# Transformation devices: Event carpets in space and space-time

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Here we extend the theory of space-time or event cloaking into that based on the carpet or ground-plane reflective surface. Further, by recasting and generalizing a scalar acoustic wave model into a mathematically covariant form, we also show how transformation theories for optics and acoustics can be combined into a single prescription. The single prescription, however, still respects the fundamental differences between electromagnetic and acoustic waves, which then provide us with an existence test for any desired transformation device: Are the required material properties (the required constitutive parameters) physically permitted by the wave theory for which it is being designed? While electromagnetism is a flexible theory permitting almost any transformation-device ($T$-device) design, we show that the acoustic model used here is more restricted.

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## I. INTRODUCTION

The concept of electromagnetic cloaking has now been with us for more than 7 years [1,2], but has been recently revitalized by the introduction of the concept of space-time cloaking [3,4]. Here, in light of the many variants of spatial cloaking that now exist—ordinary cloaks, carpet cloaks [5], exterior cloaks [6]—we extend the possible implementation to include space-time *carpet* cloaks. Of course, cloaking is not the only application of transformation theories, as evidenced not only by spatial illusion devices [7–9], but also ideas for space-time illusion devices [4] that appear to trick—but only trick—causality [10]. Other applications such as beam control [11,12], geodesic lenses [13,14], hyperbolic materials [15,16], and cosmological models [4,17–19] are also possible.

Like a spatial carpet cloak, the space-time "event" carpet cloak is one sided and reflective and is less singular than a traditional spatial cloaking implementation. Rather than hide a region of space in perpetuity, however, the event carpet cloak allows the region to always appear visible, even though a finite segment of its time line is forced to take place in darkness. This dark period means that events therein are edited out of visible history. However, those missing events are covered up by clever manipulation of the light signals that communicate to observers their information about events. Alternatively, the transformation can be adapted to temporarily include extra events, rather than exclude them; thus creating a space-time "periscope." We might even relax our preference for invisibility devices, and construct one that appears bigger on the inside than the outside: a "tardis"-like device.

In this paper we deliberately use *first-order* equations to model our wave mechanics [20] so that a pair of them are needed (in concert with constitutive or state equations of some kind) to generate wave behavior. If desired, the first-order equations can be substituted inside one another to give a familiar second-order form, but, in fact, the first-order formulation is both less restrictive and more general. Indeed, we see that it is suggestive of a generalization to a *transformation mechanics* valid for any wave equations expressible in such a form, being also related to a "transformation media" concept addressing the material properties required by transformation devices ($T$ devices). Notably, here we apply the event carpet cloak and periscope transformations to both electromagnetism (EM) and a simple pressure-acoustics ($p$-acoustics) model, demonstrating how specific wave models allow or restrict the set of possible $T$ devices.

We introduce the "wave mechanics" models for EM and our $p$ acoustics in Sec. II and show how the two can be unified. This then allows us to unify transformation optics, transformation $p$ acoustics, and indeed any other compatible wave transformation theory into a general transformation mechanics. After this, we specialize to ground-plane $T$ devices. In Sec. III we investigate those based on the (spatial) carpet cloak, and in Sec. IV we generalize them into space-time carpet $T$ devices. In Sec. V we explain the operation of space-time carpet cloaks, and propose a scheme for implementing one for EM waves, before concluding in Sec. VI.

## II. WAVES AND TRANSFORMATIONS

In this paper we do not restrict ourselves to a single type of wave mechanics in which to construct ground-plane $T$ devices. Instead, we show that a unified procedure is possible, encompassing both EM and a simple $p$ acoustics model and indeed any linear wave mechanics that can be described within a theory containing two second-rank field tensors and one fourth-rank constitutive tensor. We start with the ordinary vector calculus descriptions and then show how they can be reexpressed in the tensor form and also describe how the tensor descriptions can be abbreviated into a more accessible matrix form. This process is, of course, well known for EM, but here we use an unconventional tensorization so that it more easily matches up with the $p$-acoustic description and other possible generalizations, as well as with straightforward matrix algebra versions. A simpler one-dimensional (1D) introduction to this description can be seen in [4]. After the wave theory descriptions, we show how transformations intended to implement specific $T$ devices affect the differential equations and constitutive relations.

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### A. Electromagnetism

Maxwell's equations [21] are often written as a set of vector differential equations with two pairs of field types, and two constitutive relations. These are

$$\partial_t D(\vec{r},t) = \vec{\nabla} \times \vec{H}(\vec{r},t) - \vec{J}_f(\vec{r},t), \qquad \vec{\nabla} \cdot \vec{D} = \rho_f, \quad (2.1)$$

$$\partial_t B(\vec{r},t) = -\vec{\nabla} \times \vec{E}(\vec{r},t), \quad \vec{\nabla} \cdot \vec{B} = 0, \quad (2.2)$$

with constitutive relations

$$\vec{B} = \boldsymbol{\mu}\vec{H} - \boldsymbol{\mu}\boldsymbol{\beta}\vec{E} \rightarrow \vec{H} = \boldsymbol{\eta}\vec{B} + \boldsymbol{\beta}\vec{E}, \quad (2.3)$$

$$\vec{D} = \boldsymbol{\epsilon}'\vec{E} + \boldsymbol{\mu}\boldsymbol{\alpha}\vec{H} \quad \rightarrow \quad \vec{D} = \boldsymbol{\epsilon}\vec{E} + \boldsymbol{\alpha}\vec{B}, \quad (2.4)$$

where the permittivity matrix is $\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} - \boldsymbol{\alpha}\boldsymbol{\mu}\boldsymbol{\beta}$ and $\boldsymbol{\eta}$ is the inverse of the more common permeability matrix $\boldsymbol{\mu}$. Although in many circumstances the left hand (LH) form is preferred, for the tensor form discussed next the right hand (RH) form is used instead. The matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ contain information about magnetoelectric coupling and follow $\boldsymbol{\beta} = \boldsymbol{\alpha}^\dagger$ [22].

Since we are interested primarily in freely propagating EM fields, we assume that that there are no free electric charges (i.e., $\rho_f = 0$) and, as is usual, we have already assumed in Eq. (2.2) that there are no magnetic charges or currents.

In tensor form the differential equations have a remarkably simple form. In order to emphasize their similarities, we chose to use the usual $\mathsf{G}$ tensor density (i.e., $G^{\alpha\beta}$) but the dual of the $\mathsf{F}$ tensor (i.e., $*\mathsf{F}$, with components $*F^{\alpha\beta} = \frac{1}{2}F_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}$), where $\varepsilon^{\mu\nu\alpha\beta}$ is the antisymmetrization symbol. The equations can then be written as

$$\partial_\alpha *F^{\alpha\beta} = 0, \quad \partial_\beta G^{\alpha\beta} = J^\alpha. \quad (2.5)$$

Using $\mathsf{G}$ and $*\mathsf{F}$ means that we must then use a mixed form for the constitutive tensor $\chi$ in the constitutive relations, with

$$G^{\alpha\beta} = \tfrac{1}{2}\chi^{\alpha\beta}_{\gamma\delta} *F^{\gamma\delta}. \quad (2.6)$$

This use of $*F^{\alpha\beta}$, $G^{\alpha\beta}$, and $\chi^{\alpha\beta}_{\gamma\delta}$, both generalizes better and maps onto the vector calculus representation more simply.

Note that since $J^\alpha$ (and its vector counterpart $\vec{J}$) is a current density, this means that $\mathsf{G}$ is also a density. Likewise, although the usual EM tensor $\mathsf{F}$ is not a density, its dual $*\mathsf{F}$ is. Thus, Eqs. (2.5) must be transformed as densities. Consequently, the mixed-form constitutive tensor $\chi^{\alpha\beta}_{\gamma\delta}$ is *not* a density.

In terms of generating wave solutions, the constitutive relations connect the two tensor differential equations together into a single system and so allow (wave) solutions to be found (i.e., exist). This is most easily seen using the vector form, where we can use the constitutive relations to allow us to substitute one of the Maxwell curl equations into the other, giving us one of the four [23] possible electromagnetic second-order wave equations.

Matrix representations of the antisymmetric $*\mathsf{F}$ and $\mathsf{G}$ can be written out in $[t,x,y,z]$ coordinates, as

$$[*F^{\alpha\beta}] = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}, \quad (2.7)$$

$$[G^{\alpha\beta}] = \begin{bmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{bmatrix}. \quad (2.8)$$

Here the $\partial_\alpha$ has become the row vector $[\partial_t,\partial_x,\partial_y,\partial_z]$, and the current density $J^\alpha$ also becomes a row vector.

The constitutive tensor $\chi^{\alpha\beta}_{\gamma\delta}$ is rank 4 so that a direct matrix representation is too big to display. Nevertheless, its permittivity entries convert $E_i$ from $*F^{\gamma\delta}$ into $D_i$ from $G^{\alpha\beta}$, thus they will have the same units as permitivitty (i.e., as $\epsilon$). Likewise, the permeability entries convert $B_i$ from $*F^{\gamma\delta}$ into $H_i$ from $G^{\alpha\beta}$; thus, they will have the same units as inverse permeability (i.e., as $\mu^{-1} = \eta$).

We can follow the usual method of achieving a convenient matrixlike compacted representation for $*F^{\alpha\beta}$ and $G^{\alpha\beta}$, using the fact that each tensor has only six unique entries. For all combinations of indices, we select out the relevant row and column coordinate pairs $tx,ty,tz,yz,zx,xy$ in sequence, so we can write $*F^{\alpha\beta}$ and $G^{\alpha\beta}$ as column vectors $[*F^A]$ and $[G^B]$ in turn, being

$$[*F^A] = [B_x,B_y,B_z,-E_x,-E_y,-E_z]^T, \quad (2.9)$$

$$[G^B] = [D_x,D_y,D_z,H_x,H_y,H_z]^T. \quad (2.10)$$

As a result the rank-4 constitutive tensor can also be compacted and so be represented by a $6 \times 6$ matrix. Again the first (upper) index $A$ spans the matrix rows, so that

$$\left[\chi^A_B\right] = \begin{bmatrix} \boldsymbol{\alpha} & -\boldsymbol{\epsilon} \\ \boldsymbol{\eta} & -\boldsymbol{\beta} \end{bmatrix}, \quad (2.11)$$

so that the constitutive relation now has a matrix form,

$$\left[G^A\right] = \left[\chi^A_B\right][*F^B], \quad (2.12)$$

which is particularly convenient for manual calculation. Normally when using this kind of representation, the all-upper index tensor $\chi^{\alpha\beta\gamma\delta}$ is used, but here we are using the mixed tensor form $\chi^{\alpha\beta}_{\gamma\delta}$. This means that our matrix expression looks different; notably a simple isotropic nonmagnetoelectric medium with a diagonal $\chi^{\alpha\beta\gamma\delta}$ (matrix) is now block off-diagonal.

Note that the matrix $[\chi]$ only includes half of all the allowed elements of $\chi^{\alpha\beta}_{\gamma\delta}$, but the missing elements are all duplicates of included ones.[1] However, each element of $[\chi]$ nevertheless independently links any component of $[F]$ to any other of $[G]$; there is no duplication or *a priori* requirement that certain constitutive parameters must be equal or otherwise closely related. As a result, each constitutive property of an electromagnetic material can (at least mathematically) be adjusted independently, meaning that any well specified $T$ device might be constructed.

[1] This is why no 1/2 appears in the *matrix* constitutive relations. However, we must still account for this doubling up when evaluating transformations later.

### B. Pressure acoustics

Here we introduce a simplified pressure-acoustic ($p$-acoustic) theory, equivalent to one describing linear (perturbative) acoustic waves on a stationary background fluid, although it also encompasses many other types of wave which contain a scalar component. For linearized acoustic waves [24] in pressure $p$ and fluid velocity $\vec{u}$, we have constitutive parameters $\bar{\kappa}$ (compressibility) and $\bar{\rho}$ (mass density). As well as the traditional quantities, we also specify both a scalar $\bar{P}$, and a momentum density $\vec{V}$, giving a total of four quantities that in combination emphasize both the similarities and the differences compared with the description of EM; this can be compared to, e.g., the treatment by Sklan [25]. It also means that $p$ acoustics is also a more general model than the comparable traditional ones, which typically reduce the equations back down to $p$ and $\vec{u}$, or even just a second-order wave equation in $p$.

Here, however, in order to maximize the similarity with the EM notation, we substitute the scalar $\bar{P}$ with a number density $P$ and the fluid velocity $\vec{u}$ with a velocity density $\vec{v}$, and the constitutive parameters become an inverse energy $\kappa$ and a particle mass $\rho$. Since the background fluid is stationary, for small-amplitude waves the $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term that would usually appear in acoustic wave equations is second order and hence can be neglected.

For scalar quantities $P$ and $p$ and vectors $\vec{V}$ and $\vec{v}$, the wave equation pair, here used to model $p$ acoustics, is

$$\partial_t P(\vec{r},t) = -\vec{\nabla}\cdot\vec{v}(\vec{r},t) + Q_P(\vec{r},t), \tag{2.13}$$

$$\partial_t \vec{V}(\vec{r},t) = -\vec{\nabla}p(\vec{r},t) + \vec{Q}_V(\vec{r},t), \tag{2.14}$$

$$\vec{\nabla}\times\vec{V} = \vec{\Sigma}, \quad \vec{\nabla}\times\vec{v} = \vec{\sigma}, \tag{2.15}$$

where in simple cases $P = \kappa p$ is a number density related to the pressure $p$ by an inverse energy $\kappa$, and the momentum density $\vec{V} = \boldsymbol{\rho}\vec{v}$ is related to the velocity-field density $\vec{v}$ by a matrix of mass parameters $\boldsymbol{\rho} = [\rho^i_j]$. There are two allowed types of source, a particle number (density) source $Q_P$ and a momentum (density) source $\vec{Q}_V$.

The differential equation, Eq. (2.13), is related to the conservation of particle number (and conservation of mass) in a microscopic acoustic model, and Eq. (2.14) ensures conservation of momentum. The third equation shows the $p$ acoustics analog of charge, and, just as for the EM case, where we are interested primarily in freely propagating fields, we set these to zero ($\vec{\Sigma} = 0, \vec{\sigma} = 0$).

The most general constitutive relations for coupling between the scalar and vector fields, can be written

$$P = \kappa^{-1}p - \kappa^{-1}\vec{\alpha}\cdot\vec{v} \rightarrow p = \kappa P + \vec{\alpha}\cdot\vec{v}, \tag{2.16}$$

$$\vec{V} = \vec{\beta}\kappa^{-1}p + \boldsymbol{\rho}'\cdot\vec{v} \rightarrow \vec{V} = \vec{\beta}P + \boldsymbol{\rho}\cdot\vec{v}, \tag{2.17}$$

where $\boldsymbol{\rho}\cdot\vec{v} = \boldsymbol{\rho}'\cdot\vec{v} + \vec{\beta}(\vec{\alpha}\cdot\vec{v})/\kappa$. Although in many circumstances the first (LH) form might be used, for the tensor form discussed next the second (RH) form is instead preferred. The vector $\vec{\alpha}$ parameterizes some kind of a velocity $\rightarrow$ pressure coupling, and $\vec{\beta}$ parameterizes a pressure $\rightarrow$ momentum density coupling.

Here the physical meanings of $\vec{v},\vec{V}$, and $p,P$ mean that both Eq. (2.13) and Eq. (2.14) transform like densities, but the constitutive relations Eqs. (2.16) and (2.17) do not, just as for EM.

Just as for EM, we can now embed $P,\vec{v}$ into a tensor density, which, to ensure compatible notation, we call $*\mathsf{F}$, with contravariant components $*F^{\alpha\beta}$. We also embed the density fields $p,\vec{V}$ into a tensor density $\mathsf{G}$ with components $G^{\alpha\beta}$ and populate a constitutive tensor $\chi$ with $\kappa,\boldsymbol{\rho},\vec{\alpha},\vec{\beta}$ appropriately. The $p$-acoustic differential equations now appear in the same form as for EM, being

$$\partial_\alpha *F^{\alpha\beta} = J^\beta, \quad \partial_\beta G^{\alpha\beta} = K^\alpha, \tag{2.18}$$

with source terms $J^\alpha, K^\alpha$. The constitutive relations are

$$G^{\alpha\beta} = \chi^{\alpha\beta}_{\gamma\delta} *F^{\gamma\delta}. \tag{2.19}$$

Here the only differences from EM are the internal structure—how the tensors are populated—and no factor of one half in the constitutive relations. Again, the $\mathsf{G}$ and $*\mathsf{F}$ tensor densities and their differential equation must transform like densities, but the constitutive tensor $\chi$ does not.

In $[t,x,y,z]$ coordinates the tensor $*F^{\alpha\beta}$ and density tensor $G^{\alpha\beta}$ can be written out using field matrices

$$[*F^{\alpha\beta}] = \begin{bmatrix} P & 0 & 0 & 0 \\ v_x & 0 & 0 & 0 \\ v_y & 0 & 0 & 0 \\ v_z & 0 & 0 & 0 \end{bmatrix},$$

$$[G^{\alpha\beta}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ V_x & p & 0 & 0 \\ V_y & 0 & p & 0 \\ V_z & 0 & 0 & p \end{bmatrix}, \tag{2.20}$$

with source terms $J^0 = Q_P$ and $\vec{K} = [K^i] = \vec{Q}_V$. However, both $\vec{J} = [J^i] = 0$ and $K^0 = 0$.

Here the constitutive tensor $\chi^{\alpha\beta}_{\gamma\delta}$ relates the $P$ element from $*F^{\gamma\delta}$ to the $p$ from $G^{\alpha\beta}$, and is thus $\sim\kappa$; likewise, the $v_i$ elements from $*F^{\gamma\delta}$ are related to $V_i$ from $G^{\alpha\beta}$ by $\sim\rho$.

Note, however, that while this conveniently encodes the model of pressure acoustics we consider here, it is not the most general that might be formulated. The most general theory would be to consider field tensors that are symmetric, to contrast with the EM representation using antisymmetric field tensors. This would give not two sets of four amplitudes (i.e., $P$ and $v_i$, with $p$ and $V_i$), but two sets of ten amplitudes. However, we leave discussion of such details to a later work, including describing how pentamode acoustics [26] can be represented in this way.

As with EM, the tensor constitutive representation can be compacted. Here we use the fact that the elements $G^{xx}$, $G^{yy}$, and $G^{zz}$ are all just $p$ and so write a compact vector for $\mathsf{G}$ using this knowledge; we do similarly for $*\mathsf{F}$. The resulting field column vectors $[G^A]$, $[*F^B]$ are

$$[G^A] = [p, V_x, V_y, V_z]^T, \tag{2.21}$$

$$[*F^B] = [P, v_x, v_y, v_z]^T. \tag{2.22}$$

The matrix representation of $\chi^{\alpha\beta}_{\gamma\delta}$ has the upper index denoting rows, and the lower index denoting columns, so that the $4\times4$

constitutive matrix is

$$\left[\chi^{\alpha\beta}_{\gamma\delta}\right] = \left[\chi^{A}_{B}\right] = \begin{bmatrix} \kappa & \vec{\alpha}^{T} \\ \vec{\beta} & \boldsymbol{\rho} \end{bmatrix}; \tag{2.23}$$

the matrix constitutive relation is

$$[G^{A}] = \left[\chi^{A}_{B}\right][*F^{B}] \tag{2.24}$$

$$\begin{bmatrix} G^{ww} \\ G^{xt} \\ G^{yt} \\ G^{zt} \end{bmatrix} = \begin{bmatrix} \kappa & \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \beta_x & \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \beta_x & \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix} \begin{bmatrix} *F^{tt} \\ *F^{xt} \\ *F^{yt} \\ *F^{zt} \end{bmatrix}. \tag{2.25}$$

Note that we *must* remember that the $G^{ww}$ element can be freely substituted by any of $G^{xx}$, $G^{yy}$, or $G^{zz}$; this acts as an implicit constraint on allowed transformations, which are only allowed to transform $x$, $y$, and $z$ equivalently.[2] In a scalar wave theory such as $p$ acoustics, all nonisotropic *wave* behavior has to derive from the constitutive parameters $\vec{\alpha}$, $\vec{\beta}$, or $\boldsymbol{\rho}$, and not from a transformation.

### C. Transformations

With both of our preferred sorts of waves having been expressed using the same mathematical machinery, we can now investigate the effect of coordinate transformation in either case simultaneously, leaving any more specific details to a later stage. Deforming transformations (or, technically, diffeomorphisms [27]) of the coordinates (and hence of the matrix representations of the field and constitutive tensors) are simple to handle. Transformations between the original coordinates $x^{\alpha}$ and new ones $x^{\alpha'}$ as are easily written as

$$T^{\alpha'}_{\alpha} = \left[\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right], \tag{2.26}$$

so that the field density tensors $\mathsf{G}$ and $*\mathsf{F}$ transform as

$$G^{\alpha'\beta'} = \Delta^{-1} T^{\alpha'}{}_{\alpha} T^{\beta'}{}_{\beta} G^{\alpha\beta} = L^{\alpha'\beta'}_{\alpha\beta} G^{\alpha\beta}, \tag{2.27}$$

$$*F^{\gamma'\delta'} = \Delta^{-1} T^{\gamma}_{\gamma'} T^{\delta}_{\delta'} *F^{\gamma\delta} = M^{\gamma\delta}_{\gamma'\delta'} *F^{\gamma\delta}. \tag{2.28}$$

The factor $\Delta = |\det(T)|$ occurs because the fields are represented by tensor *densities* rather than a pure tensor. Using the modulus of the determinant ensures that parity transformations changing the handedness still preserve positive volumes.

Now we can address the transformation of the constitutive tensor $\chi$. If for EM we set $a = 2$, and for $p$ acoustics we set $a = 1$, then

$$G^{\alpha'\beta'} = \frac{1}{a} \chi^{\alpha'\beta'}_{\gamma'\delta'} *F^{\gamma'\delta'} \tag{2.29}$$

$$\text{and} \quad \chi^{\alpha'\beta'}_{\gamma'\delta'} = T^{\alpha'}_{\alpha} T^{\beta'}_{\beta} \chi^{\alpha\beta}_{\gamma\delta} T^{\gamma}_{\gamma'} T^{\delta}_{\delta'}. \tag{2.30}$$

Here we have seen how coordinate transformations can be actualized by seeing how they affect the constitutive tensor. However, there is also another part to the wave mechanics: the differential equations.

If our coordinate transformation is not just a deformation of our existing *Cartesian* coordinate system, but, e.g., a change to cylindrical or polar coordinates, then the wave equations change their form. Although such radical coordinate conversions can be very useful, they are a reparametrization of the host space-time and not a straightforward deformation, hence the appearance of extra factors of $r$ when converting from Cartesians to cylindrical coordinates. For brevity, we do not address this case here and focus instead on the deforming transformations used to produce particular $T$ devices.

Since the two field tensors might be compacted differently, as happens for $p$ acoustics, we need to allow for distinct compacted deformation matrices, one for $*\mathsf{F}$ and another for $\mathsf{G}$. The compact deformation matrix equation for the vector $[*F^{B}]$ depends on a compacted $M^{\alpha'\beta'}_{\alpha\beta}$ and is

$$[*F^{B'}] = \left[M^{B'}_{B}\right][*F^{B}], \tag{2.31}$$

but for the $[G^{A}]$ we instead compact $L^{\alpha'\beta'}_{\alpha\beta}$ to get

$$[G^{A'}] = \left[L^{A'}_{A}\right][G^{A}]. \tag{2.32}$$

The matrix representation of $\chi^{\alpha\beta}_{\gamma\delta}$ also transforms and is

$$[\chi'] = \left[\chi^{A'}_{B'}\right] = \left[L^{A'}_{A}\right]\left[\chi^{A}_{B}\right]\left[M^{B'}_{B}\right]^{-1}. \tag{2.33}$$

So if we want to physically implement the transformation $T$ as a $T$ device, for any possible states of the field tensors $*\mathsf{F}$ and $\mathsf{G}$, we need to change the host material so that its constitutive makeup is not simply the reference value $\chi$, but that of the transformed constitutive makeup $\chi'$. In what follows we implement this for both purely spatial and the more exotic space-time cloaks in a ground-plane reference geometry.

In EM we would typically start (as we do below) with a reference material and solution based on a vacuum or a uniform static nondispersive refractive index and—for carpet $T$ devices—a light-reflective surface. For acoustics we would pick a stationary and featureless elastic medium or fluid with an acoustically reflective surface. After applying the deformation to the reference constitutive parameters, we will have determined how the desired alterations to the wave mechanics (the field propagation) can be induced by appropriate changes to the propagation medium.

## III. SPATIAL CARPET $T$ DEVICES

"Carpet" or ground-plane transformation devices are essentially surfaces that have been modified; in their original conception they were engineered mirrors that appeared to give simple reflections but were actually hiding some predefined region [5]. More generally, though, it is not required for the surface to be a mirror; the transformation respecifies the properties inside a region where the light propagates, not what happens outside that region. To make it clearer how our general prescription can be applied, we first apply it to the familiar spatial carpet cloak, although generalized to incorporate other related $T$ devices.

We start with a planar carpet that lies flat on the $y$-$z$ plane at $x = 0$ and decide to transform a localized region of space—the

[2] A more explicit version of the $[G^{A}]$ vector might have six elements, the first three of which would represent $G^{xx}$, $G^{yy}$, or $G^{zz}$ in turn, but all being equal to $p$, and with $[\chi^{A}_{B}]$ being a $6 \times 4$ matrix, the top three rows being identical. We must take care to allow for this multiplicity when transforming $[G^{A}]$.

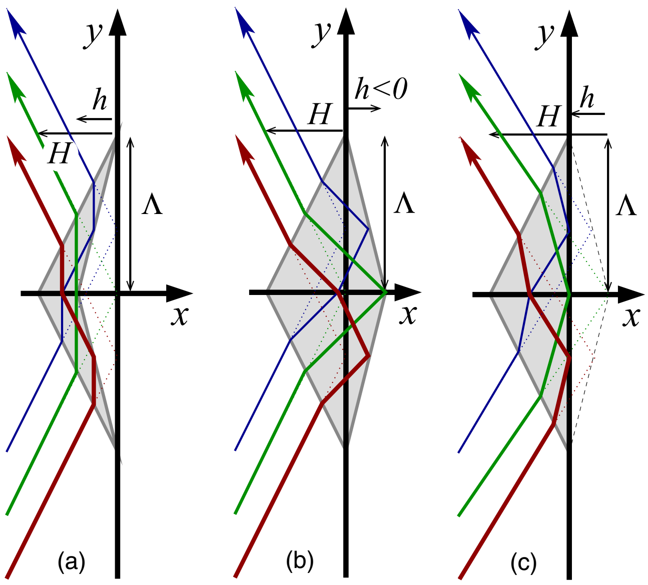


FIG. 1. (Color online) The (a) spatial carpet cloak, (b) spatial periscope, (c) spatial tardis. Continuous lines show the actual ray trajectories and dotted lines show the apparent (illusory) ray trajectories. The periscope (b) needs to be embedded in a high-index host medium to allow the internal rays to cover the greater distance to the back surface than to the apparent surface.

cloak "halo"—reaching no higher than $x = -H$ above the plane and no further than $y = \pm\Lambda$ sideways; as shown by the shaded regions in Fig. 1. First, we can choose to restrict the space probed by incoming waves to that between the maximum height $H$ and some lower height $x = h$, but transform the material so that the *apparent* space is expanded and extends all the way from $x = 0$ to $x = -H$. This $T$ device is the usual carpet cloak with an offset $h$ and scaling $R = (H - h)/H < 1$ [see Fig. 1(a)].

Second, we might expand the actual space probed by incoming waves to that between the maximum height $H$ and to a penetration depth below the plane of $x = h$, but transform the material to shrink the *apparent* space to only that between $x = 0$ to $x = -H$. This $T$ device is a cloaked "periscope" with a negative offset $h < 0$ and scaling $R = (H - h)/H > 1$. See Fig. 1(b), where we see that the periscope allows an observer to remain *below* the ground plane while allowing them an unrestricted view of their surroundings, just as if they were exposed above it.

Third, we can define the actual space probed by incoming waves to be just the same as if there were a flat plane, but nevertheless transform the material so that the *apparent* space extends below the plane by $h$. This $T$ device might be called a "tardis," since it presents the illusion that it contains a space bigger than it is on the outside; it uses an offset $h > 0$ and scaling $R = (H - h)/H < 1$. Its diagram is shown in Fig. 1(c), where the transformation is the reverse of that for the periscope, but in appearance it acts like a carpet cloak for when the observer expects to see a *dimple* on the plane.

Last, we could define the space probed by incoming waves to be just the same as if there were a flat plane, but nevertheless transform the material so that the *apparent* space was shrunk back as if there were a bump. This $T$ device might be called an "antitardis," but for brevity we do not discuss such a device here. Its diagram is not shown on Fig. 1, but note that its transformation is the reverse of that for the carpet cloak but in appearance it would act like a periscope for when the observer expects to see a *bump* on the plane.

The deformation that stretches a uniform medium (unprimed coordinates) into these $T$-device designs (primed coordinates) is applied (only) inside the shaded halo regions on Fig. 1 and keeps $t' = t, y' = y, z' = z$, but deforms $x$ so that

$$x' = Rx - C\frac{\Lambda - y\,\mathrm{sgn}(y)}{\Lambda}h = Rx - Ch + rsy, \tag{3.1}$$

where the ratio of actual depth $(H - h)$ to apparent depth $(H)$ is $R = (H - h)/H$, $r = Ch/\Lambda$, and $s = \mathrm{sgn}(y)$. Here $C$ is set to 1 for the carpet cloak and periscope, but for the tardis (where $R > 1$) it needs to be $C = R$. Using a general viewpoint applicable to any of these three $T$ devices, we see that $R < 1$ apparently expands the wave-accessible space, as needed for a cloak or tardis, while $R > 1$ compresses it, as needed for a periscope or antitardis.

The effect of this deformation from Eq. (3.1) is given by $T_\alpha^{\alpha'}$, which specifies the differential relationships between the primed and unprimed coordinates, where $\alpha \in \{t,x,y,z\}$ and $\alpha' \in \{t',x',y',z'\}$. If we let rows span the first (upper) index, and columns the second (lower) index, then

$$T_\alpha^{\alpha'} = \left[\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R & rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.2}$$

and note that $\det(T_\alpha^{\alpha'}) = R$. For some column 4-vector $V^\alpha$, we have that,

$$\text{e.g.,}\quad \begin{bmatrix} V^{t'} \\ V^{x'} \\ V^{y'} \\ V^{z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R & rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^t \\ V^x \\ V^y \\ V^z \end{bmatrix}. \tag{3.3}$$

It is worth noting that inside the carpet cloak halo, where the cloak affects the path of the rays, but not inside the cloak core, where objects are hidden, the deformation consists solely of a constant rescaling of $x$; the rest of the cloak design is defined by its spatial layout, reversed properties in the upper and lower halves, and its boundaries.

### A. EM carpets

We expressed the EM constitutive tensor using a compact $6 \times 6$ matrix form; therefore, we also use compact $6 \times 6$ deformation (transformation) matrices $[L],[M]$, being versions of the product $L_\alpha^{\alpha'} L_\beta^{\beta'}$. Note that the use of the unconventional choice of (the dual) $*F^{\alpha\beta}$ and (the mixed) $\chi_{\gamma\delta}^{\alpha\beta}$ allows us to calculate using straightforward matrix multiplication.

The two-dimensional matrix representation of the transformation matrix, as compressed for EM, is best written as an explicit transformation between the field 6-vectors to avoid error. Thus, the compressed column 6-vector $[G^A]$ is transformed using[3]

$$[L^{A'}_{A}] = \frac{1}{R}\begin{bmatrix} R & +rs & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -rs & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix}, \tag{3.4}$$

and where the matrix $[M]$ that transforms $[*F^B]$ is the same. To transform $[\chi]$, we need the inverse of $[M]$, which is

$$[M^{A'}_{A}]^{-1} = \begin{bmatrix} 1 & -rs & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & +rs & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

Starting from simple uniform background medium described only by $\epsilon$ and $\mu$—perhaps a vacuum with $\epsilon = \epsilon_0$ and $\mu = \mu_0$—the transformed constitutive matrix defining the EM carpet $T$ device is then

$$[\chi^{A'}_{B'}] = \frac{\epsilon}{R}\begin{bmatrix} 0 & 0 & 0 & -(R^2+r^2) & -rs & 0 \\ 0 & 0 & 0 & -rs & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c^2 & -rsc^2 & 0 & 0 & 0 & 0 \\ -rsc^2 & c^2(R^2+r^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & R^2c^2 & 0 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

So we see a need for birefringence, as defined by the off-diagonal permittivity elements $-\epsilon rs/R$, and the off-diagonal permeability contributions $-\epsilon rsc^2/R$. To get a $\mu$ matrix, we need to invert the $\eta$ submatrix:

$$\frac{[\mu]}{\mu_0} = \begin{bmatrix} 1/R & -rs/R & 0 \\ -rs/R & 1+r^2/R & 0 \\ 0 & 0 & R \end{bmatrix}^{-1} = \frac{1}{R}\begin{bmatrix} R^2+r^2 & +rs & 0 \\ +rs & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

Note how the structure of the (sub)matrix $[\mu]$ matches up with that of the (sub)matrix $[\epsilon]$, as we would expect as our transformation process demands a preservation of impedance matching. Numerical examples indicating how the EM fields are distorted by the different carpet $T$ devices are shown in Fig. 2.

As is well known, the requirement for impedance matching can be relaxed. Since the simple nature of the transformation of Eq. (3.1) leads to a constitutive matrix with constant values, this means that these carpet devices can be made for a specified polarization using the natural birefringence of two correctly shaped and oriented pieces of calcite, as already demonstrated for a carpet cloak [28,29].

### B. $p$-Acoustics carpets

For $p$-acoustic transformations, we need two compactified transformation matrices, with correctly arranged ordering. This is less straightforward than in EM, where rows and columns are indexed by the same list of coordinate pairs. The vector $[G^A]$ is most compactly indexed by $ww, xt, yt, zt$, where "$w$" stands in for one of $x$, $y$, or $z$. Where this means that some elements may have multiple values, we indicate this using (e.g.) $\{R,1,1\}$ for $x$, $y$, and $z$ choices, respectively. To transform this we need $[L^{A'}_{A}]$, which is[4]

$$[L^{A'}_{A}] = \frac{1}{R}\begin{bmatrix} \{R^2+r^2,1,1\} & 0 & 0 & 0 \\ 0 & R & rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.8}$$

[3]Remember that for EM, each element of $[L]$ not only includes the obvious contributions to $G^{\alpha'\beta'}$ from $G^{\alpha\beta}$, but also has a sign-flipped one from $G^{\beta\alpha}$. This accounts for the $-rs$ entry in $[L]$.

[4]Since $w$ represents multiple coordinate choices, the $w'w' = x'x'$ element of $[L]$, i.e., $L^{w'w'}_{ww} = R^2 + r^2$, results from a sum over all possible $ww \in \{xx, yy, zz\}$, in combination with $G^{xx} = G^{yy} = p$.

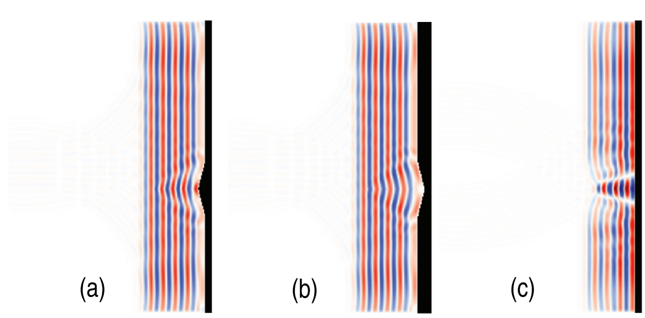


FIG. 2. (Color online) EM finite difference time domain (FDTD) snapshots of the (a) spatial carpet cloak, (b) spatial periscope, and (c) spatial tardis. The cloak and periscope panels (a),(b) show a burst of exactly perpendicular plane waves just as the leading edge hits the mirror surface. This makes it clear how the cloaking transformation has ensured that the wave front conforms perfectly to the mirror profile. In (c), we see the burst of plane waves *after* reflection, where the extra virtual space created by the tardis transformation has imprinted a dent on the wave front.

To transform $[*F^B]$, compactly indexed by $tt,xt,yt,zt$, we need $[M_B^{B'}]$ (and also its inverse),

$$[M_B^{B'}] = \frac{1}{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R & rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.9}$$

Starting from the simplest acoustic medium, i.e., one for which $\vec{\alpha} = \vec{\beta} = \vec{0}$ and $\boldsymbol{\rho} \to \rho$, the transformed constitutive matrix defining the $p$-acoustic carpet $T$ device is then

$$[\chi_{B'}^{A'}] = \rho \begin{bmatrix} \{(R^2+r^2),1,1\}c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

since $c^2 = \kappa/\rho$.

Here we can immediately see that there is no way of constructing a general carpet cloak (or related $T$ device) with $p$-acoustic waves. This is because the transformed energy parameter $\kappa'$ is required to have mutually incompatible values, i.e., both $(R^2 + r^2)\rho c^2$ and $\rho c^2$ at the same time. We can avoid this incompatibility by restricting ourselves to only 1D $x$-axis propagation, so that the waves need experience only one of these $\kappa'$ values. Although such a 1D implementation might technically match the original conception of a carpet cloak, it is merely a static waveguide that appears longer than it really is. However, as we see below, a linear *space-time* carpet $T$ device can be more interesting.

## IV. SPACE-TIME CARPET $T$ DEVICES

The basic design used here is geometrically related to that described above, except that one of the two spatial coordinates is replaced with the time coordinate. This means that waves are not diverted around the event in a spatial sense; instead, the leading waves (in an optical cloak, the "illumination") are sped up and the latter slowed down, creating a "dark" wave-free interval in which timed events are obscured [3,4]. After the event, the leading part of the illumination (whether sound or light) is slowed and the latter sped up until the initial uniform, seamless illumination has been perfectly reconstructed. For an acoustics implementation, we should think of sonar-using creatures such as bats or dolphins who rely on incoming sound waves to locate and observe events; an acoustic event cloak creates a quiet region from which no sound reflections can be heard, all without leaving any telltale distortion of either the background soundscape or deliberate sonar "pings."

We assume a carpet that lies flat on the $y$-$z$ plane at $x = 0$ and decide to cloak a region $h$ deep and $\tau$ in duration, with the cloak's influence extending out as far as $H + h$ above the carpet. The deformation that stretches a uniform and static medium (unprimed coordinates) into these $T$-device designs (primed coordinates) is applied (only) inside the shaded "halo" regions on Fig. 3 and keeps $t' = t, y' = y, z' = z$, but deforms $x$ so that

$$x' = \frac{H-h}{H}x - C\frac{\tau - t\,\mathrm{sgn}(t)}{\tau}h = Rx - Ch + rst, \tag{4.1}$$

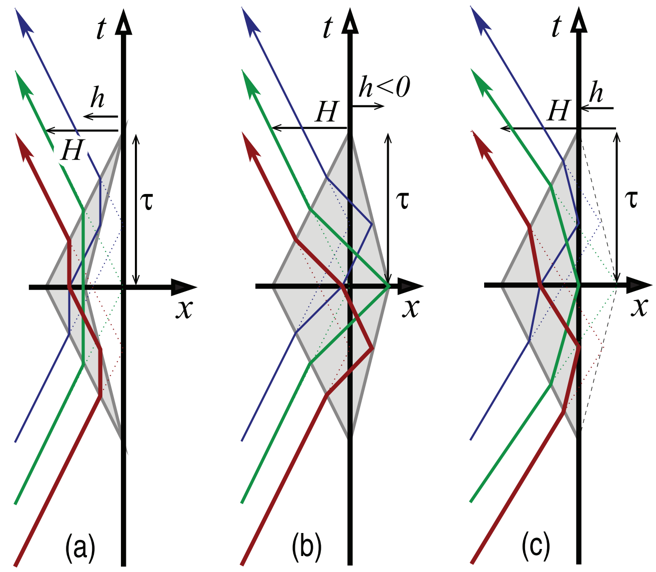


FIG. 3. (Color online) The (a) space-time "event" carpet cloak, (b) space-time periscope, and (c) space-time tardis. Continuous lines show the actual ray trajectories and dotted lines show the apparent (illusory) ray trajectories. The periscope (b) needs to be embedded in a high-index host medium to allow the internal rays to cover the greater distance to the back surface than to the apparent surface. For the carpet cloak diagram (a), we have exaggerated the transformation so that some rays are temporarily stopped, but note that any realistic implementation will involve much weaker speed modulations.

where $R = (H - h)/H$ is the ratio of actual depth to apparent depth, and now $r = Ch/\tau$ and $s = \mathrm{sgn}(t)$. Here C is usually set to 1; except for the space-time tardis, where it is $R$.

As in the spatial case, as well as the cloak shown on Fig. 3(a), this deformation can also represent other $T$ devices, such as a *space-time periscope*, as shown on Fig. 3(b), and a *space-time tardis*, as shown on Fig. 3(c), or perhaps even a space-time antitardis. The space-time periscope allows an observer to remain always *below* the ground plane while temporarily allowing them an unrestricted view of their surroundings, just as if they briefly put their head out above it; it is specified using $h < 0$. Thus, for example, a sensor can be briefly exposed to take readings while usually remaining in shelter below the plane. The tardis gives the onlooker the temporary impression of a space bigger than it actually is, with the ground plane appearing to temporarily recede, this requires that $C = R$. Although in this carpet implementation the space-time tardis is the same as a space-time carpet cloak for a dimple, in a noncarpet radial implementation the visual effect is decidedly more startling.

Using a general viewpoint applicable to any of these three $T$ devices, we see that $H$ is the apparent depth detectable by an observer, positive-valued $h$ apparently expands the accessible space, and negative $h$ apparently shrinks it.

To ensure that the deformed solution includes no waves traveling faster than the maximum wave speed $c$, we also specify that the aspect ratio $(H - h)/c\tau$ is less than 1. As seen in Fig. 3(a), this deformation pushes a small triangular space-time region—the hidden cloaked region—of the wave illumination away from the carpet plane. However, note that if *only* the spatial extent is considered, the entire ground plane itself moves out and then back to temporarily hide a slab of space.

Despite all this trickery, as the (incoming) waves approach the carpet plane from the (below) left, any deviations from ordinary straight-line propagation that occur within the shaded region are compensated for, so that when (outgoing) waves emerge traveling towards the (top) left, they appear to have reflected from a flat carpet plane. Note that the cloaking device reverses the speed changes at $t = 0$: In the figure, at early times $t < 0$, incoming waves are slowed or stopped and outgoing waves are unaffected, but at late times $t > 0$, the outgoing waves are slowed or stopped, with incoming waves traveling normally. For the practical situation, i.e., cloaks where $(H - h) > h$, waves are not stopped but slowed.

The effect of this deformation is given by the transformation matrix $T_\alpha^{\alpha'}$, which specifies the differential relationships between the deformed (primed) and reference (unprimed) coordinates, where $\alpha \in \{t,x,y,z\}$ and $\alpha' \in \{t',x',y',z'\}$. If we let rows span the first (upper) index and columns span the second (lower) index, then

$$T_\alpha^{\alpha'} = \left[\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ rs & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.2}$$

and note that $\det(T_\alpha^{\alpha'}) = R$. For some column 4-vector $V^\alpha$, we have that,

$$\text{e.g.,} \quad \begin{bmatrix} V^{t'} \\ V^{x'} \\ V^{y'} \\ V^{z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ rs & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^t \\ V^x \\ V^y \\ V^z \end{bmatrix}. \tag{4.3}$$

Just as for the spatial carpet $T$ devices in Sec. III, the deformation consists solely of a constant rescaling of $x$ and so the transformed constitutive parameters again have fixed values. However, in contrast to the spatial case, in these space-time $T$ devices the constitutive parameters toggle between fixed values, and boundaries *move*. This is discussed in Sec. V.

### A. EM event carpets

We expressed the EM constitutive tensor using a compact $6 \times 6$ matrix form; therefore, we also use compact $6 \times 6$ deformation (transformation) matrices $[L],[M]$, being versions of the product $L_\alpha^{\alpha'} L_\beta^{\beta'}$. Note that the use of the unconventional choice of (the dual) $*F^{\alpha\beta}$ and (the mixed) $\chi_{\gamma\delta}^{\alpha\beta}$ allows us to calculate using straightforward matrix multiplication.

The two-dimensional matrix representation of the transformation matrix, as compressed for EM, is best written as an explicit transformation between the field 6-vectors to avoid error. The compressed column 6-vector $[G^A]$ is transformed using[5]

$$[L_A^{A'}] = \frac{1}{R} \begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -rs & 0 & R & 0 \\ 0 & +rs & 0 & 0 & 0 & R \end{bmatrix} \tag{4.4}$$

and where the matrix $[M]$ that transforms $[*F^B]$ is the same. To transform $[\chi]$, we need the inverse of $[M]$, which is

$$[M_B^{B'}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & +rs & 0 & 1 & 0 \\ 0 & -rs & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

Starting from simple background medium described only by scalar $\epsilon$ and $\mu$, the transformed constitutive matrix defining the EM space-time carpet $T$ device is then

$$[\chi_{B'}^{A'}] = \frac{\epsilon}{R} \begin{bmatrix} 0 & 0 & 0 & -R^2 & 0 & 0 \\ 0 & 0 & -rs & 0 & -1 & 0 \\ 0 & +rs & 0 & 0 & 0 & -1 \\ c^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & R^2c^2 - r^2 & 0 & 0 & 0 & +rs \\ 0 & 0 & R^2c^2 - r^2 & 0 & -rs & 0 \end{bmatrix}. \tag{4.6}$$

Here we see the slowing and/or speeding behavior of an ordinary space-time cloak [3], but with a direction sensitivity caused by magnetoelectric effects $\pm rs\epsilon/R$ in the material. This direction-sensitivity reverses at $t = 0$, so that while the incoming light travels slower than outgoing light when $t < 0$, the incoming light travels faster for $t > 0$. An implementation of this design is discussed in Sec. V.

### B. *p*-Acoustic event carpets

For *p*-acoustic transformations, we need two compactified transformation matrices, with correctly arranged ordering. Again, and unlike EM, the compacted $[G^A]$ is indexed by $ww,xt,yt,zt$, where "$w$" stands in for one of $x$, $y$, or $z$; and any multiple valued elements are indicated using a setlike notation. We need $[L_A^{A'}]$ to transform $[G^A]$, and it is

$$[L_A^{A'}] = \frac{1}{R} \begin{bmatrix} \{R^2,1,1\} & \{rsR,0,0\} & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.7}$$

[5]Remember that for EM, each element of $[L]$ not only includes the obvious contributions to $G^{\alpha'\beta'}$ from $G^{\alpha\beta}$, but also has a sign-flipped one from $G^{\beta\alpha}$.

The compact $[*F^A]$, indexed by $tt,xt,yt,zt$, is transformed using $[M_B^{B'}]$

$$[M_B^{B'}] = \frac{1}{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ rs & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.8}$$

where

$$[M_B^{B'}]^{-1} = [M_{B'}^{B}] = \begin{bmatrix} R & 0 & 0 & 0 \\ -rs & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}. \tag{4.9}$$

Hence, since $c^2 = \kappa/\rho$, and as usual starting with a simple background medium described only by $\kappa$ and $\rho$, the transformed constitutive matrix $[\chi']$ defining the $p$-acoustic space-time carpet $T$ device is

$$[\chi_{B'}^{A'}] = \rho\begin{bmatrix} \left\{R^2 - \frac{r^2}{c^2},1,1\right\}c^2 & \{rs,0,0\} & 0 & 0 \\ -rs & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.10}$$

Again we can see that there is no way of constructing a general space-time carpet cloak (or related $T$ device) with $p$-acoustic waves, which we can see from the fact that the modulus $\kappa'$ and $\alpha_x'$ cross coupling are both required to have two incompatible values. In contrast to the spatial carpet cloak, a one-dimensional (in $x$) space-time carpet cloak can make sense; just as the one-dimensional space-time EM cloak makes sense [3,30–32]. However, the necessarily bidirectional nature of the design means that we would require a controllable coupling $\alpha_x' = rs\rho$ enabling the velocity field $v_x \equiv *F^{tx}$ to affect the pressure $p \equiv G^{xx}$, as well as a $\beta_x' = -rs\rho$ enabling the density $P \equiv *F^{tt}$ to affect the momentum density $V_x \equiv G^{xt}$.

## V. IMPLEMENTATION

Unlike the standard space-time cloak [3] and related $T$ devices [4], these space-time carpet $T$ devices cannot easily be simplified by choosing a preferred direction. Although they also use speed control of the illumination to operate, a significant complication is that a carpet device is unavoidably bidirectional due to the reflection at the ground plane. Each location within the cloak halo needs to set the speed of incoming illumination and outgoing illumination differently, as can be seen in Fig. 3. This makes it more demanding to implement than either a purely spatial carpet device or an ordinary space-time cloak.

Conceptually, the necessary properties could be achieved by making the space-time carpet $T$ device using a moving medium, but that would require material speeds comparable to the wave speed. Given the problems of sourcing and sinking the medium at its moving boundaries, this is unlikely to be straightforward; although we presume that it would be less difficult for water waves than for EM. Because of this, the more obvious approach would seem to be to design a switchable metamaterial system.

As a simple case, consider a space-time carpet cloak added to the end of a 1D waveguide terminated by a reflector, as

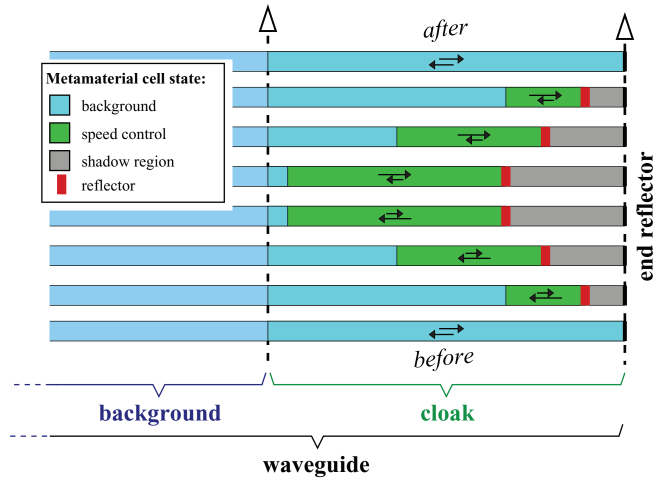


FIG. 4. (Color online) A linear (1 + 1D) waveguide acting as a space-time carpet cloak. Each horizontal bar represents the waveguide at a moment in time, with early times at the bottom and later times at the top. The changing state of the waveguide's metamaterial cells is indicated by the different colors (shades), with the small arrows indicating the direction-sensitive speed contrast.

depicted in Fig. 4. If we break up the cloaking segment of the waveguide into a chain of metamaterial cells, we find that each cell will need to support either three or four states:

(1) a state that mimics or matches the properties of the rest of the waveguide, i.e., the "background";

(2) a state that has a direction-dependent speed as defined by the cloak design, with incoming illumination slowed and outgoing illumination sped up;

(3) a complementary state with the reverse direction-dependent speeds to state 2;

(4) a state matching the behavior of the end of the waveguide, which in most cases would be imagined to be a reflector. This capability is only needed for cells that will at some stage form part of the inner boundary of the cloaked region.

For an EM carpet, magnetoelectric effects are required in the design presented in Sec. IV. However, this means of achieving direction-dependent speeds is usually hard to engineer, even having relaxed the requirement for impedance matching. As an alternative, we could distinguish ingoing from outgoing light by altering its frequency by some convenient mechanism (as for, e.g., [30]) or by using the polarization. In a 1D waveguide carpet we could control ingoing light by making it plane polarized in $y$ (hence controlling speed using $\epsilon_y$) and, with the help of an engineered polarization-switching reflection [33], control the outgoing speed with $z$ polarization and $\epsilon_z$. The necessary polarization conversion at the entry and exit of the carpet can be applied using a Faraday rotator or comparable device. However, using plane-polarized light means that the reflective state of those carpet cells needs to also be polarization switching. Since the engineering of a four-state metamaterial cell is already a challenge, we would prefer to avoid making one of those states more complicated than necessary.

Therefore, we could instead use chiral metamaterial cells with incoming light distinguished by being (e.g.) RH circularly

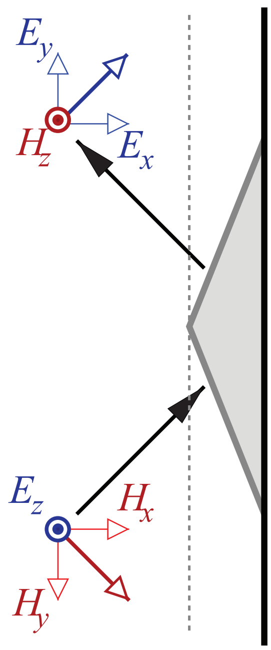


FIG. 5. (Color online) Incoming and outgoing fields for a 1 + 2D EM space-time carpet cloak implementation, but drawn as if for a spatial carpet cloak for clarity. In the space-time carpet of Sec. IV, the reflective carpet pushes out to $x = -L$ (dashed line) and back for all $y$ equally. Incoming fields are controlled by $\epsilon_z$ acting on $E_z$, while the outgoing fields are controlled by $\mu_z$ acting on $H_z$; the field polarization is switched on reflection from the carpet.

polarized and outgoing light being LH circularly polarized. This scheme means that an ordinary mirrorlike cell state is sufficient to reflect the outgoing light into the correct (opposite) circular polarization. A further advantage is that it can be implemented as a purely dielectric device, so avoiding any need to engineer a controllable magnetic response.

For an all-angle 2D space-time carpet $T$ device, the implementation gets harder; we can no longer rely on using circularly polarized light and a chiral medium. However, if we revert to linear polarization and manage to implement the polarization-switching reflector state, it can be achieved by means of a kind of birefringence that creates the speed contrast between incoming and outgoing illumination. Here, as depicted in Fig. 5, we control the incoming illumination by making the electric field component $z$-polarized and modulating the speed using $\epsilon_z$. Since the magnetic field component is then in the $xy$ plane, we set $\mu_x$ and $\mu_y$ to a background value, so that the inward speed will remain angle-independent. Next, the outgoing illumination is controlled by making the magnetic field component $z$ polarized and modulating the speed using $\mu_z$. Since the electric field component is then in the $xy$ plane, we set $\epsilon_x$ and $\epsilon_y$ to a background value so that the outward speed will remain angle independent.

While this will certainly be challenging to construct, one important simplification is that the dynamic $\epsilon_z$ and $\mu_z$ properties required are matched in strength to each other, which will hopefully simplify the necessary technology.

In $p$ acoustics, the scalar nature of the wave's pressure component prohibits us from accessing a similar trick, and so there is no alternative but to consider how we might achieve the unconventional coupling where the velocity field would be able to cause changes to the pressure. Once again, although the general nature of the mathematical formalism has enabled us to unify the transformation aspects of our chosen $T$ device, the specific physics of a given type of wave nevertheless restricts what can be done in principle, as well as in practice.

## VI. SUMMARY

We have derived the material parameters needed for new kind of event cloak—the space-time carpet cloak—and shown what material properties are required to implement it. By unifying the mathematical treatment of EM and a scalar wave theory such as pressure acoustics, we have also shown the way towards a means of unifying transformation theories for different wave types. The unified transformation mechanics theory was applied to a range of spatial $T$ devices based on the ordinary carpet cloak, e.g., the periscope and the tardis. As noted in the original event cloak paper [3], a covariant wave notation greatly facilitates the design of space-time $T$ devices [4], which is further emphasized here, with a scalar wave theory being matched up with the usual covariant formulation of EM and the subsequent ease of comparison and contrast between the spatial carpet $T$ devices from Sec. III and the space-time carpet $T$ devices from Sec. IV. We also proposed (in Sec. V) a scheme laying out the required metamaterial properties needed when implementing an electromagnetic space-time carpet $T$ device.

## ACKNOWLEDGMENTS

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