scheme, we restring it to the case with elements of the matrices (8) equal, i.e., $\rho'_{nm} = \rho_{nm}$. In this case, the concurrences of the respective outcome states (9a) and (9b) become

$$\mathcal{C}_{AB}^{\phi}(\Delta) = \frac{2\max\{0, g(\Delta) - (\rho_{11}\rho_{33} + \rho_{22}\rho_{44})\}}{(\rho_{11}+\rho_{33})^2 + (\rho_{22}+\rho_{44})^2}, \tag{10a}$$

$$\mathcal{C}_{AB}^{\psi}(0) = \frac{\max\{0, g(0) - 2\sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}\}}{(\rho_{11}+\rho_{33})(\rho_{22}+\rho_{44})}, \tag{10b}$$

where $g(\varphi) = \sqrt{(|\rho_{14}|^2 + |\rho_{23}|^2)^2 - 4|\rho_{14}|^2|\rho_{23}|^2\sin^2(\varphi)}$, $\Delta = \theta_{14} - \theta_{23}$, and θ_{nm} is the phase of $\rho_{nm} = |\rho_{nm}|e^{i\theta_{nm}}$. The concurrence $\mathcal{C}_{AB}^{\psi}(0)$ does not depend on the phases θ_{nm} and it is higher than $\mathcal{C}_{AB}^{\phi}(\Delta)$ for all values of Δ. Besides having asymmetrically distributed the entanglement over the outcomes, the probability $2P_\psi$ of achieving $\mathcal{C}_{AB}^{\psi}(0)$ is smaller than $2P_\phi$ of obtaining $\mathcal{C}_{AB}^{\phi}(\Delta)$. The concurrence $\mathcal{C}_{AB}^{\phi}(\Delta)$ reaches its maximal value at $\Delta = 0$ and the minimum for $\Delta = \pi/2$. Applying the local unitary operation $|0\rangle\langle 0| + e^{i(\theta_{14}-\theta_{23})/2}|1\rangle\langle 1|$ to $C1$ and the $C2$, before performing the measurement procedure, the phases of the off-diagonal elements vanish, which leads to $\Delta = 0$. In this case the concurrences (10) become

$$\mathcal{C}_{AB}^{\phi} = \frac{2\max\{0, |\rho_{14}|^2 + |\rho_{23}|^2 - (\rho_{11}\rho_{33} + \rho_{22}\rho_{44})\}}{(\rho_{11}+\rho_{33})^2 + (\rho_{22}+\rho_{44})^2}, \tag{11a}$$

$$\mathcal{C}_{AB}^{\psi} = \frac{\max\{0, |\rho_{14}|^2 + |\rho_{23}|^2 - 2\sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}\}}{(\rho_{11}+\rho_{33})(\rho_{22}+\rho_{44})}. \tag{11b}$$

From these expressions we find that the four outcome states (9) are entangled if the matrix elements of the input states satisfy the inequality

$$|\rho_{14}|^2 + |\rho_{23}|^2 > \rho_{11}\rho_{33} + \rho_{22}\rho_{44}, \tag{12}$$

whereas the entanglement is present only in two outcome states (9b) if the following inequalities hold:

$$\rho_{11}\rho_{33} + \rho_{22}\rho_{44} \geqslant |\rho_{14}|^2 + |\rho_{23}|^2 > 2\sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}. \tag{13}$$

Otherwise the four outcomes states are separable.

By making a detailed analysis of the inequalities (4), (6), (7), (13), and (12) we can asseverate what follows:

(a) If the input states are not entangled, then $|\rho_{14}| \leqslant \sqrt{\rho_{22}\rho_{33}}$ and $|\rho_{23}| \leqslant \sqrt{\rho_{11}\rho_{44}}$. Multiplying the respective terms of these inequalities with those for positivity [Eqs. (4)] leads to $|\rho_{14}|^2 \leqslant \sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}$ and $|\rho_{23}|^2 \leqslant \sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}$. Adding them we obtain that $|\rho_{14}|^2 + |\rho_{23}|^2 \leqslant 2\sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}}$, which means that the right-hand side inequality of expression (13) is not fulfilled. In other words, if the input states lack entanglement, then the four outcome states are separable.

Additionally, the fact that the concurrence $\mathcal{C}_{\rm in}$ of the input states is different from zero does not guarantee that the outcomes states are entangled. Specifically, by considering that $\mathcal{C}_{\rm in} > 0$, we find the following:

(b) Only the two outcome states $\hat\rho_{AB}^{\psi_\pm}$ have entanglement different from zero if

$$\mathcal{C}_{\rm min}^{\rm th} < \mathcal{C}_{\rm in} \leqslant \mathcal{C}_{\rm max}^{\rm th}.$$

(c) The entanglement is present in the four outcome states $\hat\rho_{AB}^{\psi_\pm}$ and $\hat\rho_{AB}^{\phi_\pm}$ if

$$\mathcal{C}_{\rm in} > \mathcal{C}_{\rm max}^{\rm th}.$$

The two threshold concurrence values are given by

$$\mathcal{C}_{\rm min}^{\rm th} = 2\Big(\sqrt{2\sqrt{\rho_{11}\rho_{22}\rho_{33}\rho_{44}} - \min\{|\rho_{14}|^2, |\rho_{23}|^2\}} - \min\{\sqrt{\rho_{11}\rho_{44}}, \sqrt{\rho_{22}\rho_{33}}\}\Big), \tag{14}$$

$$\mathcal{C}_{\rm max}^{\rm th} = 2\Big(\sqrt{\rho_{11}\rho_{33} + \rho_{22}\rho_{44} - \min\{|\rho_{14}|^2, |\rho_{23}|^2\}} - \min\{\sqrt{\rho_{11}\rho_{44}}, \sqrt{\rho_{22}\rho_{33}}\}\Big). \tag{15}$$

These threshold values are decreasing functions of the off-diagonal element $\min\{|\rho_{14}|^2, |\rho_{23}|^2\}$, which does not affect the amount of $\mathcal{C}_{\rm in}$. Therefore we can realize that when the input states have entanglement, the off-diagonal term $\min\{|\rho_{14}|^2, |\rho_{23}|^2\}$ now plays an important roll for achieving the task of having entanglement in the outcomes. In fact, it allows one to decrease the threshold values and to increase the outcome's entanglement. Consequently, the outcome concurrence (11) reaches the maximum value when there is maximal coherence inside both subspaces $\mathcal{H}_{00,11}$ and $\mathcal{H}_{01,10}$, namely, $|\rho_{14}| = \sqrt{\rho_{11}\rho_{44}}$ and $|\rho_{23}| = \sqrt{\rho_{22}\rho_{33}}$. Therefore, in this case and for fixed diagonal elements, the highest outcome concurrence is

$$\mathcal{C}^{\psi}_{AB,\,\max} = \frac{(\sqrt{\rho_{11}\rho_{44}} - \sqrt{\rho_{22}\rho_{33}})^2}{(\rho_{11}+\rho_{33})(\rho_{22}+\rho_{44})}, \tag{16}$$

and the smallest outcome concurrence becomes

$$\mathcal{C}^{\phi}_{AB,\,\max} = \max\left\{0, \frac{2(\rho_{11}-\rho_{22})(\rho_{44}-\rho_{33})}{(\rho_{11}+\rho_{33})^2 + (\rho_{22}+\rho_{44})^2}\right\}. \tag{17}$$

Here, we easily note that $\mathcal{C}^{\psi}_{AB,\,\max}$ is higher and $\mathcal{C}^{\phi}_{AB,\,\max}$ is smaller than the input value $\mathcal{C}_{\rm in}$. Note that (17) is different from zero when $\rho_{11} > \rho_{22}$ and $\rho_{44} > \rho_{33}$ or $\rho_{11} < \rho_{22}$ and $\rho_{44} < \rho_{33}$, which are conditions for having initial entanglement, in agreement with inequality (12) evaluated for maximal coherence.

Our result can be illustrated in the following simple example. When the inputs are Werner states, $\hat{\rho}(\gamma) = (1-\gamma)I/4 + \gamma|\psi^+\rangle\langle\psi^+|$, which is nonseparable for $\gamma > 1/3$, the two threshold concurrence values are equal to $\sqrt{(1-\gamma^2)/2} - (1-\gamma)/2$. In this case the outcome states are entangled for $\gamma > 1/\sqrt{3}$. This agrees with the result obtained in Ref. [8], where the inputs are Werner states.

In summary, we find that this process swaps the X form of the input states without conditions. When one of the inputs is a Bell state, then the other input X state is swapped fully to the remote qubits. The input entanglements are, in general, partially distributed in the four possible outcome states under certain conditions. When the input states are equal, we obtain two threshold concurrence values which have to be overcome by the input state entanglement in order for the outcome states to be nonseparable. In addition, in this case we find that there are two possible amounts of outcome entanglement: one is greater and the other is less than the input entanglement. The probability of obtaining the greatest outcome entanglement is smaller than the probability of attaining the least. Finally, we would like to emphasize that the asymmetric redistribution of the entanglement holds also in the case with pure states, but the *threshold-concurrence-values* effect occurs only for mixed X states and is thus a consequence of the nonzero decoherence of the input states.

The authors are thankful for support from FONDECyT (Grant No. 1120695), CONICyT, and DAAD RISE Worldwide (Grant No. CL-PH-1999).

[1] E. Knill and R. Laflamme, Phys. Rev. Lett. **81**, 5672 (1998); L. Roa, J. C. Retamal, and M. Alid-Vaccarezza, *ibid.* **107**, 080401 (2011); D. Bruss and C. Machiavello, Found. Phys. **33**, 1617 (2003); L. Roa, M. Alid-Vaccarezza, C. Jara-Figueroa, and A. B. Klimov, Phys. Lett. A **378**, 941 (2014).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993); K. Banaszek, Phys. Rev. A **62**, 024301 (2000); L. Roa, A. Delgado, and I. Fuentes-Guridi, *ibid.* **68**, 022310 (2003); W. L. Li, C. F. Li, and G. C. Guo, *ibid.* **61**, 034301 (2000); Li-Yi Hsu, *ibid.* **66**, 012308 (2002); C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).

[3] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998); S. Hill and W. K. Wootters, *ibid.* **78**, 5022 (1997).

[4] L. Henderson and V. Vedral, J. Phys. A: Math. Gen. **34**, 6899 (2001); H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).

[5] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993); B. Yurke and D. Stoler, Phys. Rev. A **46**, 2229 (1992); H. Weinfurter, Europhys. Lett. **25**, 559 (1994).

[6] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004); **97**, 140403 (2006); M. F. Santos, P. Milman, L. Davidovich, and N. Zagury, Phys. Rev. A. **73**, 040305(R) (2006); Qi-liang He and Jing-bo Xu, J. Opt. Soc. Am. B **30**, 251 (2013); F. Lastra, C. E. López, L. Roa, and J. C. Retamal, Phys. Rev. A **85**, 022320 (2012); L. Roa, A. Krügel, and C. Saavedra, Phys. Lett. A **366**, 563 (2007); R. Lo Franco, B. Bellomo, S. Maniscalco, and G. Compagno, Int. J. Mod. Phys. B **27**, 1345053 (2013); B. Bellomo, G. Compagno, A. D'Arrigo, G. Falci, R. Lo Franco, and E. Paladino, Phys. Rev. A **81**, 062309 (2010); R. Lo Franco, A. D'Arrigo, G. Falci, G. Compagno, and E. Paladino, Phys. Scr. **T147**, 014019 (2012); B. Bellomo, G. Compagno, R. Lo Franco, A. Ridolfo, and S. Savasta, Int. J. Quantum Inf. **9**, 1665 (2011); B. Aaronson, R. Lo Franco, and G. Adesso, Phys. Rev. A **88**, 012120 (2013); B. Aaronson, R. Lo Franco, G. Compagno, and G. Adesso, New J. Phys. **15**, 093022 (2013); B. Bellomo, R. Lo Franco, S. Maniscalco, and G. Compagno, Phys. Scr. **T140**, 014014 (2010).

[7] C. Miao, M. Yang, and Z.-L. Cao, Int. J. Theor. Phys. **52**, 1780 (2013).

[8] W. Song, M. Yang, and Z. L. Cao, Phys. Rev. A **89**, 014303 (2014).

[9] T. Yu and J. H. Eberly, Quant. Inf. Comput. **7**, 459 (2007); S. M. Hashemi Rafsanjani, M. Huber, C. J. Broadbent, and J. H. Eberly, Phys. Rev. A **86**, 062303 (2012); S. M. Hashemi Rafsanjani and S. Agarwal, arXiv:1204.3912v1.