# Analytic model for the description of above-threshold ionization by an intense short laser pulse 

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#### Abstract

We present an analytic model for the description of above-threshold ionization (ATI) of an atom by an intense, linearly polarized short laser pulse. Our treatment is based upon a description of ATI by an infinitely long train of short laser pulses whereupon we take the limit that the time interval between pulses becomes infinite. In the quasiclassical approximation, we provide detailed quantum-mechanical derivations, within the time-dependent effective range (TDER) model, of the closed-form formulas for the differential probability $\mathcal{P}(\mathbf{p})$ of ATI by an intense, short laser pulse that were presented briefly by Frolov et al. [Phys. Rev. Lett. 108, 213002 (2012)] and that were used to describe key features of the high-energy part of ATI spectra for H and He atoms in an intense, few-cycle laser pulse, using a phenomenological generalization of the physically transparent TDER results to the case of real atoms. Moreover, we extend these results here to the case of an electron bound initially in a $p$ state; we also take into account multiple-return electron trajectories. The ATI amplitude in our approach is given by a coherent sum of partial amplitudes describing ionization by neighboring optical cycles near the peak of the intensity envelope of a short laser pulse. These results provide an analytical explanation of key features in short-pulse ATI spectra, such as the left-right asymmetry in the ionized electron angular distribution, the multiplateau structures, and both large-scale and fine-scale oscillation patterns resulting from quantum interferences of electron trajectories. Our results show that the shape of the ATI spectrum in the middle part of the ATI plateau is sensitive to the spatial symmetry of the initial bound state of the active electron. This sensitivity originates from the contributions of multiple-return electron trajectories. Our analytic results are shown to be in good agreement with results of numerical solutions of the time-dependent Schrödinger equation for He and Ar atoms. Comparison of our results with those of quantitative rescattering theory is also discussed.


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## I. INTRODUCTION

Above-threshold ionization (ATI) is one of the most fundamental phenomena in strong-field (intense laser-atom) physics and has been intensively investigated for more than three decades. This process consists of the ionization of an atom with absorption of a larger number ( $n$ ) of laser field photons than the minimum number $\left(n_{0}\right)$ necessary for ionization. From the first experimental observation of a set of ATI peaks in 1979 [1] up to the present, one can delineate three main stages in experimental and theoretical ATI investigations. In the first stage (up to the early 2000s), the key features of ATI spectra for the case of an essentially monochromatic electromagnetic field, corresponding to long (tens of femtoseconds) laser pulses, were well established (see, e.g., Refs. [2,3]). These key features are the plateaulike shape of the high-energy part of the ATI spectrum (i.e., the ATI electron peaks corresponding to absorption of different numbers of photons have nearly equal magnitudes) and the insensitivity of the electron energy at the plateau cutoff, $E_{\text {cut }} \approx 10 u_{p}$, to the particular atomic $\operatorname{target}$ [where $u_{p}=e^{2} F^{2} /\left(4 m \omega^{2}\right)$ is the average energy of freeelectron oscillations in a laser electric field $\mathbf{F}(t)=\mathbf{F} \cos \omega t$ of frequency $\omega$ ]. The second stage of ATI investigations involved the use of supershort laser pulses (i.e., having few

[^0]optical cycles), with each such pulse having a stabilized carrier-envelope phase (CEP) $\phi$. The most spectacular feature of short-pulse ATI spectra is the left-right asymmetry in the angular distribution of ATI electrons having the same energy but opposite momenta, $\mathbf{p}$ and $-\mathbf{p}$. This asymmetry is most pronounced for ejection of electrons along the direction of laser polarization and strongly depends on the CEP $[4,5]$. Finally, besides its intrinsic interest as a source of high-energy electrons in laser atom interactions, recently a third stage of ATI investigations that has attracted much attention is its use as a tool for imaging field-free atomic and molecular structures (cf., e.g., Ref. [6]) by means of the so-called ATI-based spectroscopy.

Qualitatively, the key features of ATI, including the case of short-pulse ATI [7], can be understood in terms of a semiclassical, three-step (or rescattering) scenario for ATI [3,7]: (i) an initially bound electron tunnels to the continuum in an oscillating laser field, (ii) is accelerated away from the atom by the laser field (acquiring a kinematic momentum along the direction of laser polarization), and (iii) is accelerated back to the atomic core (after the field changes sign), from which it scatters, changing the direction of its momentum and producing the observed angular distribution of ATI electrons. This scenario is supported by both numerical solutions of the time-dependent Schrödinger equation (TDSE) and semianalytical quantum models, such as the improved strong-field approximation $[3,8]$, in which the atomic potential $U(r)$ is taken into account perturbatively, in a Born-like approximation. Estimates of the temporal integrals for the

ATI amplitudes in this approximation provide a transparent quasiclassical description of ATI dynamics in terms of electron trajectories in a laser pulse [3,7,9-11].

However, a perturbative account of the atomic potential is inadequate to extract from ATI spectra information on field-free atomic dynamics, such as the differential cross section for elastic electron scattering from the atomic core, which is the final step of the three-step (rescattering) scenario. ATI spectroscopy is based upon the phenomenological factorization of the ATI yield in terms of an "electron wave packet" and the exact (non-Born) cross section for elastic electron scattering [12,13]. This factorization is the core assumption of quantitative rescattering (QRS) theory for ATI $[13,14]$ and is very useful for analyzing signatures of atomic dynamics in ATI spectra (cf., e.g., Refs. [15-18]). For a monochromatic field interacting with an electron bound initially in a short-range potential $U(r)$, this factorization was justified theoretically in Ref. [19] within the time-dependent effective range (TDER) theory $[20,21]$ and that factorization was generalized to the case of a monochromatic field interacting with a neutral atom. For a one-dimensional zero-range potential model, analytic derivations of the ATI yield for an arbitrary shape of the laser pulse have been performed in Ref. [22] using an adiabatic approach. For three-dimensional finite-range potentials, a rigorous adiabatic theory has been developed in Ref. [23]. This theory is valid for both low- and high-energy (rescattering) parts of ATI spectra, and comparisons with exact solutions of the TDSE were presented for the case of ultrashort halfor single-cycle laser pulses. For the high-energy part of the ATI plateau, closed-form analytic formulas for the amplitude and differential probability of ATI initiated by a few-cycle laser pulse were presented recently in Ref. [24] within TDER theory (i.e., for the case of a short-range potential). The phenomenological generalization of these results to the case of neutral atoms was then used to describe key features of few-cycle pulse ATI spectra for both the H and He atoms [24].

The purpose of this paper is to derive in detail the analytic results presented only briefly in Ref. [24] and to employ the theory for a complete analysis of short-pulse ATI for atoms having outer $s$ and $p$ electrons. After a brief formulation of the problem in Sec. II, in Sec. III we extend our TDER theory for description of ATI in a monochromatic field to the case of a periodic (but nonmonochromatic) field with a period $\mathcal{T}$. In Sec. IV we present the TDER result for the differential probability of short-pulse ATI, obtained as the limit of the ATI rate for the periodic field as $\mathcal{T} \rightarrow \infty$, and discuss its generalization to the case of a neutral atom. In Sec. V A we discuss some general features of short-pulse ATI spectra. In Sec. V B we compare our analytic results with results of numerical solutions of the TDSE; good agreement is demonstrated in the high-energy part of the ATI plateau for both He and Ar atoms. In Sec. V C 1 we provide an analytic explanation for a number of key features of short-pulse ATI spectra, such as their multiplateau structure, their left-right asymmetry, and their large-scale and fine-scale oscillation patterns that are shown to stem from interference phenomena. Moreover, we show how the fine-scale oscillations reduce to the ATI spectrum for a monochromatic laser field as the number of optical cycles in the laser pulse increases. The contributions of multiple-return electron trajectories to
short-pulse ATI spectra are discussed in Sec. V D. Comparison of our results with those of the QRS theory is discussed in Sec. VE. In Sec. VI we summarize our results and present some conclusions. Finally, in the Appendix we present the lengthy analytic derivation of our TDER theory for the ATI amplitude for a bound electron in a periodic field.

## II. GENERAL FORMULATION

We use the dipole approximation for interaction of an atomic electron with a laser pulse:

$$
\begin{equation*}
V(\mathbf{r}, t)=|e| \mathbf{F}(t) \cdot \mathbf{r}, \quad \mathbf{F}(t)=-\frac{1}{c} \frac{\partial \mathbf{A}(t)}{\partial t} \tag{1}
\end{equation*}
$$

where $\mathbf{F}(t)=\mathbf{e}_{z} F(t)$ is the electric field of a laser pulse with linear polarization and $\mathbf{A}(t)=\mathbf{e}_{z} A(t)$ is the corresponding vector potential. We assume that the active electron is initially in the bound state $\psi_{\kappa l 0}(\mathbf{r})=\varphi_{\kappa l}(r) Y_{l 0}(\hat{\mathbf{r}})$ with energy $E_{0}=$ $-\hbar^{2} \kappa^{2} /(2 m)$, angular momentum $l$, and angular momentum projection $m_{l}$. We consider the cases $l=0$ and $l=1$ and, for simplicity, $m_{l}=0$. In order to describe accurately the interaction of a short laser pulse with an atomic system, we employ the approach in Ref. [25], i.e., the interaction with a short laser pulse of duration $\tau$ is considered as the limiting case of the interaction with an infinite train of short pulses separated in time by a period $\mathcal{T}$ with $\mathcal{T}>\tau$. For a finite $\mathcal{T}$, the interaction with the (periodic in time) pulse train can be described accurately within the framework of the quasistationary quasienergy state (QQES) approach [26], in which the ionization amplitude follows from the asymptotic form of the QQES wave function at large distances [19,27]. Once the pulse train problem is solved, the result for a short pulse follows from the limit $\mathcal{T} \rightarrow \infty\left(\omega_{\tau}=2 \pi / \mathcal{T} \rightarrow 0\right)$. In particular, the ionization probability in this approach is given by the limit [25]

$$
\begin{equation*}
\mathcal{P}(\mathbf{p})=\frac{d^{2} P}{d E_{p} d \Omega_{\mathbf{p}}}=\frac{2 \pi}{\hbar} \lim _{\omega_{\tau} \rightarrow 0} \frac{\Gamma(\mathbf{p})}{\omega_{\tau}^{2}}, \tag{2}
\end{equation*}
$$

where $\mathcal{P}(\mathbf{p})$ is the doubly differential probability to detect the photoelectron in the energy interval ( $E_{p}, E_{p}+d E_{p}$ ) and in the solid angle between $\Omega_{\mathbf{p}}$ and $\Omega_{\mathbf{p}}+d \Omega_{\mathbf{p}}, \Gamma(\mathbf{p})$ is the ionization rate in the pulse-train field (see Sec. II C of Ref. [25] for details). The energy $E_{p}=p^{2} /(2 m)$ is the energy of the ionized electron with momentum $\mathbf{p}$.

In order to obtain the explicit expression for the probability $\mathcal{P}(\mathbf{p})$ for a short laser pulse in Eq. (2), in Sec. III we derive the rate $\Gamma(\mathbf{p})$ for a pulse train described by the vector potential $\mathbf{A}_{\tau}(t)=\mathbf{e}_{z} A_{\tau}(t)$ and the corresponding electric field $\mathbf{F}_{\tau}(t)=$ $\mathbf{e}_{z} F_{\tau}(t)$, where

$$
\begin{equation*}
F_{\tau}(t)=-\frac{1}{c} \frac{\partial A_{\tau}(t)}{\partial t} \tag{3}
\end{equation*}
$$

## III. TDER RESULTS FOR A TRAIN OF PULSES

## A. Exact TDER results for a periodic field

In order to obtain an analytic description of the nonlinear interaction of a bound electron with a strong, periodic in time laser field $\mathbf{F}_{\tau}(t)$, we use the TDER theory [20,21], which is applicable to the case of a bound electron in a short-range potential $U(r)$ (of radius $r_{c}$ ). This theory combines the QQES
(or Floquet) approach (for describing the electron's interaction with a periodic laser field) and effective range theory [28] (for describing a weakly bound electron in a short-range potential). In effective range theory the interaction of the electron with the potential $U(r)$ is described in terms of the scattering phase in the $l$-wave channel, $\delta_{l}(E)$, which is parametrized by the scattering length $a_{l}$ and the effective range $r_{l}$ [28].

The key ingredients of the TDER theory are the complex quasienergy $\epsilon$ [which reduces to $E_{0}$ for $\mathbf{F}_{\tau}(t)=0$ ] and a periodic function,

$$
\begin{equation*}
f_{\epsilon}^{(l)}(t)=\sum_{n} f_{n}^{(l)} e^{-i n \omega_{\tau} t} \tag{4}
\end{equation*}
$$

which determines the behavior of the time-periodic QQES wave function $\Phi_{\epsilon}(\mathbf{r}, t)$ at small $r(\kappa r \ll 1)[20,21]$ :

$$
\begin{align*}
& \int \Phi_{\epsilon}(\mathbf{r}, t) Y_{l 0}^{*}(\hat{\mathbf{r}}) d \Omega \sim \sum_{n}\left[r^{-l-1}+\cdots\right. \\
& \left.\quad+B_{l}\left(\epsilon+n \hbar \omega_{\tau}\right)\left(r^{l}+\cdots\right)\right] f_{n}^{(l)} e^{-i n \omega_{\tau} t} \tag{5}
\end{align*}
$$

where the coefficient $B_{l}\left(\epsilon+n \hbar \omega_{\tau}\right)$ is parametrized according to effective range theory [28]:

$$
\begin{align*}
\frac{[(2 l+1)!!]^{2}}{2 l+1} B_{l}(E) & =k^{2 l+1} \cot \delta_{l}(E)  \tag{6a}\\
\approx-1 / a_{l}+r_{l} k^{2} / 2, \quad k & =\sqrt{2 m E} / \hbar . \tag{6b}
\end{align*}
$$

Note that the key role of the periodic function $f_{\epsilon}^{(l)}(t)$ in TDER theory and the different levels of approximation for its evaluation that are necessary for different strong-field processes have been discussed previously: for high-order harmonic generation (HHG), see Refs. [29,30]; for ATI, see Ref. [19]; for a general discussion of the importance of its Fourier coefficients [cf. Eq. (4)] for both HHG and ATI, see Refs. [21,31]. Note that whereas in HHG the atomic potential is only taken into account in the initial state, in ATI it must be taken into account also in the final state in order to properly treat rescattering.

Outside the atomic core $\left(r>r_{c}\right)$, the QQES wave function can be expressed as a convolution of the periodic function $f_{\epsilon}^{(l)}(t)$ and the retarded Green's function for a free electron in the electric field $\mathbf{F}_{\tau}(t), G\left(\mathbf{r}, t ; 0, t^{\prime}\right)$ [21]:

$$
\begin{align*}
& \Phi_{\epsilon}(\mathbf{r}, t) \\
&=-C_{\kappa l} \frac{\sqrt{(2 l+1) \kappa \pi} \hbar^{2}}{m}\left(-\frac{i}{\hbar \kappa}\right)^{l} \int_{-\infty}^{t} e^{i \epsilon\left(t-t^{\prime}\right) / \hbar} \\
& \times f_{\epsilon}^{(l)}\left(t^{\prime}\right)\left\{\mathbf{e}_{z} \cdot\left[\frac{m \mathbf{r}}{\left(t-t^{\prime}\right)}+\boldsymbol{\alpha}\left(t^{\prime} ; t, t^{\prime}\right)\right]\right\}^{l} G\left(\mathbf{r}, t ; 0, t^{\prime}\right) d t^{\prime} \tag{7}
\end{align*}
$$

where

$$
\begin{equation*}
\boldsymbol{\alpha}\left(\xi ; t, t^{\prime}\right)=\mathbf{e}_{z} \alpha\left(\xi ; t, t^{\prime}\right) \tag{8}
\end{equation*}
$$

$$
\begin{equation*}
\alpha\left(\xi ; t, t^{\prime}\right)=\frac{|e|}{c}\left[A_{\tau}(\xi)-\int_{t^{\prime}}^{t} A_{\tau}(\tau) d \tau /\left(t-t^{\prime}\right)\right] \tag{9}
\end{equation*}
$$

For $\mathbf{F}_{\tau}(t)=0$, Eq. (7) reduces to the expression for the boundstate wave function $\psi_{\kappa l 0}(\mathbf{r})$ in the region $r>r_{c}$, for which

$$
\begin{align*}
& U(r)=0: \\
&  \tag{10}\\
& \quad \psi_{\kappa l 0}(\mathbf{r})=-i^{l} \kappa^{3 / 2} C_{\kappa l} h_{l}(i \kappa r) Y_{l 0}(\hat{\mathbf{r}}),
\end{align*}
$$

where $C_{\kappa l}$ is its (dimensionless) asymptotic coefficient and $h_{l}(x)$ is the spherical Hankel function of the first kind.

As for the case of a monochromatic laser field [20,21], by matching the expression (7) to the boundary condition (5), one obtains a homogeneous integrodifferential equation for the function $f_{\epsilon}^{(l)}(t)$ and the complex quasienergy $\epsilon$. We present this integrodifferential equation here directly in terms of the Fourier coefficients $f_{n}^{(l)}$ of $f_{\epsilon}^{(l)}(t)$ [cf. Eq. (4)] as this form of the equation is the most useful one for our further analyses:

$$
\begin{align*}
& \sum_{n=-\infty}^{\infty} \mathcal{R}_{l}\left(\epsilon+n \hbar \omega_{\tau}\right) f_{n}^{(l)} e^{-i n \omega_{\tau} t} \\
& \quad=\sum_{m=-\infty}^{\infty} e^{-i m \omega_{\tau} t} f_{m}^{(l)} \mathcal{M}_{l}\left(\epsilon+m \hbar \omega_{\tau}, t\right) \tag{11}
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{R}_{l}(E)= & \frac{[(2 l+1)!!]^{2}}{2 l+1} B_{l}(E)-i\left(2 m E / \hbar^{2}\right)^{l+1 / 2},  \tag{12}\\
\mathcal{M}_{0}(\epsilon, t)= & \sqrt{\frac{m}{2 \pi i \hbar}} \int_{-\infty}^{t} \frac{e^{i \epsilon\left(t-t^{\prime}\right) / \hbar}}{\left(t-t^{\prime}\right)^{3 / 2}}\left[e^{i S\left(t, t^{\prime}\right) / \hbar}-1\right] d t^{\prime},  \tag{13}\\
\mathcal{M}_{1}(\epsilon, t)= & -3 i \sqrt{\frac{m^{3}}{2 \pi i \hbar^{3}}} \int_{-\infty}^{t} \frac{e^{i \epsilon\left(t-t^{\prime}\right) / \hbar}}{\left(t-t^{\prime}\right)^{5 / 2}}\left[e^{i S\left(t, t^{\prime}\right) / \hbar}-1\right] d t^{\prime} \\
+ & 3 \sqrt{\frac{m}{2 \pi i \hbar^{5}}} \int_{-\infty}^{t} \frac{e^{i \epsilon\left(t-t^{\prime}\right) / \hbar+i S\left(t, t^{\prime}\right) / \hbar} \mathcal{P}\left(t, t^{\prime}\right)}{\left(t-t^{\prime}\right)^{3 / 2}} d t^{\prime},  \tag{14}\\
& S\left(t, t^{\prime}\right)=-\frac{1}{2 m} \int_{t^{\prime}}^{t} \alpha\left(\xi ; t, t^{\prime}\right)^{2} d \xi,  \tag{15}\\
& \mathcal{P}\left(t, t^{\prime}\right)=\alpha\left(t^{\prime} ; t, t^{\prime}\right) \alpha\left(t ; t, t^{\prime}\right) . \tag{16}
\end{align*}
$$

Expansion of the matrix elements $\mathcal{M}_{l}(\epsilon, t)$ in Fourier series reduces Eq. (11) to a system of homogeneous linear equations (i.e., to an eigenvalue problem) for the Fourier coefficients $f_{n}^{(l)}$ and the complex quasienergy $\epsilon$. As Eq. (12) shows, the basic equation (11) of the TDER theory involves the exact coefficient $B_{l}(E)$ given by Eq. (6a). Thus the major assumption of the TDER model is that the bound state interacts with the continuum only through a single $l$-wave channel, while the parametrization (6b) for the scattering phase $\delta_{l}(E)$ is not mandatory and we use it only for the sake of simplicity. Since this parametrization is valid only for low-energy collisions [28], for high energies the relation (6a) for $B_{l}(E)$ with known $\delta_{l}(E)$ (which is a parameter of the problem) should be used. Note also that the TDER model for an arbitrary periodic field can be generalized to account for two continuum channels (providing a "two-state TDER model") in a way similar to the generalization for a monochromatic field that was derived in Ref. [30].

## B. Quasiclassical result for the function $\boldsymbol{f}_{\epsilon}^{(l)}(\boldsymbol{t})$

Rather than obtaining exact numerical solutions of Eq. (11), we seek to obtain approximate analytical solutions for the function $f_{\epsilon}^{(l)}(t)$ by approximating $\epsilon$ by $E_{0}$ and using the quasiclassical approximation to estimate the matrix elements $\mathcal{M}_{l}\left(\epsilon=E_{0}, t\right)$. These approximations are valid in the tunneling regime, i.e., the amplitude of the laser field $(F)$ is assumed to be small compared to the characteristic field $F_{0}=\sqrt{2 m\left|E_{0}^{3}\right|} /(|e| \hbar)$, the carrier frequency of the laser pulse $(\omega)$ is assumed to be smaller then $\hbar /\left|E_{0}\right|$, and $\gamma_{K}=$ $\hbar \omega /\left(|e| F \kappa^{-1}\right) \ll 1$, where $\gamma_{K}$ is the Keldysh parameter. In this case, the integrands of $\mathcal{M}_{l=0,1}\left(E_{0}, t\right)$ in Eqs. (13) and (14) are highly oscillating functions of the time $t^{\prime}$, so that the Fourier coefficients of $\mathcal{M}_{l=0,1}\left(E_{0}, t\right)$ are exponentially small. Thus, as the first step of our quasiclassical analysis of $f_{E_{0}}^{(l)}(t)$, we perform the integration over $t^{\prime}$ in Eqs. (13) and (14) using the stationary phase approximation. The result for both $\mathcal{M}_{0}\left(E_{0}, t\right)$ and $\mathcal{M}_{1}\left(E_{0}, t\right)$ can be presented in a unified form:

$$
\begin{equation*}
\mathcal{M}_{l}\left(E_{0}, t\right)=(2 l+1) \sum_{v} \frac{m e^{i \mathcal{S}\left(t, t_{v}^{\prime}\right) / \hbar}}{\sqrt{\mathcal{D}}\left(t-t_{v}^{\prime}\right)^{3 / 2}}\left(\frac{\mathcal{P}\left(t, t_{v}^{\prime}\right)}{\hbar^{2}}\right)^{l} \tag{17}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathcal{S}\left(t, t_{v}^{\prime}\right)=E_{0}\left(t-t_{v}^{\prime}\right)+S\left(t, t_{v}^{\prime}\right),  \tag{18}\\
\mathcal{D}=|e| F_{\tau}\left(t_{v}^{\prime}\right) \alpha\left(t_{v}^{\prime} ; t, t_{v}^{\prime}\right) \tag{19}
\end{gather*}
$$

and the stationary phase points $t_{v}^{\prime}$ are given by the equation

$$
\begin{equation*}
\frac{\alpha^{2}\left(t_{v}^{\prime} ; t, t_{v}^{\prime}\right)}{2 m}=E_{0}, \quad v=1,2, \ldots \tag{20}
\end{equation*}
$$

Note that we only consider the contributions of the stationary phase points $t_{v}^{\prime} \equiv t_{v}^{\prime}(t)$ for which the imaginary part of the second derivative of $S\left(t, t^{\prime}\right)$ in $t^{\prime}$ is positive, $\operatorname{Re} \mathcal{D}>0$ (since otherwise the integral would diverge). Note also that for the case $l=1$, the contribution of the first term on the right in Eq. (14) is neglected since in the lowfrequency limit its contribution is $\omega$ times smaller than the contribution of the second term, which gives the result in Eq. (17).

In the quasiclassical limit, the function $f_{\epsilon}^{(l)}(t)$ is only slightly disturbed by the laser field from its unperturbed [for $\left.\mathbf{F}_{\tau}(t)=0\right]$ value (cf., e.g., Refs. [21,32]). Thus in the lowest approximation we replace $f_{E_{0}}^{(l)}(t)$ by unity (and its Fourier coefficients by $\left.f_{n}^{(l)}=\delta_{n, 0}\right)$. The next-order correction to the function $f_{E_{0}}^{(l)}(t)$ can be found by iterative solution of Eq. (11), substituting $f_{m}^{(l)}=\delta_{m, 0}$ in the right-hand side of Eq. (11). We thus obtain the Fourier coefficients $f_{n}^{(l)}$ with $n \neq 0$ in terms of the Fourier coefficients of $\mathcal{M}_{l}\left(E_{0}, t\right)$. The corresponding result for the function $f_{E_{0}}^{(l)}(t)$ has the form

$$
\begin{equation*}
f_{E_{0}}^{(l)}(t) \approx 1+\frac{1}{\mathcal{T}} \sum_{n \neq 0} \int_{0}^{\mathcal{T}} e^{i n \omega_{\tau}(\xi-t)} \frac{\mathcal{M}_{l}\left(E_{0}, \xi\right)}{\mathcal{R}_{l}\left(E_{0}+n \hbar \omega_{\tau}\right)} d \xi \tag{21}
\end{equation*}
$$

The main contribution to the integral in (21) comes from the vicinity of the stationary phase points of the integrand. Taking into account Eq. (17), the equation for these stationary phase
points is

$$
\begin{equation*}
E_{0}+n \hbar \omega_{\tau}=\frac{\alpha^{2}\left(\xi ; t, t_{v}^{\prime}\right)}{2 m} \tag{22}
\end{equation*}
$$

Using Eq. (22), within the quasiclassical approximation we can replace the integrand in Eq. (21) by a function $\tilde{\mathcal{M}}_{l}(\xi)$ that is independent of $n$, as follows:

$$
\begin{align*}
& \frac{\mathcal{M}_{l}\left(E_{0}, \xi\right)}{\mathcal{R}_{l}\left(E_{0}+n \hbar \omega_{\tau}\right)} \rightarrow \tilde{\mathcal{M}}_{l}(\xi) \\
& \quad \tilde{\mathcal{M}}_{l}(\xi)=\sum_{v} \frac{(2 l+1) m e^{i \mathcal{S}\left(\xi, t_{v}^{\prime}\right) / \hbar}\left[\mathcal{P}\left(\xi, t_{v}^{\prime}\right) / \hbar^{2}\right]^{l}}{\sqrt{\mathcal{D}}\left(\xi-t_{v}^{\prime}\right)^{3 / 2} \mathcal{R}_{l}\left[\alpha^{2}\left(\xi ; t, t_{v}^{\prime}\right) /(2 m)\right]} \tag{23}
\end{align*}
$$

Taking into account the definition of the $\delta$ function,

$$
\begin{equation*}
\delta(\xi-t)=\frac{1}{\mathcal{T}} \sum_{n} e^{i n \omega_{\tau}(\xi-t)} \tag{24}
\end{equation*}
$$

the integral in Eq. (21) [after making the substitution (23)] can be evaluated to obtain the following expression for the function $f_{E_{0}}^{(l)}(t)$ :

$$
\begin{equation*}
f_{E_{0}}^{(l)}(t) \approx 1+\tilde{\mathcal{M}}_{l}(t) \tag{25}
\end{equation*}
$$

## C. Detachment amplitude for a periodic field

For known $\epsilon$ and $f_{\epsilon}^{(l)}(t)$, the exact TDER amplitude for above-threshold detachment (ATD) with absorption of $n$ photons, each with energy $\hbar \omega_{\tau}$, can be obtained similarly to the case of a monochromatic field [19]:

$$
\begin{align*}
\mathcal{A}^{(l)}\left(\mathbf{p}_{n}\right)= & \frac{C_{\kappa l}(-i)^{l}}{\mathcal{T} \hbar^{l} \kappa^{l-1 / 2}} \sqrt{\frac{(2 l+1)}{4 \pi}} \int_{-\mathcal{T} / 2}^{\mathcal{T} / 2} e^{i S\left(\mathbf{p}_{n}, t\right) / \hbar} \\
& \times\left[\mathbf{e}_{z} \cdot \mathbf{P}_{n}(t)\right]^{l} f_{\epsilon}^{(l)}(t) d t \tag{26}
\end{align*}
$$

where

$$
\begin{gather*}
S\left(\mathbf{p}_{n}, t\right)=\int^{t}\left[\mathbf{P}_{n}^{2}(t) /(2 m)-\epsilon\right] d t  \tag{27}\\
\mathbf{P}_{n}(t)=\mathbf{p}_{n}+\frac{|e|}{c} \mathbf{A}_{\tau}(t), \quad \mathbf{p}_{n}=\boldsymbol{n} p_{n}  \tag{28}\\
p_{n}=\sqrt{n \hbar \omega_{\tau}+\epsilon-u_{p}}, \quad u_{p}=\frac{1}{\mathcal{T}} \int_{-\mathcal{T} / 2}^{\mathcal{T} / 2} A_{\tau}^{2}(t) d t \tag{29}
\end{gather*}
$$

and where the unit vector $\boldsymbol{n}$ indicates the momentum direction of the detached electron. The differential detachment rate for the electron with momentum $\mathbf{p}_{n}$ is given by the expression [19]

$$
\begin{equation*}
\Gamma\left(\mathbf{p}_{n}\right) \equiv \frac{d W\left(\mathbf{p}_{n}\right)}{d \Omega_{\mathbf{p}_{n}}}=\frac{1}{m}\left|\sqrt{p_{n}} \mathcal{A}^{(l)}\left(\mathbf{p}_{n}\right)\right|^{2} \tag{30}
\end{equation*}
$$

Substituting the quasiclassical result (25) for the function $f_{\epsilon}^{(l)}(t)$ into Eq. (26), we obtain the amplitude $\mathcal{A}^{(l)}\left(\mathbf{p}_{n}\right)$ as a sum of two terms:

$$
\begin{equation*}
\mathcal{A}^{(l)}\left(\mathbf{p}_{n}\right)=\mathcal{A}^{(K)}\left(\mathbf{p}_{n}\right)+\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right) \tag{31}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{A}^{(K)}\left(\mathbf{p}_{n}\right)= & \frac{C_{\kappa l}(-i)^{l}}{\mathcal{T} \hbar^{l} \kappa^{l-1 / 2}} \sqrt{\frac{(2 l+1)}{4 \pi}} \\
& \times \int_{-\mathcal{T} / 2}^{\mathcal{T} / 2} e^{i S\left(\mathbf{p}_{n}, t\right) / \hbar}\left[\mathbf{e}_{z} \cdot \mathbf{P}_{n}(t)\right]^{l} d t, \tag{32}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)= & \frac{C_{\kappa l}(-i)^{l}}{\mathcal{T} \hbar^{l} \kappa^{l-1 / 2}} \sqrt{\frac{2 l+1}{4 \pi}} \sum_{\nu} \int_{-\mathcal{T} / 2}^{\mathcal{T} / 2} e^{i S\left(\mathbf{p}_{n}, t\right) / \hbar} \\
& \times \frac{\left[\mathbf{e}_{z} \cdot \mathbf{P}_{n}(t)\right]^{l} \mathcal{M}_{l}\left(E_{0}, t\right) d t}{\mathcal{R}_{l}\left[\alpha^{2}\left(t ; t, t_{v}^{\prime}\right) /(2 m)\right]} \tag{33}
\end{align*}
$$

In these latter two equations $\mathbf{p}_{n}=\boldsymbol{n} \sqrt{n \hbar \omega_{\tau}+E_{0}-u_{p}}$ and $\epsilon$ is replaced by $E_{0}$. The amplitude (32) gives the strong-field approximation (or Keldysh) result for the amplitude $\mathcal{A}^{(l)}\left(\mathbf{p}_{n}\right)$ and describes only the low-energy part of the ATD spectrum (see, e.g., the review [33]). Rescattering effects originate from the term $\tilde{\mathcal{M}}_{l}(t)$ in Eq. (25); thus the amplitude (33) contributes to the high-energy (rescattering) part of the ATD spectrum (see, e.g., the review [3]). In what follows, we will focus only on the rescattering amplitude $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$.

## D. Quasiclassical result for $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$

An analytical evaluation of the rescattering amplitude $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$ in Eq. (33) in the quasiclassical limit can be performed in a way similar to that for a monochromatic field [19]. The technical details of this evaluation can be found in the Appendix. Here we present only the final analytic result, expressed in terms of the classical ionization $\left(t_{i}^{(j)}\right)$ and rescattering $\left(t_{f}^{(j)}\right)$ times that satisfy the following two coupled equations [3]:

$$
\begin{align*}
A_{\tau}\left(t_{i}^{(j)}\right)-\frac{1}{t_{f}^{(j)}-t_{i}^{(j)}} \int_{t_{i}^{(j)}}^{t_{f}^{(j)}} A_{\tau}(\tau) d \tau & =0,  \tag{34a}\\
2 F_{\tau}\left(t_{f}^{(j)}\right)+\frac{1}{c} \frac{A_{\tau}\left(t_{f}^{(j)}\right)-A_{\tau}\left(t_{i}^{(j)}\right)}{t_{f}^{(j)}-t_{i}^{(j)}} & =0 . \tag{34b}
\end{align*}
$$

These equations can be obtained from the analysis of classical electron trajectories in a laser field [34]. The first equation, Eq. (34a), implies that at the moment of ionization, $t=t_{i}^{(j)}$, the electron leaves the atomic system with zero velocity and moves in the laser field along a closed classical trajectory, starting and ending at times $t_{i}^{(j)}$ and $t_{f}^{(j)}$. At the time of rescattering, $t=t_{f}^{(j)}$, the electron backscatters from the atomic core and moves away from it. Equation (34b) implies that the electron has the maximum energy after backscattering.

As the quasiclassical analysis in the Appendix shows, the ATD amplitude $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$ can be presented as a sum of partial amplitudes (corresponding to a given set of times $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ for the $j$ th closed electron trajectory), each of which can be parametrized in accordance with the three-step scenario of

ATD/ATI [34], as follows:

$$
\begin{equation*}
\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)=\sqrt{i} \frac{\hbar \omega_{\tau}}{e^{2} \sqrt{a}} \sum_{j} A_{j} \tag{35}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{j}=a_{\tau}^{(j)} a_{\mathcal{W}}^{(j)} f_{l}\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right) e^{i \varphi_{j} / \hbar}, \tag{36}
\end{equation*}
$$

$$
\begin{gather*}
\varphi_{j}=\varphi\left(t_{i}^{(j)}, t_{f}^{(j)}\right)=S\left(\mathbf{p}_{n}, t_{f}^{(j)}\right)-\int_{t_{i}^{(j)}}^{t_{f}^{(j)}}\left[\mathcal{E}\left(t_{i}^{(j)}, t\right)-E_{0}\right] d t  \tag{37}\\
\mathcal{E}\left(t_{i}^{(j)}, t\right)=\frac{e^{2}\left[A_{\tau}\left(t_{i}^{(j)}\right)-A_{\tau}(t)\right]^{2}}{2 m c^{2}} \tag{38}
\end{gather*}
$$

We discuss each of the factors in Eq. (36) for the partial amplitude $A_{j}$ in turn.

The first factor, $a_{\tau}^{(j)}$, describes the tunneling step and is given by the expression

$$
\begin{align*}
a_{\tau}^{(j)} & \equiv a_{\tau}\left(t_{i}^{(j)}\right) \\
& =C_{\kappa l}\left(-\sigma_{j}\right)^{l} \sqrt{\frac{(2 l+1) F_{\mathrm{at}}}{4 \pi \tilde{F}_{j}}} \exp \left(-\frac{\hbar^{2} \kappa^{3}}{3 m|e| \tilde{F}_{j}}\right) \tag{39}
\end{align*}
$$

where $F_{\mathrm{at}}=|e| / a^{2}, a=\hbar^{2} /\left(m e^{2}\right)$ is the Bohr radius, and $\tilde{F}_{j}=\sigma_{j} F_{\tau}\left(t_{i}^{(j)}\right)>0$. This latter condition shows that $\sigma_{j}=+1$ $\left(\sigma_{j}=-1\right)$ for those pairs $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ for which $F_{\tau}\left(t_{i}^{(j)}\right)>0$ $\left[F_{\tau}\left(t_{i}^{(j)}\right)<0\right]$. Thus the sign of $\sigma_{j}$ determines the direction of electron propagation after tunneling: along the direction of the vector $\mathbf{e}_{z}$ for $\sigma_{j}=-1$ and along the vector $-\mathbf{e}_{z}$ for $\sigma_{j}=+1$.

The second factor, $a_{\mathcal{W}}^{(j)}$, in the partial amplitude (36) does not depend on the atomic structure and describes the propagation of the liberated electron in the laser field along the $j$ th closed classical trajectory until the rescattering event:

$$
\begin{equation*}
a_{\mathcal{W}}^{(j)} \equiv a_{\mathcal{W}}\left(t_{i}^{(j)}, t_{f}^{(j)}\right)=\frac{\operatorname{Ai}\left(\xi_{j}\right)}{\zeta_{j}^{1 / 3}\left[\left(t_{f}^{(j)}-t_{i}^{(j)}\right) \omega_{\mathrm{at}}\right]^{3 / 2}} \tag{40}
\end{equation*}
$$

where $\operatorname{Ai}(x)$ is the Airy function and

$$
\begin{gather*}
\xi_{j} \equiv \xi\left(t_{i}^{(j)}, t_{f}^{(j)}\right)=\frac{\Delta E_{\max }^{(j)}}{E_{\mathrm{at}} \zeta_{j}^{1 / 3}},  \tag{41}\\
\Delta E_{\max }^{(j)} \equiv \Delta E_{\max }\left(t_{i}^{(j)}, t_{f}^{(j)}\right)=\frac{\mathcal{P}_{j}^{2}}{2 m}-E_{\max }^{(j)},  \tag{42}\\
E_{\max }^{(j)}=\mathcal{E}\left(t_{i}^{(j)}, t_{f}^{(j)}\right)-2 \frac{F_{\tau}\left(t_{f}^{(j)}\right)}{F_{\tau}\left(t_{i}^{(j)}\right)}\left|E_{0}\right|,  \tag{43}\\
\zeta_{j} \equiv \zeta\left(t_{i}^{(j)}, t_{f}^{(j)}\right)=-\frac{\dot{F}_{\tau}\left(t_{f}^{(j)}\right)\left\{\mathbf{e}_{z} \cdot\left[\mathbf{p}_{n}+|e| \mathbf{A}_{\tau}\left(t_{i}^{(j)}\right) / c\right]\right\}}{2 F_{\mathrm{at}}^{2}|e|} \\
+\frac{F_{\tau}^{2}\left(t_{f}^{(j)}\right)}{F_{\mathrm{at}}^{2}}\left(4 \frac{F_{\tau}\left(t_{f}^{(j)}\right)}{F_{\tau}\left(t_{i}^{(j)}\right)}-3\right),  \tag{44}\\
\mathcal{P}_{j} \equiv \mathcal{P}\left(t_{f}^{(j)}\right)=\mathbf{p}_{n}+\frac{|e|}{c} \mathbf{A}_{\tau}\left(t_{f}^{(j)}\right) \tag{45}
\end{gather*}
$$

Note that in the above equations $E_{\mathrm{at}}=\hbar \omega_{\mathrm{at}}=e^{2} / a \approx$ 27.21 eV and $I_{\mathrm{at}}=c e^{2} /\left(8 \pi a^{4}\right) \approx 3.51 \times 10^{16} \mathrm{~W} / \mathrm{cm}^{2}$. Our analysis of the partial amplitudes $A_{j}$ takes into account two closed trajectories, along which the electron starts to move at the same time $t_{i}^{(j)}$, but rescatters at the times $t<t_{f}^{(j)}$ (short trajectory) and $t>t_{f}^{(j)}$ (long trajectory), respectively. For $\xi_{j}<0$, the Airy function in $a_{\mathcal{W}}^{(j)}$ oscillates and describes the interference between these short and long trajectories. At $\xi=-1.019$ these two trajectories merge into a single (extremal) trajectory, corresponding to the times $t_{i}^{(j)}, t_{f}^{(j)}$. For $\xi>0$, the factor $a_{\mathcal{W}}^{(j)}$ decreases exponentially. The sign of $\xi_{j}$ is governed by $\Delta E_{\text {max }}^{(j)}$, which is the difference between the energy, $\mathcal{P}_{j}^{2} /(2 m)$, of the electron with momentum $\mathbf{p}_{n}$ in the laser field, and the maximum electron energy, $E_{\text {max }}^{(j)}$, gained from the laser field during the traveling time $\Delta t=t_{f}^{(j)}-t_{i}^{(j)}$. Note that the last term in Eq. (43) gives a quantum correction to the classical electron energy $\mathcal{E}\left(t_{i}^{(j)}, t_{f}^{(j)}\right)$ for the case of an arbitrary periodic field (for the case of a monochromatic field, this correction was derived in Ref. [35]).

The sum in Eq. (35) is taken over those roots $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ of the system (34) that ensure the positivity of $\zeta_{j}$, given by Eq. (44). (The positivity of $\zeta_{j}$ is discussed in the Appendix.) Since we consider only those high-energy electrons for which $\left|\left(\mathbf{e}_{z} \cdot \mathbf{p}_{n}\right)\right|>(|e| / c) \max \left|A_{\tau}\left(t_{f}^{(j)}\right)\right|$, then for electrons detached into the "left" hemisphere $\left[\left(\mathbf{e}_{z} \cdot \mathbf{p}_{n}\right)<0\right.$ ], the time $t_{f}^{(j)}$ should be chosen so that $\dot{F}_{\tau}\left(t_{f}^{(j)}\right)>0$, while for electrons detached into the "right" hemisphere $\left[\left(\mathbf{e}_{z} \cdot \mathbf{p}_{n}\right)>0\right]$, the rescattering time $t_{f}^{(j)}$ should ensure that $\dot{F}_{\tau}\left(t_{f}^{(j)}\right)<0$.

The third factor in the partial amplitude (36), $f_{l}\left(\tilde{\mathcal{P}}_{j}, \boldsymbol{\mathcal { P }}_{j}\right)$, is the amplitude for elastic electron scattering in effective range theory:

$$
\begin{equation*}
f_{l}\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)=\frac{(2 l+1)\left(\tilde{\mathcal{P}}_{j} \cdot \mathcal{P}_{j}\right)^{l}}{\mathcal{R}_{l}\left(\mathcal{P}_{j}^{2} / 2 m\right) \hbar^{2 l}}, \quad l=0,1 \tag{46}
\end{equation*}
$$

where $\tilde{\mathcal{P}}_{j}=s_{\alpha} \mathbf{e}_{z}\left|\mathcal{P}_{j}\right|$ and $s_{\alpha}=\operatorname{sgn}\left[A_{\tau}\left(t_{f}^{(j)}\right)\right]=\operatorname{sgn}\left[\dot{F}_{\tau}\left(t_{f}^{(j)}\right)\right]$. For electrons detached into the left hemisphere, the vector $\tilde{\mathcal{P}}_{j}$ is directed along the positive $\mathbf{z}$ axis, while for electrons detached into the right hemisphere, it is directed along the negative $\mathbf{z}$ axis.

The shortcoming of effective range theory for $l=0,1$ is that the forward and backward electron scattering amplitudes in Eq. (46) differ from each other only by a phase factor $(-1)^{l}$. For this reason, it is unclear whether the backscattering amplitude enters the partial amplitude $A_{j}$. To clarify the situation, we have considered ATD employing a more advanced "two-state" TDER model, which takes into account two scattering phase shifts in the $L$-wave continuum channels, i.e., $L=0$ and $L=1$ [30]. Within this model, we found that the third factor in the parametrization (36) is given by the following scattering amplitude:

$$
\begin{align*}
f_{l}\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right) & =\frac{1}{2 i k}\left[\left(e^{2 i \delta_{0}}-1\right)+3\left(e^{2 i \delta_{1}}-1\right) \cos \theta\right] \\
& =\mathcal{R}_{0}^{-1}\left(\mathcal{P}_{j}^{2} / 2 m\right)+3 \frac{\left(\tilde{\mathcal{P}}_{j} \cdot \mathcal{P}_{j}\right)}{\mathcal{R}_{1}\left(\mathcal{P}_{j}^{2} / 2 m\right) \hbar^{2}} \tag{47}
\end{align*}
$$

In the two-state TDER model the forward scattering and backscattering amplitudes differ, so that within this model we confirm that only the backscattering amplitude appears in our analysis of the rescattering amplitude.

## E. Detachment rate for a periodic field

Substituting the ATD amplitude (35) into Eq. (30), it is convenient to present the ATD rate $\Gamma\left(\mathbf{p}_{n}\right)$ as follows:

$$
\begin{equation*}
\Gamma\left(\mathbf{p}_{n}\right)=\left(\frac{\omega_{\tau}}{\omega}\right)^{2}\left[\Gamma_{\mathrm{dir}}\left(\mathbf{p}_{n}\right)+\Gamma_{\mathrm{int}}\left(\mathbf{p}_{n}\right)\right] \tag{48}
\end{equation*}
$$

where we have introduced the carrier frequency $\omega$ of a short laser pulse $\mathbf{F}(t)$.

The first ("direct") term in Eq. (48) is a sum of partial rates $\Gamma_{j}$ corresponding to the partial amplitudes $A_{j}$ in Eq. (36):

$$
\begin{gather*}
\Gamma_{\mathrm{dir}}\left(\mathbf{p}_{n}\right)=\sum_{j} \Gamma_{j}\left(\mathbf{p}_{n}\right)  \tag{49}\\
\Gamma_{j}\left(\mathbf{p}_{n}\right)=\frac{p_{n}}{m a^{3}}\left(\frac{\hbar \omega}{E_{\mathrm{at}}}\right)^{2}\left|A_{j}\right|^{2}=\mathcal{I}_{j} \mathcal{W}_{j} \sigma\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right), \tag{50}
\end{gather*}
$$

where $\Gamma_{j}$ is parametrized as a product of three factors: an ionization factor $\mathcal{I}_{j}$, a propagation factor $\mathcal{W}_{j}$, and an elastic backscattering cross section $\sigma\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)$. [Note that the factor $\omega^{2}$ in Eq. (50) compensates its inverse in Eq. (48).]

The ionization factor is expressed in terms of the decay rate $\Gamma_{\text {st }}$ for a weakly bound electron in an "effective" dc field with strength $\tilde{F}_{j} \mathbf{e}_{z}$ [36]:

$$
\begin{equation*}
\mathcal{I}_{j}=\left(\frac{\hbar \omega}{E_{\mathrm{at}}}\right)^{2} a^{-1}\left|a_{\tau}^{(j)}\right|^{2}=\frac{m}{\pi \hbar \kappa} \tilde{\gamma}_{j}^{2} \Gamma_{\mathrm{st}}\left(\tilde{F}_{j}\right), \tag{51}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{\mathrm{st}}(F)=\frac{\left|E_{0}\right|}{\hbar}(2 l+1) C_{\kappa l}^{2} \frac{F}{2 F_{0}} e^{-2 F_{0} /(3 F)} \tag{52}
\end{equation*}
$$

$F_{0}=(\kappa a)^{3} F_{\mathrm{at}}, \tilde{\gamma}_{j}=\hbar \omega \kappa /\left(|e| \tilde{F}_{j}\right)$, and $\tilde{F}_{j}=\left|F_{\tau}\left(t_{i}^{(j)}\right)\right|$. The propagation factor and the scattering cross section are given respectively by

$$
\begin{gather*}
\mathcal{W}_{j}=\frac{p_{n} \omega_{\mathrm{at}}}{\hbar}\left|a_{\mathcal{W}}^{(j)}\right|^{2}=\frac{p_{n}}{\hbar} \frac{\mathrm{Ai}^{2}\left(\xi_{j}\right)}{\zeta_{j}^{2 / 3}\left[t_{f}^{(j)}-t_{i}^{(j)}\right]^{3} \omega_{\mathrm{at}}^{2}}  \tag{53}\\
\sigma\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)=\left|f_{l}\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)\right|^{2} \tag{54}
\end{gather*}
$$

The interference between amplitudes $A_{j}$ with different $j$, corresponding to different pairs of ionization and rescattering times, is described by the "interference" term $\Gamma_{\text {int }}\left(\mathbf{p}_{n}\right)$ in Eq. (48):

$$
\begin{gather*}
\Gamma_{\mathrm{int}}\left(\mathbf{p}_{n}\right)=\sum_{j \neq j^{\prime}} s_{j, j^{\prime}} \sqrt{\Gamma_{j}\left(\mathbf{p}_{n}\right) \Gamma_{j^{\prime}}\left(\mathbf{p}_{n}\right)} \cos \Phi_{j, j^{\prime}},  \tag{55}\\
\Phi_{j, j^{\prime}}=\left(\varphi_{j}-\varphi_{j^{\prime}}\right) / \hbar+\psi\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)-\psi\left(\tilde{\mathcal{P}}_{j^{\prime}}, \mathcal{P}_{j^{\prime}}\right), \tag{56}
\end{gather*}
$$

where the phase $\varphi_{j}$ is given by Eq. (37), $s_{j, j^{\prime}}=$ $\operatorname{sgn}\left[\operatorname{Ai}\left(\xi_{j}\right) \operatorname{Ai}\left(\xi_{j^{\prime}}\right)\right]\left(\sigma_{j} \sigma_{j^{\prime}}\right)^{l}(= \pm 1)$, and $\psi\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)$ is the phase of the electron scattering amplitude:

$$
f\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)=\left|f\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)\right| e^{i \psi\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)}
$$

## IV. DETACHMENT PROBABILITY FOR A SHORT PULSE AND GENERALIZATION TO THE ATOMIC CASE

To obtain the differential probability $\mathcal{P}(\mathbf{p})$ for a short pulse, we substitute $\mathbf{p}_{n} \rightarrow \mathbf{p}$ into Eq. (48) for the rate $\Gamma\left(\mathbf{p}_{n}\right)$ (where $\mathbf{p}$ is the momentum of an electron detached by a short pulse) and replace $\mathbf{A}_{\tau}(t)$ and $\mathbf{F}_{\tau}(t)$ by the vector potential $\mathbf{A}(t)$ and the electric-field vector $\mathbf{F}(t)$ for a given short pulse. Then, taking into account the explicit form for $\Gamma(\mathbf{p})$ [cf. Eq. (48)], the limit in Eq. (2) is calculated straightforwardly and gives the result for $\mathcal{P}(\mathbf{p})$ announced in Ref. [24]:

$$
\begin{equation*}
\mathcal{P}(\mathbf{p})=\frac{2 \pi}{\hbar \omega^{2}}\left[\Gamma_{\mathrm{dir}}(\mathbf{p})+\Gamma_{\mathrm{int}}(\mathbf{p})\right] \tag{57}
\end{equation*}
$$

where the sums over $j$ in Eqs. (49) and (55) for $\Gamma_{\mathrm{dir}}(\mathbf{p})$ and $\Gamma_{\mathrm{int}}(\mathbf{p})$ are over the solutions $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ of the classical equations (34) for a given short pulse [i.e., substituting there $A_{\tau}(t) \rightarrow A(t)$ and $\left.F_{\tau}(t) \rightarrow F(t)\right]$.

Since all three factors in the parametrization of $\Gamma_{\mathrm{dir}}(\mathbf{p})$ according to Eqs. (49) and (50) have a transparent physical meaning, the analytic result (57) can be generalized to describe the high-energy part of the short-pulse ATI spectrum of a neutral atom. Moreover, since the propagation factor $\mathcal{W}_{j}$ is essentially independent of the atomic dynamics, this generalization can be performed by replacing only the other two factors in Eq. (50) by their corresponding atomic counterparts for the particular atom considered. Specifically, in order to generalize our result (57) for $\mathcal{P}(\mathbf{p})$ to the case of an atom, we must replace the dc detachment rate $\Gamma_{\text {st }}(\tilde{F})$ in Eq. (51) by the result for a potential having an asymptotic Coulomb field [36]:

$$
\begin{equation*}
\Gamma_{\mathrm{st}}(\tilde{F})=\frac{\left|E_{0}\right|}{\hbar}(2 l+1) C_{\kappa l}^{2}\left(\frac{2 F_{0}}{\tilde{F}}\right)^{2 \nu-1} e^{-2 F_{0} /(3 \tilde{F})} \tag{58}
\end{equation*}
$$

where $v=Z /(\kappa a), Z$ is the charge of the atomic core, and $C_{\kappa l}$ is the asymptotic coefficient for an initial bound state in a potential with an asymptotic Coulomb field:

$$
\begin{equation*}
\left.\psi_{\kappa, l, m=0}(\mathbf{r})\right|_{\kappa r \gg 1}=C_{\kappa l} \sqrt{\kappa} r^{-1}(\kappa r)^{v} e^{-\kappa r} Y_{l 0}(\hat{\mathbf{r}}) \tag{59}
\end{equation*}
$$

Also, the elastic scattering cross section $\sigma$ and the phase of the scattering amplitude $\psi$ should be replaced by the corresponding results for electron-ion scattering. As an example, for the hydrogen atom we have

$$
\begin{align*}
\sigma\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)= & \frac{m^{2} e^{4}}{\mathcal{P}_{j}^{4}}\left(1-\left(\tilde{\mathcal{P}}_{j} \cdot \mathcal{P}_{j}\right) / \mathcal{P}_{j}^{2}\right)^{-2},  \tag{60}\\
\psi\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)= & 2 \arg \Gamma\left(1+\frac{i \hbar}{\mathcal{P}_{j} a}\right) \\
& -\frac{\hbar}{\mathcal{P}_{j} a} \ln \left[\frac{1-\left(\tilde{\mathcal{P}}_{j} \cdot \mathcal{P}_{j}\right) / \mathcal{P}_{j}^{2}}{2}\right] . \tag{61}
\end{align*}
$$

## V. DISCUSSION AND NUMERICAL RESULTS

Before presenting our numerical results, we discuss first some general properties of the "partial" rates $\Gamma_{j}(\mathbf{p})$ [cf. Eq. (50)] for a given set of ionization $\left(t_{i}^{(j)}\right)$ and rescattering $\left(t_{f}^{(j)}\right)$ times.

## A. General features of $\boldsymbol{\Gamma}_{\boldsymbol{j}}(\mathbf{p})$

The key ingredients of our analytic results for $\mathcal{P}(\mathbf{p})$ in Eq. (57) are the partial rates $\Gamma_{j}(\mathbf{p})$, whose parametrization (50) is the same as for the case of a monochromatic field [19], i.e., it is the product of three factors: a tunneling factor $\left(\mathcal{I}_{j}\right)$, a propagation factor $\left(\mathcal{W}_{j}\right)$, and the cross section for elastic electron scattering $(\sigma)$. Since the tunneling factor does not depend on the momentum $\mathbf{p}$ of the ionized electron but depends strongly on the ionization time $t_{i}^{(j)}$ [cf. Eq. (51)], this factor determines the absolute contribution of $\Gamma_{j}(\mathbf{p})$ to the ionization probability $\mathcal{P}(\mathbf{p})$. In particular, this factor filters out those $\Gamma_{j}(\mathbf{p})$ (or electron trajectories corresponding to the set $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ ), for which the instantaneous value of the laser intensity at the moment of ionization $t_{i}^{(j)}$ is small compared to the peak intensity of the laser pulse.

The propagation factor $\mathcal{W}_{j}$ describes the dynamics of a free electron in the laser field from the ionization time $t_{i}^{(j)}$ up to the rescattering event at time $t_{f}^{(j)}$. This dynamics is described in terms of classical trajectories. The explicit form (53) for the propagation factor shows that $\mathcal{W}_{j}$ becomes exponentially small for $\xi_{j}>\alpha_{0}$, where $\alpha_{0} \equiv-1.019$ is the argument of the Airy function at which the long and short trajectories merge into a single extremal trajectory [see the discussion in the text below Eq. (45)]. Thus the equality $\xi_{j}=\alpha_{0}$ or

$$
\begin{equation*}
\frac{1}{2 m}\left[\mathbf{p}+\mathbf{e}_{z} \frac{|e|}{c} A\left(t_{f}^{(j)}\right)\right]^{2}-E_{\max }^{(j)}=\alpha_{0} E_{\mathrm{at}} \zeta_{j}^{1 / 3} \tag{62}
\end{equation*}
$$

determines a border between classically allowed and classically forbidden regimes for transfer of the energy $E=$ $\mathbf{p}^{2} /(2 m)$ from the laser field to the ionized electron. For $u_{p}=e^{2} F^{2} /\left(4 m \omega^{2}\right) \gg \hbar \omega$ (i.e., in the low-frequency limit), Eq. (62) can be simplified, because the terms on the left in Eq. (62) are of order of $u_{p}$, while the term on the right is of order $E_{\mathrm{at}}\left(F / F_{\mathrm{at}}\right)^{2 / 3}$ [cf. Eq. (44)], where $F$ is the peak value of $F(t)$. Neglecting this latter term, Eq. (62) reduces to

$$
\begin{equation*}
\frac{1}{2 m}\left[p_{\|}+\frac{|e|}{c} A\left(t_{f}^{(j)}\right)\right]^{2}+\frac{1}{2 m} p_{\perp}^{2}=E_{\max }^{(j)} \tag{63}
\end{equation*}
$$

where $p_{\|}=p \cos \Theta, p_{\perp}=p \sin \Theta$, and $\Theta$ is the angle between the momentum direction of the ionized electron and the laser polarization direction $\mathbf{e}_{z}$. Equation (63) shows that two (classically allowed and classically forbidden) regimes are separated in the plane of $p_{\|}$and $p_{\perp}$ by the circle with radius $\sqrt{2 m E_{\text {max }}^{(j)}}$ (cf. Fig. 1). The cutoff energy of the electron, $E_{\text {cut }}^{(j)}$, as a function of the angle $\Theta$ can be obtained by solving the quadratic Eq. (63) for $p=p_{\text {cut }}^{(j)}$ and choosing the largest root to obtain

$$
\begin{align*}
E_{\mathrm{cut}}^{(j)}(\Theta)= & \frac{\left[\mathbf{p}_{\mathrm{cut}}^{(j)}\right]^{2}}{2 m}=2 u_{p}\left(a_{j}^{2} \cos 2 \Theta+\varepsilon_{j} / 2\right. \\
& \left.+\left|a_{j}\right| \sqrt{2 \varepsilon_{j} \cos ^{2} \Theta-a_{j}^{2} \sin ^{2} 2 \Theta}\right) \tag{64}
\end{align*}
$$

where $a_{j}=A\left(t_{f}^{(j)}\right) / A_{0}, A_{0}=c F / \omega$, and $\varepsilon_{j}=E_{\max }^{(j)} / u_{p}$. As shown in Eq. (64), $E_{\text {cut }}^{(j)}$ scales linearly with $u_{p}$. In the low-frequency limit, $E_{\text {max }}^{(j)} \approx \mathcal{E}\left(t_{i}^{(j)}, t_{f}^{(j)}\right.$ ) [ignoring the small quantum correction on the right in Eq. (43)] so that $E_{\text {cut }}^{(j)}$


FIG. 1. (Color online) Sketch of the cutoff circle for a given $\Gamma_{j}$ and $\left(\mathbf{e}_{z} \cdot \mathbf{p}_{\text {cut }}^{(j)}\right)>0$ (see text for details). The kinetic momentum of the electron after rescattering, $\mathcal{P}_{j}$, is given by the orange vector; the vector potential at the return time $t_{f}^{(j)}$ is given by the blue vector, and the cutoff momentum $\mathbf{p}_{\text {cut }}^{(j)}$ is given for two directions, indicated by black and gray vectors.
becomes insensitive to the atomic target [34]. For $\Theta=0^{\circ}$, Eq. (64) gives the well-known classical result [7]:

$$
\begin{align*}
E_{\mathrm{cut}}^{(j)}\left(0^{\circ}\right) & =2\left(\left|a_{j}\right|+\sqrt{\frac{\varepsilon_{j}}{2}}\right)^{2} u_{p} \\
& \approx \frac{e^{2}}{2 m c^{2}}\left[\left|A\left(t_{f}^{(j)}\right)\right|+\left|A\left(t_{f}^{(j)}\right)-A\left(t_{i}^{(j)}\right)\right|\right]^{2}=e_{j} u_{p} \tag{65}
\end{align*}
$$

where $e_{j}$ is the magnitude of $E_{\mathrm{cut}}^{(j)}\left(0^{\circ}\right)$ in units of $u_{p}$. (Note that $e_{j}=10.007$ for the cutoff energy of the ATI plateau in a monochromatic field [34].) The second equality in Eq. (65) follows from Eqs. (38) and (43) and has a clear physical interpretation: the term $\left|A\left(t_{f}^{(j)}\right)-A\left(t_{i}^{(j)}\right)\right|$ is the momentum gained by the electron from the time of ionization to the time of rescattering, while $\left|A\left(t_{f}^{(j)}\right)\right|$ is the momentum gain following rescattering. Clearly Eq. (65) shows that the largest cutoff energy is obtained when $A\left(t_{f}^{(j)}\right)$ and $A\left(t_{i}^{(j)}\right)$ have opposite signs. The cutoff energy decreases with increasing $\Theta$ [for $\left.\left(\mathbf{e}_{z} \cdot \mathbf{p}\right)>0\right]$ and $\tilde{\Theta}=\pi-\Theta\left[\right.$ for $\left.\left(\mathbf{e}_{z} \cdot \mathbf{p}\right)<0\right]$ [34]. [Note that Eq. (64) is invariant with respect to the replacement $\Theta \rightarrow \tilde{\Theta}$.] This fact is clear from the geometrical point of view (cf. Fig. 1): the length of the vector $\mathbf{p}_{\text {cut }}^{(j)}$ becomes smaller for larger angles between $\mathbf{p}_{\text {cut }}^{(j)}$ and $\mathbf{e}_{z}$.

A pair of classical times $\left(t_{i}^{(j)}, t_{f}^{(j)}\right)$ determines the starting $\left(t_{i}^{(j)}\right)$ and ending $\left(t_{f}^{(j)}\right)$ times of the electron motion in a laser field along a closed extreme trajectory, which provides a local maximum $\left(E_{\text {cut }}^{(j)}\right)$ of the electron's energy gain in the vicinity of the time $t=t_{f}^{(j)}$. If the time $t$ differs slightly from $t_{f}^{(j)}$, the electron gains an energy $E$ that is smaller than $E_{\text {cut }}^{(j)}$. Moreover, it can attain this energy by moving along either of two different trajectories (a "short" trajectory for $t<t_{j}^{(j)}$ and a "long" trajectory for $t>t_{j}^{(j)}$ ). Since both of these trajectories (or pathways) correspond to the same energy $E$, their interference causes an oscillatory pattern in
the ATI spectrum [37,38]. Mathematically, these oscillations are described by oscillations of the Airy function in Eq. (53) for $\xi_{j}<\alpha_{0}$. The condition for constructive and destructive interferences can be found from an equation similar to Eq. (62), $\xi_{j}=\alpha_{n}$, where $\alpha_{n}$ for odd $n$ determines the positions of the zeros of the Airy function (destructive interference), while for even $n$ it gives the positions of the maxima of $\mathrm{Ai}^{2}\left(\xi_{j}\right)$ (constructive interference). For $n>1$, an analytic approximation, $\alpha_{n}=0.25[3(2 n+1)]^{2 / 3}$, is valid with high accuracy.

## B. Comparison with TDSE results

In our TDSE calculations we define the laser pulse by the following vector potential:

$$
\begin{equation*}
A(t)=-\frac{c F}{\omega} f(t) \sin (\omega t+\phi) \tag{66}
\end{equation*}
$$

where the pulse envelope is defined by $f(t)=\sin ^{2}(t \pi / \tau)$ for $t \in(0, \tau)$ and zero otherwise; $\tau=2 \pi N / \omega$, where $N$ is the number of optical cycles in the pulse; and $F, \omega$, and $\phi$ are respectively the peak value of the electric field, the carrier frequency, and the CEP. Our results are for the He and Ar atoms. For He , we use the same one-electron potential as in Ref. [39]. The asymptotic coefficient $C_{\kappa l}$ and binding energy $\left|E_{0}\right|$ obtained from this potential are 1.98 and 0.903 a.u. respectively, which are close to the recommended values in Ref. [40]: 1.99, 0.903 a.u. For Ar we use the one-electron potential of Ref. [41], which gives $C_{\kappa l}=1.23$ and $E_{0}=$ 0.582 a.u., while the recommended values in Ref. [40] are $1.90,0.579$ a.u. Our analytic results obtained from Eq. (57) (using the aforementioned one-electron potentials to calculate the elastic-scattering amplitudes for He and Ar ) are compared with TDSE results for ATI spectra in an intense mid-IR field with the experimentally available wavelengths $\lambda=1.25 \mu \mathrm{~m}$ $(\hbar \omega=1.03 \mathrm{eV})$ and $1.5 \mu \mathrm{~m}(\hbar \omega=0.83 \mathrm{eV})$ [42-46].

The details of our numerical solution of the TDSE can be found in Refs. [39,47]. In brief, we expand the wave function $\Psi(\mathbf{r}, t)$ in spherical harmonics and the corresponding radial wave functions are discretized using the finite difference method. The wave function $\Psi(\mathbf{r}, t)$ is propagated in time using the split-operator method with a time step of $0.01 \mathrm{a} . \mathrm{u}$. The maximum radial grid point is taken to be 6000 a.u. with a grid spacing of $0.1 \mathrm{a} . \mathrm{u}$. to avoid reflection of the fastest electronic wave packets at the box edge. All the parameters are carefully chosen so that all results are fully converged. The typical maximum angular momentum $L$ is taken to be $L_{\max } \approx 300$. The ionization probability for an electron with asymptotic momentum $\mathbf{p}$ is obtained by projecting the wave function $\Psi\left(\mathbf{r}, t_{f}\right)$ (after the laser pulse is turned off) onto the scattering states (incoming Coulomb waves) $\Psi_{\mathbf{p}}{ }^{-}$of the field-free Hamiltonian, i.e.,

$$
\begin{equation*}
\mathcal{P}(\mathbf{p})=\left|\left\langle\Psi_{\mathbf{p}}^{-}(\mathbf{r}) \mid \Psi\left(\mathbf{r}, t_{f}\right)\right\rangle\right|^{2} \tag{67}
\end{equation*}
$$

In Figs. 2-5 we compare TDSE and analytic results for He ( $s$ state) and $\operatorname{Ar}$ ( $p$ state) subjected to an intense mid-IR laser pulse: for He , the peak intensity is $I=2 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}$ and $\lambda=1.5 \mu \mathrm{~m}$, while for $\mathrm{Ar}, I=4 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}$ and $\lambda=$ $1.25 \mu \mathrm{~m}$. The analytic results are found to be in excellent agreement with the TDSE results, reproducing the shape of


FIG. 2. (Color online) ATI spectra for He subjected to a laser pulse defined by Eq. (66) with peak intensity $I=2 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}$, $\lambda=1.5 \mu \mathrm{~m}(\hbar \omega=0.83 \mathrm{eV}), N=4$, and $\phi=0$. Thick solid (black) lines: analytic result (57); thin solid (blue) lines: TDSE results. $\Theta$ is the angle between the direction of linear laser polarization $\mathbf{e}_{z}$ and the ionized electron momentum $\mathbf{p}$.
the high-energy part of the ATI plateau, as well as both the large-scale and fine-scale oscillations of the ATI spectra in a short laser pulse. The shape of the ATI spectrum strongly depends on the number of cycles in the pulse, the electron ejection angle $\Theta$, and the CEP $\phi$. For example, for fixed $\Theta=0^{\circ}$ and $\phi=0^{\circ}$, a single-plateau structure is formed for small $N$ [cf. Figs. 2(a) and 4(a)], while a two-plateau structure appears for larger $N$ [cf. Figs. 3(a), 4(c) and 4(e)].

We have found good agreement between the TDSE and analytic results for electron energies $E \gtrsim 5 u_{p}$, where $u_{p}=$ $e^{2} F^{2} /\left(4 m \omega^{2}\right)\left(u_{p}=1.54\right.$ a.u. for Figs 2 and 3 and $u_{p}=$ 2.14 a.u. for Figs. 4 and 5), while for $E<5 u_{p}$ we observe noticeable discrepancies. We attribute the origin of these discrepancies to the inaccuracy of our analytic results for low-electron energies. Indeed, the analytic results depend on the cubic approximation in Eq. (A22) for the phase function $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ in Eq. (A6), which is justified only near the points of coalescence of the short and long trajectories (cf. Sec. V A). These points are associated with cutoff energies for the partial rates $\Gamma_{j}$, so that the inaccuracy in estimating $\Gamma_{j}$ increases with


FIG. 3. (Color online) The same as in Fig. 2 but for $N=6$ and $\phi=\pi / 2$.


FIG. 4. (Color online) ATI spectra for Ar subjected to a laser pulse defined by Eq. (66) with peak intensity $I=4 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}$, $\lambda=1.25 \mu \mathrm{~m}(\hbar \omega=1.03 \mathrm{eV})$, and $\phi=0$. Thick solid (black) lines: analytic result (57); thin solid (blue) lines: TDSE results.
decreasing electron energy $E$. The accuracy of our analytic results also decreases with increasing offset of the ionized electron momentum $\mathbf{p}$ from the polarization axis of the laser pulse. Our comparisons of the analytic and the TDSE results show that reasonable agreement can be achieved for angles $\Theta<45^{\circ}$ for $\left(\mathbf{p} \cdot \mathbf{e}_{z}\right)>0$ and for $\Theta>135^{\circ}$ for $\left(\mathbf{p} \cdot \mathbf{e}_{z}\right)<0$, where $\Theta$ is the angle between $\mathbf{p}$ and $\mathbf{e}_{z}$. This drawback of our analytical analysis originates from the approximate calculation of the second derivative of the phase function $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ [cf. Eq. (A9)], which is not valid for ( $\mathbf{p} \cdot \mathbf{e}_{z}$ ) close to zero.

## C. Multiplateau structure, left-right asymmetry, and interference features in short-pulse ATI spectra

The major differences of short-pulse ATI spectra from those for a monochromatic field are (i) the pronounced multiplateau structure of short-pulse ATI spectra; (ii) the breakdown of the left-right symmetry in short-pulse ATI spectra [4,7]; and (iii) the presence of both large-scale and fine-scale oscillations in short-pulse ATI spectra [7]. We discuss below how all these features can be described within our analytic theory.


FIG. 5. (Color online) The same as in Fig. 4 but for $\phi=\pi / 2$.

TABLE I. Parameter values for the trajectories $j$ having the highest plateau cutoff energies for laser pulses with different numbers of cycles $N$ and CEPs $\phi$. For each trajectory $j, t_{i}^{(j)}$ and $t_{f}^{(j)}$ are the initial (ionization) and final (rescattering) times, $f_{j}=F\left(t_{i}^{(j)}\right) / F$ and $a_{j}=A\left(t_{f}^{(j)}\right) / A_{0}$ (where $\left.A_{0}=c F / \omega\right)$ are the scaled electric field and vector potential at those times, $\varepsilon_{j}=E_{\max }^{(j)} / u_{p}$ and $e_{j}=E_{c u t}^{(j)}\left(0^{\circ}\right) / u_{p}$ are scaled energies (in units of the ponderomotive potential $u_{p}$ ), and $\mathcal{I}_{j}$ is the ionization factor [cf. Eqs. (50) and (51)]. ( $n$ ) $\equiv 10^{n}$.

| ${ }^{j}$ | $\omega t_{i}^{(j)}$ | $\omega t_{f}^{(j)}$ | $f_{j}$ | $a_{j}$ | $\varepsilon_{j}$ | $e_{j}$ | $\mathcal{I}_{j}$ (a.u.) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N=4, \phi=0,(\mathbf{e} \cdot \mathbf{p})>0$ |  |  |  |  |  |  |  |
| 1 | 9.72 | 13.95 | $-0.87$ | -0.95 | 2.91 | 9.33 | 4.50(-11) |
| $N=4, \phi=0,(\mathbf{e} \cdot \mathbf{p})<0$ |  |  |  |  |  |  |  |
| 1 | 12.90 | 16.96 | 0.94 | 0.69 | 2.07 | 5.83 | 2.66(-10) |
| $N=6, \phi=\pi / 2,(\mathbf{e} \cdot \mathbf{p})>0$ |  |  |  |  |  |  |  |
| 1 | 14.40 | 18.71 | -0.85 | -0.99 | 2.96 | 9.73 | 3.36(-11) |
| 2 | 20.73 | 24.88 | -0.92 | -0.74 | 2.15 | 6.34 | 1.84(-10) |
| 3 | 14.25 | 24.99 | -0.86 | -0.75 | 1.44 | 5.13 | 4.11(-11) |
| $N=6, \phi=\pi / 2,(\mathbf{e} \cdot \mathbf{p})<0$ |  |  |  |  |  |  |  |
| 1 | 17.56 | 21.80 | 0.95 | 0.92 | 2.89 | 9.03 | 3.56(-10) |
| 2 | 14.25 | 21.89 | -0.86 | 0.93 | 1.41 | 6.27 | 4.11(-11) |

## 1. Multiplateau structure of short pulse ATI spectra

Each of the partial rates $\Gamma_{j}$ in Eqs. (49) and (55) is associated with some particular ionization and rescattering event, which happen respectively at times $t_{i}^{(j)}$ and $t_{f}^{(j)}$. Since the ionization factor $\mathcal{I}_{j}$ and the cutoff energy $E_{\text {cut }}^{(j)}\left(\Theta=0^{\circ}\right)=$ $e_{j} u_{p}$ can be expressed in terms of $t_{i}^{(j)}$ and $t_{f}^{(j)}$ [cf. Eqs. (51) and (65)], we may also associate $\Gamma_{j}$ with the ionization factor $\mathcal{I}_{j}$ [cf. Eqs. (50) and (51)] and the cutoff energy $E_{\text {cut }}^{(j)}\left(\Theta=0^{\circ}\right)=e_{j} u_{p}$. Let us assume that there is a set of $\Gamma_{j}$ for which $\mathcal{I}_{j_{1}}<\mathcal{I}_{j_{2}}<\cdots<\mathcal{I}_{j_{n}}$, while $e_{j_{1}}>e_{j_{2}}>\cdots>e_{j_{n}}$ [cf. the definition of $e_{j}$ in Eq. (65)]. Obviously, each term of this set contributes only in a particular range of energies in the ATI spectrum, thereby forming a multiplateau structure. Indeed, for energies $E>E_{j_{2}}$, only $\Gamma_{j_{1}}$ contributes and forms a plateau up to the energy $E_{j_{1}}$, while for the energy interval $E_{j_{2}}>E>E_{j_{3}}$, two partial rates $\Gamma_{j}$ contribute: $\Gamma_{j_{1}}$ and $\Gamma_{j_{2}}$. Since $\mathcal{I}_{j_{2}}>\mathcal{I}_{j_{1}}$, the contribution of $\Gamma_{j_{1}}$ is suppressed by $\Gamma_{j_{2}}$, so that $\Gamma_{j_{2}}$ determines the shape of the second plateau.

An example of multiplateau structure in a short-pulse ATI spectrum is presented in Figs. 3(a) and 3(c). Table I shows that for $N=6$ and the positive scalar product $\left(\mathbf{p} \cdot \mathbf{e}_{z}\right)$, there are three contributing terms $\Gamma_{j}$. The first term (with $j=1$ ) gives a longer but lower plateau, while the term with $j=2$ gives a shorter but higher plateau. The term with $j=3$ contributes for energies $E<5 u_{p}$ and is associated with multiple returns, which we will discuss in Sec. V D. In Figs. 3(b), 3(d), and 3(f) we present results for a negative scalar product ( $\mathbf{p} \cdot \mathbf{e}_{z}$ ). In this case there are two contributing partial rates $\Gamma_{j}$. However, owing to the difference in ionization factors ( $\mathcal{I}_{1}>\mathcal{I}_{2}$ ), the plateau structure of $\Gamma_{1}$ masks that of $\Gamma_{2}$ (cf. Table I). The occurrence of multiplateau features in ATI spectra depends crucially on the subcycle structure of the laser pulse, which for the case of a $\sin ^{2}$-shaped pulse depends on both the number of optical cycles $N$ and the CEP $\phi$.

## 2. Left-right asymmetry of short pulse ATI spectra

For a short pulse with stabilized CEP, the angular distribution of ATI electrons, unlike that for a monochromatic field,
is asymmetric with respect to opposite directions, $\mathbf{p}$ and $-\mathbf{p}$, of the electron momentum (cf. the review [7]). Phenomenologically, this asymmetry originates from the nonequivalence of the temporal distribution of the pulse electric-field vector $\mathbf{F}(t)$ with respect to the change $\mathbf{F}(t) \rightarrow-\mathbf{F}(t)$ or $\mathbf{e}_{z} \rightarrow-\mathbf{e}_{z}$. Since a true scalar quantity $\mathcal{P}(\mathbf{p})$ involves the vector $\mathbf{e}_{z}$ only in combination with another vector of the problem, $\mathbf{p}$, the change $\mathbf{e}_{z} \rightarrow-\mathbf{e}_{z}$ is equivalent to $\mathbf{p} \rightarrow-\mathbf{p}$ [48]. Since the temporal distribution of the field $\mathbf{F}(t)$ depends significantly on the $\operatorname{CEP} \phi$, this CEP dependence provides a way for controlling the asymmetry in the ATI angular distribution [5,49]. The asymmetry of the ATI yield of electrons ejected into the left and right hemispheres centered about the $z$ axis is clearly visible in Figs. 2-5. In our analytic results, the ATI asymmetry originates from the difference in the ionization factors and the cutoff energies $E_{\text {cut }}^{(j)}(\Theta)$ for angles $\Theta$ and $\pi-\Theta$.

To demonstrate the CEP dependence of the cutoff energies and the disappearance of the ATI asymmetry for a large number $N$ of optical cycles in the pulse, in Fig. 6 we present $E_{\text {cut }}^{(j)}\left(0^{\circ}\right)$ and $E_{\text {cut }}^{(j)}\left(180^{\circ}\right)$ as functions of the CEP for different $N$. Each line in Fig. 6 is colored according to the magnitude of the corresponding ionization factor for a given $\phi$ and $j$. For example, Fig. 6(d) presents two curves, which correspond to two $\Gamma_{j}$ for $\Theta=180^{\circ}$ : the upper line corresponds to the $\Gamma_{j}$ with higher cutoff energy, while its ionization factor decreases with increasing $\phi$, as shown by the color of the line changing gradually from red to yellow; the bottom line corresponds to the $\Gamma_{j}$ with smaller cutoff energy but smaller ionization factor, which also decreases with increasing $\phi$, as shown by the color of the line changing gradually from yellow to black. As seen in Fig. 6, with increasing number $N$ of cycles in the laser pulse, the curves of the cutoff energies for $\Theta=0^{\circ}$ and $180^{\circ}$ approach each other and the left-right asymmetry of the ATI spectrum decreases. For large $N$, a huge number of partial rates $\Gamma_{j}$ contribute. However, the cutoff energies $E_{\mathrm{cut}}^{(j)}$ become insensitive to the CEP (with nearly horizontal, uniformly colored curves in Fig. 6 for $N=100$ near the $10 u_{p}$ cutoff energy); consequently the left-right ATI asymmetry vanishes. Our numerical analysis shows that for large $N$ the ionization factors of the contributing partial rates $\Gamma_{j}$ are approximately


FIG. 6. (Color online) Dependence of the cutoff energies $E_{\text {cut }}^{(j)}\left(0^{\circ}\right)$ (left panels) and $E_{\text {cut }}^{(j)}\left(180^{\circ}\right)$ (right panels) on the CEP $\phi$ for different numbers $N$ of optical cycles in the laser pulse with $I$ and $\lambda$ as in Fig. 2. Panels (a) and (b): $N=4$; (c) and (d): $N=6$; (e) and (f): $N=100$. Each $E_{\text {cut }}^{(j)}$ curve is colored according to the magnitude of the corresponding ionization factor $\mathcal{I}_{j}$ for a given $\operatorname{CEP} \phi$.
the same. However, the cutoff positions are located near $10 u_{p}$, $8 u_{p}$, etc., in agreement with the detailed trajectory analysis in Ref. [50].

## 3. Interference features and their dependence on the number $N$ of optical cycles in the laser pulse

Both large-scale and fine-scale oscillations are observable in short-pulse ATI spectra; however, they appear in different intervals of the ionized electron energy [cf. Fig. 3(a)]. Largescale oscillations are typical of the cutoff region of the ATI spectrum [cf. Fig. 3(a) for energy $E>11$ a.u.], while fine-scale oscillations appear in the middle part of the ATI plateau [cf. Fig. 3(a) for $E<11$ a.u. and Fig. 7]. The fine-scale oscillations modulate the large-scale oscillations. Note that although the distance between two neighboring fine-oscillation peaks is of the order of the energy $\hbar \omega$, these peaks cannot be associated with the well-known ATI peaks separated by


FIG. 7. (Color online) Details of fine-scale oscillation features in the ATI spectrum in Fig. 3(a).


FIG. 8. (Color online) ATI spectra for the He atom for a laser pulse with $I=2 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}, \lambda=1.2 \mu \mathrm{~m}$, and $\phi=0$. Left panels: ATI spectra for a trapezoidal pulse with a two-cycle linear ramp for switching on and off and an $(N-4)$-cycle flat-top; right panels: ATI spectra for a $\sin ^{2}$-shaped pulse with $N$ cycles. Blue (bright) thin lines: ATI spectra for the angle $\Theta=0^{\circ}$; black lines: ATI spectra for the angle $\Theta=180^{\circ}$. Thick orange lines: rescaled shape of ATI spectra for a monochromatic field. (a), (b): $N=10$; (c), (d): $N=30 ;(\mathrm{e})$, (f): $N=50$.
the photon energy, due to their dependence on both the CEP and the number of optical cycles in the pulse [cf. Figs. 8(b) 8(d), and 8(e)]. As discussed in Sec. V A, the large-scale oscillations originate from the interference of the short and long trajectories that contribute to the same partial amplitude $A_{j}$. The fine-scale oscillations originate from interference between different amplitudes $A_{j}$ and $A_{j^{\prime}}$ (with $j \neq j^{\prime}$ ) that have comparable magnitudes [cf. Eq. (55)]. If only two partial amplitudes interfere (as for the case of $N=6$ and $\phi=\pi / 2$ in Fig. 3), then a short pulse works like a double slit, making possible the interference of two different pathways to the same final state, giving rise to the fine-scale oscillations [5] (cf. Fig. 7).

With increasing $N$, the interference features become more complicated, because many partial amplitudes $A_{j}$ contribute. Starting with some large value of $N$, sharp ATI peaks develop beyond the plateau cutoff region as a result of interference between many partial amplitudes $A_{j}$. With further increases of $N$, these ATI peaks appear in the plateau region. Moreover, the phases $\varphi_{j}$ [cf. Eq. (37)] of the amplitudes $A_{j}$ depend on $j$ almost linearly. Indeed, if the laser pulse is long enough, then for some interval of $t$ the amplitude of a laser pulse can be considered to be constant. Thus the ionization times for different optical cycles over this time interval can be reduced to those for a single optical period $T=2 \pi / \omega$ by shifting them by an integer number of optical periods. Since the rescattering times are intimately connected with the ionization times, they are shifted in the same way as $t_{i}^{(j)}$, i.e.,

$$
\begin{equation*}
t_{i}^{(j)} \approx t_{i}^{(q)}+j^{\prime} T, \quad t_{f}^{(j)} \approx t_{f}^{(q)}+j^{\prime} T, \quad j^{\prime} \in[0, M] \tag{68}
\end{equation*}
$$

where $\left\{t_{i}^{(q)}, t_{f}^{(q)}\right\}$ is the $q$ th pair of ionization and recombination times for a monochromatic field $\left[t_{i}^{(q)} \in(0, T)\right]$ and $M$ is the number of periods for which the pulse field can be
approximated as a monochromatic one. Using Eq. (68), the major ingredients of the amplitude $A_{j}$ in Eq. (36) are transformed as

$$
\begin{gather*}
a_{\tau}\left(t_{i}^{(j)}\right) \rightarrow a_{\tau}\left(t_{i}^{(q)}\right),  \tag{69}\\
a_{\mathcal{W}}\left(t_{i}^{(j)}, t_{f}^{(j)}\right) \rightarrow a_{\mathcal{W}}\left(t_{i}^{(q)}, t_{f}^{(q)}\right),  \tag{70}\\
\mathcal{P}\left(t_{f}^{(j)}\right) \rightarrow \mathcal{P}\left(t_{f}^{(q)}\right),  \tag{71}\\
\varphi\left(t_{i}^{(j)}, t_{f}^{(j)}\right) \rightarrow \varphi\left(t_{i}^{(q)}, t_{f}^{(q)}\right)+\Delta_{\mathbf{p}} j^{\prime} T,  \tag{72}\\
\Delta_{\mathbf{p}}=\mathbf{p}^{2} /(2 m)+\left|E_{0}\right|+u_{p}, \tag{73}
\end{gather*}
$$

where the energy $\Delta_{\mathbf{p}}$ originates from the dc component of the integrand of $S\left(\mathbf{p}_{n}, t\right)$ [cf. Eq. (27)] for $\epsilon=E_{0}$. As a result, the rescattering amplitude $\mathcal{A}^{R}(\mathbf{p})$ within the approximation (68) can be presented as

$$
\begin{align*}
\mathcal{A}^{R}(\mathbf{p}) & =\sqrt{i} \frac{\hbar \omega_{\tau}}{e^{2} \sqrt{a}} \sum_{j^{\prime}=0}^{M} A_{\operatorname{mon}}(\mathbf{p}) e^{i \Delta_{\mathbf{p}} j^{\prime} T} \\
& =\sqrt{i} e^{i \Delta_{\mathbf{p}} N T /(2 \hbar)} \frac{\hbar \omega_{\tau}}{e^{2} \sqrt{a}} A_{\operatorname{mon}}(\mathbf{p}) \frac{\sin \left[\frac{(M+1) \Delta_{\mathbf{p}} T}{2 \hbar}\right]}{\sin \left(\frac{\Delta_{\mathbf{p}} T}{2 \hbar}\right)} \tag{74}
\end{align*}
$$

where

$$
\begin{equation*}
A_{\mathrm{mon}}(\mathbf{p})=\sum_{q} A_{q} \tag{75}
\end{equation*}
$$

is the ATI amplitude for a monochromatic field. Substituting the amplitude $\mathcal{A}^{R}(\mathbf{p})$ into Eq. (30), we then obtain from Eq. (2) the result

$$
\begin{align*}
\mathcal{P}(\mathbf{p}) & =\frac{2 \pi}{\hbar \omega^{2}} \Gamma_{\mathrm{mon}}(\mathbf{p}) \mathcal{F}(\mathbf{p})  \tag{76}\\
\mathcal{F}(\mathbf{p}) & =\frac{\sin ^{2}\left[\frac{(M+1) \Delta_{\mathrm{p}} T}{2 \hbar}\right]}{\sin ^{2}\left(\frac{\Delta_{\mathrm{p}} T}{2 \hbar}\right)} \tag{77}
\end{align*}
$$

where $\Gamma_{\text {mon }}(\mathbf{p})$ is the ionization rate for a monochromatic field and $\mathcal{F}(\mathbf{p})$ is a comb function with peaks at $p=p_{n}=$ $\sqrt{n \hbar \omega-\left|E_{0}\right|-u_{p}}$. In the limit $M \rightarrow \infty$, the function $\mathcal{F}(\mathbf{p})$ can be approximated by a sum of $\delta$ functions:

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \mathcal{F}(\mathbf{p})=\hbar \omega(M+1) \sum_{n} \delta\left(\frac{p^{2}}{2 m}-\frac{p_{n}^{2}}{2 m}\right) \tag{78}
\end{equation*}
$$

Substituting Eq. (78) into Eq. (76), we obtain the ionization probability for a long pulse in terms of the $n$-photon ionization rates:

$$
\begin{equation*}
\mathcal{P}(\mathbf{p})=T(M+1) \sum_{n} \Gamma\left(\mathbf{p}_{n}\right) \delta\left(\frac{\mathbf{p}^{2}}{2 m}-\frac{\mathbf{p}_{n}^{2}}{2 m}\right) \tag{79}
\end{equation*}
$$

The number of optical cycles in a laser pulse at which the asymptotic result (76) becomes applicable for describing the shape of short-pulse ATI spectra depends crucially on the pulse shape. As shown in Fig. 8, for a trapezoidal pulse, the asymptotic result (76) applies already for $N=10$ for both
$\Theta=0^{\circ}$ and $\Theta=180^{\circ}$, while for a $\sin ^{2}$-shaped pulse even $N=50$ is not enough to obtain reasonable agreement with the asymptotic result (76). For the trapezoidal pulse we employ $N=10$ cycles: two optical cycles for ramping the pulse on and off and six cycles on the flat-top part. We note the difference between absolute values (but not in the shape) of ATI yields for $\Theta=0^{\circ}$ and $\Theta=180^{\circ}$ for the trapezoidal pulse [cf. Fig. 8(a)]. This difference is due to the different number of optical cycles on the (six-cycle) flat-top part of the pulse that contribute significantly to the ATI yield for $\Theta=0^{\circ}$ (six cycles) and $\Theta=$ $180^{\circ}$ (five cycles). Nevertheless, the shape of both short-pulse ATI spectra coincide precisely with that for a monochromatic field.

## D. Contribution of multiple returns to short-pulse ATI spectra

As discussed already, each solution $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$ of the classical system of Eqs. (34) determines an extreme closed classical trajectory. Each trajectory $j$ can be classified in terms of its excursion time, $\Delta t_{j}=t_{f}^{(j)}-t_{i}^{(j)}$. Those trajectories for which $\Delta t_{j}<T$ are denoted "single-return" (SR) trajectories [cf. Fig. 9(a)]; otherwise, for $\Delta t_{j}>T$, they are denoted "multiple-return" (MR) trajectories [cf. Fig. 9(b)]. Moving along a MR trajectory, the electron may return to the starting


FIG. 9. (Color online) Sketch of classical single return (SR) and multiple return (MR) electron trajectories $z(t)$ (a), (b) and rescattered electron energies $E_{j}^{(r)}$ (c) as a function of time for a $\sin ^{2}$-shaped laser pulse with $N=10$ cycles and CEP $\phi=\pi / 2$. Dotted lines in (a), (b): the pulse electric field $F(t)$. (a) Solid (dot-dashed) (red) lines: SR trajectories representing electrons ionized at negative (positive) peak field amplitudes; the dashed (black) lines show an example of a pair of short and long trajectories. (b) Two-return [thin (orange) lines] and three-return [thick (blue) lines] MR trajectories; solid (dotdashed) lines are used for the two-return and three-return trajectories that begin at the first negative (positive) peak amplitude of the laser pulse. (c) The dependence of the rescattered electron energy, $E_{j}^{(r)}[\mathrm{cf}$. Eq. (65) for definition of $e_{j}$ ], on the rescattering time. Solid lines: for negative half cycles; dot-dashed lines: for positive half cycles.
point several times before rescattering at the time $t=t_{f}^{(j)}$. If the energy of the ionized electron is less than the maximum energy that the electron gains when it moves along an extreme trajectory, the extreme trajectory splits into "short" and "long" trajectories [cf. the dashed lines in Fig. 9(a)]. For any trajectory (either SR or MR), the sign of the first derivative, $\dot{F}(t)$, of the electric field at $t=t_{f}^{(j)}$ determines into which hemisphere the electron described by the partial ionization amplitude $A_{j}$ will be ejected: to the "L" or "left" hemisphere $[\Theta \in[\pi / 2, \pi)]$ for $\dot{F}\left(t_{f}^{(j)}\right)>0$ or to the "R" or "right" hemisphere $[\Theta \in(0, \pi / 2)]$ for $\dot{F}\left(t_{f}^{(j)}\right)<0$ [cf. the discussion in the text above Eq. (46) and in the Appendix].

Since the excursion time for a SR trajectory is less than the optical period $T$ of a pulse, the contribution of this trajectory to the yield of electrons into either the R or the L hemisphere is determined by the sign of the electric field $F(t)$ at the time of ionization: it is the $\mathrm{L}(\mathrm{R})$ hemisphere for $F\left(t_{i}^{(j)}\right)>0\left[F\left(t_{i}^{(j)}\right)<0\right]$. Thus if we restrict our consideration only to SR trajectories, the sign of the factor $\left(-\sigma_{j}\right)^{l}$ in Eq. (39) becomes independent of $j$. Moreover, the dependence of the interference term $\Gamma_{\text {int }}$ in Eq. (55) on the spatial symmetry of the initial state disappears because that $l$ dependence is given by the factor $s_{j j^{\prime}}$ [cf. its definition below Eq. (56)], in which both $\sigma_{j}$ and $\sigma_{j^{\prime}}$ have the same sign in the SR-trajectory approximation. (Note that this approximation becomes exact for an ultrashort pulse having only a few oscillations of the electric field, as for the case $N=4$ in Fig. 2.)

The MR trajectories may contribute to the middle part of the ATI plateau [50]. In Fig. 10 we present ATI spectra for an artificial "atom" having a binding energy $\left|E_{0}\right|=15.84 \mathrm{eV}$, $C_{\kappa l}=1$, and a constant scattering amplitude $f\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)=1$ (in order to minimize atomic potential effects). Figure 10 shows that the SR-trajectory approximation is appropriate near the cutoff of the high-energy plateau (i.e., for energies


FIG. 10. (Color online) ATI spectra for an "atom" with $\left|E_{0}\right|=$ $15.76 \mathrm{eV}, C_{\kappa l}=1$, and $f\left(\mathbf{p}_{i}, \mathbf{p}_{f}\right)=1$ for the $\sin ^{2}$-shaped pulse with peak intensity $4 \times 10^{14} \mathrm{~W} / \mathrm{cm}^{2}, \lambda=1.2 \mu \mathrm{~m}, \phi=\pi / 2$, and $N=10$.
(a) ATI spectra for electrons ejected at $\Theta=180^{\circ}$ (left hemisphere). (b) ATI spectra for $\Theta=0^{\circ}$ (right hemisphere). Black lines: $l=0$; red lines: $l=1$; blue lines: the SR-trajectory approximation. Arrows in (b) mark the electron energies $E=5.4 u_{p}$ and $5.6 u_{p}$ at which the red and black curves respectively have maxima (see text for discussion).
$E \geqslant 8.5 u_{p}$ ). For $E<8.5 u_{p}$, noticeable discrepancies are observed between the ATI spectra for $l=0$ and $l=1$, as well as between the exact results and the SR-trajectory approximation. These discrepancies originate from the contribution of MR trajectories, which may be created on either the positive or negative half cycles of the electric field [cf. Fig. 9(b)]. Depending on the excursion time, these MR trajectories can contribute to the electron yield into either the L or the R hemispheres. For instance, the solid (blue) trajectory in Fig. 9(b), a three-return trajectory created at the negative peak amplitude of the electric field, contributes to the R hemisphere, while the solid (orange) trajectory in Fig. 9(b), a two-return trajectory created at the same peak field, contributes to the L hemisphere. In Fig. 9(c), we present the electron energy after rescattering as a function of the (rescattering) time $t$. These results show explicitly that MR trajectories are expected to contribute to the middle part of an ATI spectrum: for energies $E<7 u_{p}$, both SR [red dot-dashed line in Fig. 9(a)] and MR [solid orange line in Fig. 9(b)] trajectories contribute; these trajectories were created on different (positive and negative) half cycles of the laser pulse.

According to Eq. (55), the result of interference between two partial amplitudes $A_{j}$ depends on both the sign of the electric field at the moment of ionization and the parity of the initial bound state. Indeed, the partial amplitudes $A_{j}$ and $A_{j^{\prime}}$ interfere "in phase" if the electric field has the same sign at the two ionization events, while the result of interference depends on the parity of initial state if the signs are opposite: for even $l$ ( $s$ state), they still interfere "in phase," but for odd $l$ ( $p$ state) they interfere "out of phase." Thus the shape of the middle part of the ATI plateau ( $E<8 u_{p}$ ) depends on the spatial symmetry of the initial state: for instance, if the ATI spectrum for an $s$ state has a maximum, then the ATI spectrum for a $p$ state has a minimum at the same energy [cf. the corresponding peaks and dips in the ATI spectra for $s$ and $p$ states at $E=5.4 u_{p}$ and $5.6 u_{p}$ in Fig. 10(b)].

## E. Comparison with the QRS theory

In this section we discuss the relation between the QRS theory and our analytic results. The QRS theory is based on the phenomenological factorization of the ATI yield $\mathcal{P}(\mathbf{p})$ in the high-energy plateau in terms of an electronic wave packet (EWP), $W^{Q R S}$, and the field-free cross section for elastic electron scattering (describing the scattering of an electron with initial momentum $\mathbf{p}_{i}=-\boldsymbol{e}_{z} p_{f}$ to the state with momentum $\left.\mathbf{p}_{f}=\mathbf{p}+\boldsymbol{e}_{z} \frac{|e|}{c} A_{0}\right)[12,14,51,52]$ :

$$
\begin{equation*}
\mathcal{P}^{(Q R S)}(\mathbf{p})=W^{Q R S} \sigma\left(\mathbf{p}+\boldsymbol{e}_{z} \frac{|e|}{c} A_{0}\right), \tag{80}
\end{equation*}
$$

where $A_{0}=c F / \omega$ is the global amplitude of the vector potential $\mathbf{e}_{z} A(t)$ of the pulse [cf. Eq. (66)] and $\boldsymbol{e}_{z}= \pm \mathbf{e}_{z}$, where the sign $+(-)$ depends on whether the scalar product $\left(\mathbf{e}_{z} \cdot \mathbf{p}\right)$ is positive or negative. For a monochromatic field, the QRS parametrization (80) was justified theoretically within the TDER theory near the ATI plateau cutoff energy [19]. However, as our analysis here shows, the features of atomic dynamics for short-pulse ATI are different for different optical cycles of the laser pulse, so that the total ATI amplitude is given by a sum of partial amplitudes with elastic-scattering
amplitudes having different arguments [cf. Eq. (35)]. In general, this fact prevents a factorization of the electron yield in terms of an EWP and a single elastic-scattering cross section, as in Eq. (80). Nevertheless, for a few-cycle pulse, our results show that the factorization (80) may be applicable. Indeed, if only a single partial rate $\Gamma_{j}$ contributes for a given direction $\hat{\mathbf{p}}$ of the ionized electron [i.e., indicated by the sign of $\left(\mathbf{e}_{z} \cdot \mathbf{p}\right)$ ], then the ionization probability $\mathcal{P}(\mathbf{p})$ can be parametrized in the same way as for a monochromatic field [19] [cf. the parametrization of $\Gamma_{j}$ in Eq. (50), substituting there $\left.\mathbf{p}_{n} \rightarrow \mathbf{p}\right]$. The applicability of such a parametrization may extend from the ATI plateau cutoff down to the middle part of the ATI plateau [cf. Figs. 2(a), 2(b), and 4(a)]. We note also that for the case of a few-cycle pulse, the vector potential at the time $t_{f}^{(j)}$ may not be close to its maximum value (cf. Table I for $N=4$ ), which may also cause a deviation between the parametrization (80) and our analytic results.

The QRS result (80) can be formally obtained from the analytic result (57) by replacing

$$
\begin{equation*}
\mathcal{P}_{j} \rightarrow \mathbf{p}+\boldsymbol{e}_{z} \frac{|e|}{c} A_{0} \tag{81}
\end{equation*}
$$

in the partial amplitude $A_{j}$ in Eq. (36). Using the substitution (81) and taking into account Eqs. (49) and (55), the ATI probability (57) then factorizes as in (80). The resulting factorized form of Eq. (57) then provides an explicit form for the QRS wave packet $W^{Q R S}$, which can be presented as

$$
\begin{equation*}
W^{(Q R S)}=\frac{2 \pi}{\hbar \omega^{2}}\left|\sum_{j}\left(-\sigma_{j}\right)^{l} \operatorname{sgn}\left[\mathrm{~A} i\left(\xi_{j}\right)\right] \sqrt{\mathcal{I}_{j} \mathcal{W}_{j}}\right|^{2} \tag{82}
\end{equation*}
$$

where the ionization factor $\mathcal{I}_{j}$ is given by Eq. (51) generalized to the case of a neutral atom (as discussed in Sec. IV) and the propagation factor $\mathcal{W}_{j}$ is given by Eq. (53), replacing there $\mathbf{p}_{n} \rightarrow \mathbf{p}$.

To estimate the accuracy of the replacement (81), we introduce the time $t_{0}$, which corresponds to an extremum of the vector potential $A(t)$ [i.e., $\left.\partial A(t) /\left.(\partial t)\right|_{t=t_{0}}=0\right]$ and is nearest to the rescattering time $t_{f}^{(j)}$. We also re-express the vector $\mathcal{P}_{j}$ as follows:

$$
\begin{align*}
\mathcal{P}_{j} & =\mathbf{p}+\mathbf{e}_{z} \frac{|e|}{c} A\left(t_{0}\right)-\mathbf{e}_{z} \frac{|e|}{c}\left(A\left(t_{0}\right)-A\left(t_{f}^{(j)}\right)\right) \\
& =\mathbf{p}+\boldsymbol{e}_{z} \frac{|e|}{c} A_{0}+\Delta \mathcal{P}_{j}^{(1)}+\Delta \mathcal{P}_{j}^{(2)} \tag{83}
\end{align*}
$$

where

$$
\begin{align*}
\Delta \mathcal{P}_{j}^{(1)} & =\mathbf{e}_{z} \frac{|e|}{c} A\left(t_{0}\right)-\boldsymbol{e}_{z} \frac{|e|}{c} A_{0},  \tag{84}\\
\Delta \mathcal{P}_{j}^{(2)} & =\mathbf{e}_{z} \frac{|e|}{c}\left(A\left(t_{f}^{(j)}\right)-A\left(t_{0}\right)\right) \\
& \approx \mathbf{e}_{z} \frac{|e|}{c} \frac{\partial^{2} A\left(t_{0}\right)}{\partial^{2} t_{0}} \frac{\left(t_{0}-t_{f}^{(j)}\right)^{2}}{2} \tag{85}
\end{align*}
$$

Since rescattering occurs near a maximum or minimum of the vector potential, the second correction, $\Delta \mathcal{P}_{j}^{(2)}$, is small and can be neglected. The first correction, $\Delta \mathcal{P}_{j}^{(1)}$, gives the difference between the extremum of the vector potential at the
point $t_{0}$ and its global amplitude $A_{0}$. This difference depends on the duration and CEP of the laser pulse: for many-cycle pulses, $\left|A\left(t_{0}\right)\right|$ is close to $A_{0}$ and the scattering amplitude can be expanded in series:

$$
\begin{align*}
f\left(\tilde{\mathcal{P}}_{j}, \mathcal{P}_{j}\right)= & f\left(\tilde{\mathcal{P}}_{0}, \mathcal{P}_{0}\right) \\
& +\left[\frac{f\left(\tilde{\mathcal{P}}_{0}, \mathcal{P}_{0}\right)}{\partial \tilde{\mathcal{P}}_{0}}+\frac{f\left(\tilde{\mathcal{P}}_{0}, \mathcal{P}_{0}\right)}{\partial \mathcal{P}_{0}}\right] \Delta \mathcal{P}_{j}^{(1)}, \tag{86}
\end{align*}
$$

where $\mathcal{P}_{0}=\mathbf{p}+\boldsymbol{e}_{z} \frac{|e|}{c} A_{0}$. Equation (86) shows that the accuracy of QRS theory increases if the scattering amplitude is a slowly varying function of electron energy (as for the case of a many-cycle pulse).

We have confirmed that for many-cycle pulses the deviation between the predictions of QRS theory and our results is only a few percent. However, since the pulse envelope for a few-cycle pulse varies in time much faster than for a many-cycle pulse, the accuracy of QRS theory decreases when only a few partial rates $\Gamma_{j}$ contribute to the ATI yield. In Fig. 11 we present ATI spectra for He for the same peak laser intensity and carrier frequency as in Fig. 2, but for $N=5$ and $\phi=\pi / 2$. One sees that the QRS theory works well for those electron energies for which the ATI probability can be described in terms of one SR trajectory [cf. insert in Fig. 11(b)] and for which the


FIG. 11. (Color online) Comparison of few-cycle pulse ATI spectra for He as predicted by the present TDER analytic result in Eq. (57) (solid black lines) and by the QRS result in Eq. (80) for two ionized electron ejection angles: (a) $\Theta=0^{\circ}$ and (b) $\Theta=180^{\circ}$. The laser pulse has the same intensity and wavelength as in Fig. 2, $N=5$ cycles, and a CEP $\phi=\pi / 2$. Inset in (b): ATI spectra on a log scale for the energy interval 8-16 a.u. In panel (c) we give the temporal dependence of the pulse electric field $F(t)$ (dot-dashed red line) and vector potential $A(t)$ (solid blue line). The horizontal dashed lines mark the amplitudes of $F(t)$ and $A(t)$. The vertical arrows mark the ionization and rescattering times for contributing extreme trajectories. Two SR trajectories that contribute to the ATI spectrum for $\Theta=180^{\circ}$ are shown by thick (black) solid lines.
rescattering time ensures a magnitude of the vector potential $A\left(t_{f}^{(j)}\right)$ that is close to its amplitude $A_{0}$ [cf. the trajectory that starts at $1 \frac{3}{4} T$ and finishes at $2 \frac{1}{2} T$ in Fig. 11(c)]. All other rescattering times [cf. Fig. 11(c)] correspond to magnitudes of the vector potential smaller than its amplitude $A_{0}$ so that the results of QRS theory become inaccurate [cf. Figs. 11(a) and 11(b)]. The difference between the results of QRS theory and our results is most pronounced in the middle part of the ATI plateau [cf. Fig. 11(b)], because for this interval of energies the extreme trajectories start near the (local) maximum magnitude of the electric field but rescatter when the magnitude of the vector potential is lower than its maximum $A_{0}$ [cf. the SR trajectory on the right in Fig. 11(c)].

## VI. SUMMARY AND CONCLUSIONS

In this work we have presented a quantum-mechanical derivation of closed-form analytical formulas for the differential probability $\mathcal{P}(\mathbf{p})$ of ATI produced by an intense, linearly polarized few-cycle laser pulse. Our derivations are based on considering ATI by an infinitely long train of short pulses and then taking the limit that the time between pulses becomes infinite [25]. In order to analytically evaluate this limit, we use our TDER model (for describing an electron in a short-range potential) to obtain analytically the ATI rate for a periodic (but nonmonochromatic) laser field in the quasiclassical limit. For the high-energy ATI plateau region, the resulting expression for the short-pulse ATI amplitude is given by a coherent sum of partial amplitudes $A_{j}$ [cf. Eq. (35)]. These amplitudes describe electrons ionized from a few neighboring optical cycles in the vicinity of the peak of the laser pulse intensity envelope. The index $j$ enumerates the extreme closed classical trajectories, moving along which (after being ionized by tunneling) the electron acquires the maximum energy after it rescatters from its atomic core potential. The amplitude $A_{j}$ is thus associated with the $j$ th extreme trajectory. These trajectories are determined by ionization $\left(t_{i}^{(j)}\right)$ and rescattering $\left(t_{f}^{(j)}\right)$ times, which are given by the solution of the classical equations (34). Each partial ATI amplitude $A_{j}$ has a factorized form (36) (similar to that for a monochromatic field [19]) in agreement with the known classical three-step scenario for ATI [3]. This factorization allows one to separate explicitly the atomic and laser parameters. The atomic factors are the tunneling factor (corresponding to tunneling in an effective static electric field) and the amplitude for elastic electron rescattering, while the propagation factor describes free-electron motion in the laser field between the ionization and rescattering events. This latter factor involves the Airy function and is essentially independent of the shape of the atomic potential.

Owing to the transparent physical meaning of each of the three factors that enter the factorization (36) for $A_{j}$, our analytic TDER result for the ATI probability $\mathcal{P}(\mathbf{p})$ can be straightforwardly generalized to describe short-pulse ATI by real atoms: one simply replaces the tunneling factor and the amplitude for elastic electron scattering by their atomic counterparts. Thus, for practical calculations of ATI spectra for real atoms, our theory requires only the calculation of the classical times $t_{i}^{(j)}$ and $t_{f}^{(j)}$ for a given laser pulse shape
[cf. Eq. (34)] as well as information on the elastic electron scattering amplitude for a given atomic ion.

We have tested the accuracy of our theory by comparison with TDSE calculations of short-pulse ATI spectra for He and Ar atoms. This comparison shows that the TDSE and the analytic results agree well for ATI electron energies $\gtrsim 5 u_{p}$ and ionization angles $\Theta<\pi / 4(\Theta>3 \pi / 4)$ for ionization of electrons into the right (left) hemisphere about the laser polarization axis. Also, we compared our results with results of the QRS theory [14] and obtained an explicit form (82) for the QRS theory wave packet factor, which is sensitive to the spatial symmetry of the initial electron bound state. We found that results of the QRS theory and our results are in good agreement for many-cycle pulses. However, for a few-cycle pulse, results of the QRS theory and our TDSE results agree only if a single partial amplitude $A_{j}$ is dominant, while for an ATI spectrum having two plateaus, the QRS theory results overestimate the magnitude of the shorter but more intense plateau (cf. discussion in Sec. VE).

All major features of the high-energy part of short-pulse ATI spectra are well reproduced and can be explained within the present analytic theory: the multiplateau features in short-pulse ATI spectra, the asymmetry in the yield of electrons ionized into the right $\left[\left(\mathbf{p} \cdot \mathbf{e}_{z}\right)>0\right]$ and left $\left[\left(\mathbf{p} \cdot \mathbf{e}_{z}\right)<0\right]$ hemispheres, and the large-scale and fine-scale oscillatory structures in the ATI plateau. The simplicity of the analyses of these features provides a physically transparent parametrization (36) for the partial ATI amplitudes $A_{j}$ and for the general properties of the corresponding partial ionization rates $\Gamma_{j}(\mathbf{p})$ (cf. Sec. VA). Using these results, we have shown that the appearance of multiplateau features depends crucially on the relation between the magnitudes of the ionization factors $\left(\mathcal{I}_{j}\right)$ and between the cutoff energies $\left(E_{\text {cut }}^{(j)}\right)$ for neighboring partial rates $\Gamma_{j}(\mathbf{p})$ (cf. Sec. V C 1). The left-right asymmetry in ATI spectra originates from the fact that different extreme trajectories (with different times $\left\{t_{i}^{(j)}, t_{f}^{(j)}\right\}$, magnitudes of the electric field at the moment $t_{i}^{(j)}$, etc.) contribute to the ATI yield of electrons into the right and left hemispheres.

The ATI spectra exhibit two kinds of interference phenomena, intercycle and intracycle interferences, which cause large-scale and fine-scale oscillation patterns. The large-scale oscillations originate from interference between two (short and long) trajectories, to which the $j$ th extreme trajectory splits when the ionized electron energy is less than the maximum classical energy $\mathcal{E}\left(t_{i}^{(j)}, t_{f}^{(j)}\right)$ for the given extreme trajectory. These oscillations are described in our theory in terms of the Airy function [cf. Eq. (40)]. The fine-scale oscillations originate from interference between partial amplitudes $A_{j}$ corresponding to different extreme trajectories. We have shown that the large-scale oscillations in ATI spectra are particularly prominent in the case of a few cycle pulse when only one or two extreme trajectories contribute, while fine-scale oscillations are more pronounced for many-cycle pulses. We emphasize that just as for the case of HHG produced by a short laser pulse [25], the fine-scale oscillations coalesce with increasing number $N$ of optical cycles in a pulse to form the regularly spaced ATI peaks (separated in energy by $\hbar \omega$ ) that is characteristic of a monochromatic (long) laser pulse.

Finally, we have analyzed the contributions of SR and MR trajectories to short-pulse ATI spectra. We found that the SR-trajectory approximation is applicable near the cutoff of the high-energy ATI plateau (for electron energies $E \gtrsim 8.0 u_{p}$ ) and becomes exact for a few-cycle pulse. MR trajectories contribute to the middle part of the plateau (for electron energies less than $7.5 u_{p}$ ) for the case of several-cycle pulses. Moreover, most interesting, we found that the contribution of MR trajectories is sensitive to the spatial symmetry ( $s$ or $p$ ) of the initial electron bound state. Thus the shape of the middle part of the short-pulse ATI plateau is sensitive to the symmetry of the initial electron bound state.

Concluding, we note that our results in Sec. III for ATI rates in a periodic field are general and can be used to describe the ATI plateau produced by a two-color laser field as well as to describe ATI by a short laser pulse having an elliptical polarization.

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## APPENDIX: QUASICLASSICAL DERIVATION OF EQ. (35) FOR THE RESCATTERING AMPLITUDE $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$

In this Appendix we obtain a quasiclassical estimate for the rescattering amplitude $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$ in Eq. (33). In the quasiclassical approximation, the integrand of the integral in Eq. (33) changes rapidly with $t$. Thus the main contribution to the integral (for a given $v$ ) originates from the vicinity of the stationary phase point $t=t_{v}$ given by the equation

$$
\begin{equation*}
\frac{\mathbf{P}_{n}^{2}\left(t_{\nu}\right)}{2 m}=\frac{\boldsymbol{\alpha}^{2}\left[\mathrm{t}_{v} ; \mathrm{t}_{\nu}, \mathrm{t}_{v}^{\prime}\left(\mathrm{t}_{\nu}\right)\right]}{2 m}, \tag{A1}
\end{equation*}
$$

where $\boldsymbol{\alpha}\left(\xi ; t, t^{\prime}\right)$ and $t_{v}^{\prime}(t)$ are given by Eqs. (8) and (20). Taking into account Eq. (A1), we simplify the integrand in Eq. (33) by making the replacements:

$$
\begin{align*}
\mathcal{R}_{l}\left[\frac{\boldsymbol{\alpha}^{2}\left(t ; t, t_{v}^{\prime}\right)}{2 m}\right] & \rightarrow \mathcal{R}_{l}\left[\frac{\mathbf{P}_{n}^{2}(t)}{2 m}\right]  \tag{A2}\\
\alpha\left(t ; t, t^{\prime}\right) & \rightarrow \sqrt{\mathbf{P}_{n}^{2}(t)} s_{\alpha} \tag{A3}
\end{align*}
$$

where $s_{\alpha}=\operatorname{sgn}\left[\alpha\left(t ; t, t^{\prime}\right)\right]$. As a result, we obtain

$$
\begin{align*}
\mathcal{A}_{n}^{R}\left(\mathbf{p}_{n}\right)= & \sqrt{i} \frac{(2 l+1)^{3 / 2} C_{\kappa l} m}{2 \mathcal{T} \sqrt{\pi \hbar}}\left(-\frac{\sigma}{\hbar^{2}}\right)^{l} \\
& \times \sum_{\sigma= \pm 1} \sum_{\nu} \int_{-\mathcal{T} / 2}^{\mathcal{T} / 2} \frac{e^{i \Phi\left(t, t_{l}^{\prime \sigma}\right) / \hbar}\left\{\left[\mathbf{e}_{z} \cdot \mathbf{P}_{n}(t)\right]\left|\mathbf{P}_{n}(t)\right| s_{\alpha}\right\}^{l}}{\sqrt{|e| \sigma F_{\tau}\left(t_{v}^{\prime \sigma}\right)}\left(t-t_{v}^{\prime \sigma}\right)^{3 / 2}} \\
& \times \mathcal{R}_{l}^{-1}\left[\mathbf{P}_{n}^{2}(t) /(2 m)\right] d t . \tag{A4}
\end{align*}
$$

In Eq. (A4) we separated the stationary phase points $t_{v}^{\prime}$ into two branches with respect to the sign of $\sigma$ as follows [cf. Eq. (20)]:

$$
\begin{equation*}
\alpha\left(t_{v}^{\prime \sigma} ; t, t_{v}^{\prime \sigma}\right)=-i \sigma \hbar \kappa, \quad \sigma= \pm 1 \tag{A5}
\end{equation*}
$$

and introduced the notation

$$
\begin{equation*}
\Phi\left(t, t_{v}^{\prime \sigma}\right)=S\left(\mathbf{p}_{n}, t\right)+\mathcal{S}\left(t, t_{v}^{\prime \sigma}\right) \tag{A6}
\end{equation*}
$$

The integral in Eq. (A4) can be evaluated using the method proposed by Nikishov and Ritus [53] (cf. also Ref. [19]). The general idea of this method is to expand $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ in a cubic polynomial near a point $t=t_{v}^{\sigma}$, which gives zero for the second derivative of $\Phi\left(t, t_{v}^{\prime \sigma}\right)$. In order to find this point, we calculate the first derivative of $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ :

$$
\begin{align*}
\frac{d \Phi\left(t, t_{v}^{\prime \sigma}\right)}{d t} & =\frac{\mathbf{P}_{n}^{2}(t)}{2 m}-\frac{\boldsymbol{\alpha}^{2}\left(t ; t, t_{v}^{\prime \sigma}\right)}{2 m} \\
& =\frac{1}{2 m}\left[\mathbf{p}_{n}+\frac{|e|}{c} \mathbf{A}_{\tau}\left(t_{v}^{\prime \sigma}\right)+i \mathbf{e}_{z} \sigma \hbar \kappa\right] \cdot\left[\mathbf{p}_{n}+\mathbf{Q}(t)\right] \tag{A7}
\end{align*}
$$

where

$$
\begin{align*}
\mathbf{Q}(t) & =\mathbf{e}_{z} Q(t), \\
Q(t) & =\frac{|e|}{c}\left(2 A_{\tau}(t)-\frac{1}{t-t_{v}^{\prime \sigma}} \int_{t_{v}^{\prime \sigma}}^{t} A_{\tau}(\tau) d \tau\right) \\
& =\frac{|e|}{c}\left[2 A_{\tau}(t)-A_{\tau}\left(t_{v}^{\prime \sigma}\right)\right]-i \sigma \hbar \kappa . \tag{A8}
\end{align*}
$$

The first factor in expression (A7) can be considered as a constant and thus the second derivative of $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ is given by

$$
\begin{equation*}
\frac{d^{2} \Phi\left(t, t_{v}^{\prime \sigma}\right)}{d t^{2}} \approx \frac{1}{2 m}\left[\mathbf{p}_{n} \cdot \mathbf{e}_{z}+\frac{|e|}{c} A_{\tau}\left(t_{v}^{\prime \sigma}\right)+i \sigma \hbar \kappa\right] \frac{d Q(t)}{d t} \tag{A9}
\end{equation*}
$$

Calculating the first derivative of $Q(t)$ and equating it to zero, we obtain the equation for the time $t_{v}^{\sigma}$ :

$$
\begin{gather*}
2|e| F_{\tau}\left(t_{v}^{\sigma}\right)+\frac{|e|}{c} \frac{A_{\tau}\left(t_{v}^{\sigma}\right)-A_{\tau}\left(t_{v}^{\prime \sigma}\right)}{t_{v}^{\sigma}-t_{v}^{\prime \sigma}} \\
\quad=\frac{i \sigma \hbar \kappa}{\left(t_{v}^{\sigma}-t_{v}^{\prime \sigma}\right)}\left[1-2 \frac{F\left(t_{v}^{\sigma}\right)}{F\left(t_{v}^{\prime \sigma}\right)}\right] . \tag{A10}
\end{gather*}
$$

Combining now Eq. (A10) with Eq. (A5) evaluated at $t=t_{v}^{\sigma}$, we obtain a system of two coupled equations for the times $t_{v}^{\prime \sigma}$ and $t_{\nu}^{\sigma}$, which we solve by considering the parameter $i \sigma \hbar \kappa$ as a perturbation. To zero order in $i \sigma \hbar \kappa$, this system can be written in the form

$$
\begin{align*}
A_{\tau}\left(t_{i}^{(\nu)}\right)-\frac{1}{t_{f}^{(\nu)}-t_{i}^{(\nu)}} \int_{t_{i}^{(\nu)}}^{t_{f}^{(\nu)}} A_{\tau}(\tau) d \tau & =0  \tag{A11a}\\
2 F_{\tau}\left(t_{f}^{(\nu)}\right)+\frac{1}{c} \frac{A_{\tau}\left(t_{f}^{(\nu)}\right)-A_{\tau}\left(t_{i}^{(\nu)}\right)}{t_{f}^{(\nu)}-t_{i}^{(\nu)}} & =0 \tag{A11b}
\end{align*}
$$

where $t_{i}^{(\nu)}$ and $t_{f}^{(\nu)}$ are zero-order (classical) solutions for $t_{v}^{\prime \sigma}$ and $t_{v}^{\sigma}$, respectively. Substituting into Eqs. (A5) and (A10) the times $t_{v}^{\prime \sigma}$ and $t$ in the form $t_{v}^{\prime \sigma}=t_{i}^{(\nu)}+\Delta_{v}^{\prime \sigma}, t_{v}^{\sigma}=t_{f}^{(\nu)}+\Delta_{v}^{\sigma}$ and expanding Eqs. (A5) and (A10) in series in both $\Delta_{v}^{\prime \sigma}$ and $\Delta_{v}^{\sigma}$
up to linear terms, we obtain a system of two linear equations for $\Delta_{v}^{\prime \sigma}$ and $\Delta_{v}^{\sigma}$. The solution of this system is

$$
\begin{gather*}
\Delta_{v}^{\prime \sigma}=\frac{i \hbar \kappa}{|e| \tilde{F}_{v}}, \quad \tilde{F}_{\nu}=\sigma F_{\tau}\left(t_{i}^{(\nu)}\right)  \tag{A12}\\
\Delta_{v}^{\sigma}=0 \tag{A13}
\end{gather*}
$$

With the correction (A12) taken into account, the first derivative of the function $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ at the points $t_{v}^{\prime \sigma}$ and $t_{v}^{\sigma}$ can be obtained with an accuracy up to $\sim(\hbar \kappa)^{2}$ :

$$
\begin{align*}
& \Delta E_{\max }^{(\nu)} \equiv\left.\frac{d \Phi\left(t, t_{v}^{\prime \sigma}\right)}{d t}\right|_{t=t_{f}^{(\nu)}}=\frac{\mathbf{P}_{n}^{2}\left(t_{f}^{(\nu)}\right)}{2 m}-\mathcal{E}\left(t_{i}^{(\nu)}, t_{f}^{(\nu)}\right) \\
&+2 \frac{F_{\tau}\left(t_{f}^{(\nu)}\right)}{F_{\tau}\left(t_{i}^{(\nu)}\right)}\left|E_{0}\right|  \tag{A14}\\
& \mathcal{E}\left(t_{i}^{(\nu)}, t_{f}^{(\nu)}\right)=\frac{e^{2}\left[A_{\tau}\left(t_{f}^{(\nu)}\right)-A_{\tau}\left(t_{i}^{(\nu)}\right)\right]^{2}}{2 m c^{2}} \tag{A15}
\end{align*}
$$

To evaluate the integral in Eq. (A4) we also require the third derivative of $\Phi\left(t, t_{v}^{\prime \sigma}\right)$, which we calculate to zero order in the parameter $i \sigma \hbar \kappa$ :

$$
\begin{equation*}
\left.\frac{d^{3} \Phi(t)}{d t^{3}}\right|_{t=t_{f}^{(\nu)}}=2 \zeta_{\nu} \hbar \omega_{\mathrm{at}}^{3} \tag{A16}
\end{equation*}
$$

where

$$
\begin{align*}
\zeta_{v}= & -\frac{I\left(t_{f}^{(\nu)}\right)}{2 I_{\mathrm{at}}}\left[\frac{\dot{F}_{\tau}\left(t_{f}^{(\nu)}\right)\left[\mathbf{e}_{z} \cdot \mathbf{P}_{n}\left(t_{i}^{(\nu)}\right)\right]}{|e| F_{\tau}^{2}\left(t_{f}^{(\nu)}\right)}\right. \\
& \left.+2\left\{3-4 \frac{F_{\tau}\left(t_{f}^{(\nu)}\right)}{F_{\tau}\left(t_{i}^{(\nu)}\right)}\right\}\right] \tag{A17}
\end{align*}
$$

$\omega_{\mathrm{at}}=E_{\mathrm{at}} / \hbar$, and $I\left(t_{f}^{(\nu)}\right)=c F_{\tau}^{2}\left(t_{f}^{(\nu)}\right) /(8 \pi)$ is the intensity of the laser field at the moment $t_{f}^{(\nu)}$. In our approach we consider only those extreme trajectories that correspond to local maxima of the energy gained by the rescattered electron in the laser field. Since the third derivative of the classical action is the second derivative of the electron energy in the laser field (with the opposite sign), the sign of $\zeta_{v}$ should be positive. The positivity of $\zeta_{\nu}$ allows one to separate the sets of roots $\left\{t_{i}^{(\nu)}, t_{f}^{(\nu)}\right\}$ (or closed classical trajectories) that correspond to ejection of high-energy electrons into the "right" $\left(\mathbf{e} \cdot \mathbf{p}_{n}\right)>0$ and "left" $\left(\mathbf{e} \cdot \mathbf{p}_{n}\right)<0$ hemispheres about the laser polarization axis. [Note that we consider only those electron energies for which $\left|\left(\mathbf{e}_{z} \cdot \mathbf{p}_{n}\right)\right|>(|e| / c) \max \left|A_{\tau}\left(t_{f}^{(\nu)}\right)\right|$ for any $\nu$; thus the sign of $\left[\mathbf{e} \cdot \mathbf{P}_{n}\left(t_{f}^{(\nu)}\right)\right]$ is the same as that of $\left(\mathbf{e} \cdot \mathbf{p}_{n}\right)$.] The first term in square brackets in Eq. (A17) dominates and determines
the sign of $\zeta_{v}$. Thus, the electrons ejected to the right (left) hemisphere correspond to those sets of times $\left\{t_{i}^{(\nu)}, t_{f}^{(\nu)}\right\}$ for which $\dot{F}_{\tau}\left(t_{f}^{(\nu)}\right)<0\left[\dot{F}_{\tau}\left(t_{f}^{(\nu)}\right)>0\right]$.

Since $t_{v}^{\prime \sigma}$ in general is complex [cf., e.g., Eq. (A12)], the function $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$ is also complex valued. In order to separate the real and imaginary parts of $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$, we express $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$ in the form

$$
\begin{align*}
\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right) \approx & S\left(\mathbf{p}_{n}, t_{f}^{(\nu)}\right)-\int_{t_{i}^{(\nu)}}^{t_{f}^{(\nu)}}\left[\mathcal{E}\left(t_{i}^{(\nu)}, t\right)-E_{0}\right] d t \\
& -\int_{t_{i}^{(v)}+i\left(\hbar \kappa /|| | \tilde{F})_{v}\right.}^{t_{i}^{(\nu)}}\left[\mathcal{E}\left(t_{i}^{(\nu)}, t\right)-E_{0}\right] d t \tag{A18}
\end{align*}
$$

The first two terms in Eq. (A18) are real, while the third term contributes to the imaginary part of $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$. Expanding the integrand in the third term of the series in $t-t_{i}^{(\nu)}$ up to quadratic terms and calculating the integral, we obtain

$$
\begin{equation*}
\int_{t_{i}^{(\nu)}+i(\hbar \kappa / e \tilde{F})_{v}}^{t_{i}^{(v)}}\left[\mathcal{E}\left(t_{i}^{(\nu)}, t\right)-E_{0}\right] d t \approx-\frac{i \hbar^{3} \kappa^{3}}{3 m|e| \tilde{F}_{v}} \tag{A19}
\end{equation*}
$$

The result for $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$ (with accuracy $\sim i \hbar \kappa$ ) can thus be presented in the form

$$
\begin{equation*}
\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)=\varphi_{v}+\frac{i \hbar^{3} \kappa^{3}}{3 m|e| \tilde{F}_{v}}, \tag{A20}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi_{\nu}=S\left(\mathbf{p}_{n}, t_{f}^{(\nu)}\right)-\int_{t_{i}^{(\nu)}}^{t_{f}^{(\nu)}}\left[\mathcal{E}\left(t_{i}^{(\nu)}, t\right)-E_{0}\right] d t \tag{A21}
\end{equation*}
$$

The three-term expansion of $\Phi\left(t, t_{v}^{\prime \sigma}\right)$ is thus

$$
\begin{align*}
\Phi\left(t, t_{v}^{\prime \sigma}\right) \approx & \Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)+\Delta E_{\max }^{(\nu)}\left(t-t_{f}^{(\nu)}\right) \\
& +\frac{\zeta_{\nu} \hbar \omega_{\mathrm{at}}^{3}}{3}\left(t-t_{f}^{(\nu)}\right)^{3} \tag{A22}
\end{align*}
$$

where the explicit forms for $\Delta E_{\text {max }}^{(\nu)}, \zeta_{\nu}$, and $\Phi\left(t_{v}^{\sigma}, t_{v}^{\prime \sigma}\right)$ are given by Eqs. (A14), (A16), and (A20). Substituting the expansion (A22) into the integral in Eq. (A4), replacing the pre-exponential function in this integral by its value at the points $t=t_{f}^{(\nu)}, t_{v}^{\prime \sigma}=t_{i}^{(\nu)}$, and expanding the limits of integration from $-\infty$ to $\infty$, we can perform the integration over $t$ analytically in terms of the Airy function. Separating then the roots $\left\{t_{i}^{(\nu)}, t_{f}^{(\nu)}\right\}$ for the left and right hemispheres and introducing the unified notations $\{\sigma, \nu\} \rightarrow j$, we obtain the result (35) for the amplitude $\mathcal{A}^{(R)}\left(\mathbf{p}_{n}\right)$.
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