Nonlocal and controlled unitary operators of Schmidt rank three

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Implementing nonlocal unitary operators is an important and hard question in quantum computing and cryptography. We show that any bipartite nonlocal unitary operator of Schmidt rank three on the $(d_A \times d_B)$-dimensional system is locally equivalent to a controlled unitary when d_A is at most three. This operator can be locally implemented assisted by a maximally entangled state of Schmidt rank $r = \min\{d_A^2, d_B\}$. We further show that stochastic-equivalent nonlocal unitary operators are indeed locally equivalent, and propose a sufficient condition on which nonlocal and controlled unitary operators are locally equivalent. We also provide the solution to a special case of a conjecture on the ranks of multipartite quantum states.

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I. INTRODUCTION

Implementing multipartite unitary operators is a fundamental task in quantum information theory. The operators are called *local* when they are the tensor product of unitary operators locally acting on subsystems, i.e., they have *Schmidt rank* one. Otherwise, they are called *nonlocal*. It is known that the local unitary can be implemented by local operations and classical communication (LOCC) with probability one. Recent research has been devoted to the decomposition of local unitaries into elementary operations [1], and the local equivalence between multipartite quantum states of fermionic systems under local unitaries [2–4].

Nonlocal unitary operators have a more complex structure and play a more powerful role than local unitaries in quantum computing, cryptography, and so on. Nonlocal unitaries can create quantum entanglement between distributed parties [5], and their equivalence has been studied under LOCC [6]. Nonlocal unitaries cannot be implemented by LOCC only, even if the probability is allowed to be close to zero [7]. The understanding of the forms and implementation schemes of nonlocal unitary operators is still far from complete. The simplest type of nonlocal unitary is the *controlled unitary* gates, which are of the general form $U = \sum_{j=1}^{m} P_j \otimes V_j$ acting on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where P_j are orthogonal projectors on $\mathcal{H}_A$ and V_j are unitaries on $\mathcal{H}_B$. They can be implemented by a simple nonlocal protocol [8] using a maximally entangled state of Schmidt rank m. In this sense the implementation of controlled unitaries is operational. Some other types of nonlocal unitaries are discussed in [8], but in this paper we will focus on controlled unitaries. (Note that entirely different implementations are possible if the systems are deemed to be near enough so as to allow direct quantum interactions between them, e.g., multiqubit controlled gates can be decomposed into certain elementary gates [9].) Recently, an interesting connection between nonlocal and controlled unitaries was found: they are locally equivalent when they have Schmidt rank two [10]. In this case their implementations are the same. So it is important to strengthen this connection for operationally implementing more nonlocal unitaries.

In this paper we show that any bipartite unitary operation U of Schmidt rank three on $d_A \times d_B$ system is locally equivalent to a controlled unitary when $d_A = 2,3$; see Theorems 3 and 6. This is illustrated in Fig. 1. They not only imply the method of implementing U but also simplify the structure of U. We also propose an operational method of explicitly decomposing U into the form of controlled unitaries in the end of Sec. III. As an application we can simplify the problem of deciding the stochastic local equivalence, namely SL equivalence of two bipartite unitaries of Schmidt rank three with $d_A = 2,3$. This is based on Theorem 7 that any two SL-equivalent nonlocal unitary operators are locally equivalent. Using this theorem we provide a sufficient condition by which a bipartite unitary is locally equivalent to a controlled unitary in Corollary 8. Next we show that U can be implemented by LOCC and a maximally entangled state $|\Psi_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} |ii\rangle$, where $r = \min\{d_A^2, d_B\}$ in Lemma 9. Next, we apply our result to solve a special case of a conjecture on the ranks of multipartite quantum states; see Conjecture 10.

Controlled unitary operators are one of the most easily accessible and extensively studied quantum operators. For example, the controlled NOT (CNOT) gate is essential to construct the universal quantum two-qubit gate used in quantum computing [9]. Experimental schemes of implementing the CNOT gates have also been proposed, such as cavity QED technique [11] and trapped ions [12]. Recently, CNOT gates have been proved to be decomposed in terms of a two-qubit entangled gate and single qubit phase gates, which could be implemented by trapped ions controlled by fully overlapping laser pulses [13]. Next, multiqubit graph states for one-way quantum computing are generated by a series of controlled-Z gates [14]. Third, controlled phase gates have been used to construct the mutually unbiased bases (MUBs) [15] and graph states for which the violation of multipartite Bell-type inequalities have been experimentally demonstrated [16]. These applications (and those not mentioned above) could be improved by the strengthened connection between nonlocal and controlled unitaries presented in this paper.

The rest of this paper is organized as follows. In Sec. II we introduce the preliminary knowledge, and propose Conjecture 1 as the main question in this paper. In Sec. III we prove Conjecture 1 when $d_A = 2,3$, and we propose its applications on general nonlocal unitaries in Sec. IV. Finally, we conclude in Sec. V.

II. PRELIMINARIES

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the complex Hilbert space of a finite-dimensional bipartite quantum system of Alice and Bob. We

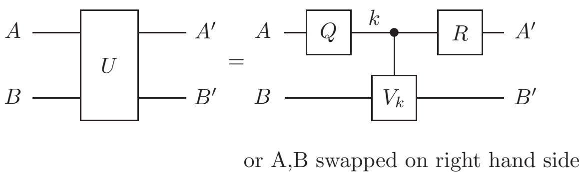

FIG. 1. Any bipartite unitary U on $d_A \times d_B$ system of Schmidt rank three is locally equivalent to a controlled unitary when $d_A = 2,3$, where the controlling side may be A or B. This is expressed as $U = (Q \otimes I)(\sum_{k=1}^{d_A} |k\rangle\langle k| \otimes V_k)(R \otimes I)$ or $U = (I \otimes Q)(\sum_{k=1}^{d_B} V_k \otimes |k\rangle\langle k|)(I \otimes R)$, where V_k, Q, and R are local unitaries. The output systems A' and B' are assumed to be of the same size as A and B, respectively.

denote by d_A,d_B the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. It is known that $\mathcal{H}$ is spanned by the computational basis $|i,j\rangle, i = 1,\dots,d_A$ and $j = 1,\dots,d_B$. We shall denote $I_k = \sum_{i=1}^{k} |i\rangle\langle i|$. For convenience, we denote $I_A = I_{d_A}$, $I_B = I_{d_B}$, and $I = I_{d_A d_B}$ as the identity operator on spaces $\mathcal{H}_A,\mathcal{H}_B$, and $\mathcal{H}$, respectively. Let $d = d_A d_B$, and $U,V \in \mathrm{U}(d)$ be two unitary matrices on the space $\mathcal{H}$. We say that U,V are equivalent under stochastic local operations, or *SL equivalent* when there are two locally invertible matrices $S_1,S_2 \in \mathrm{GL}(d_A) \times \mathrm{GL}(d_B)$ such that $U = S_1 V S_2$. In particular, we say that U,V are *locally equivalent* when S_1,S_2 are unitary.

A unitary matrix U on $\mathcal{H}$ has Schmidt rank n if there is a decomposition $U = \sum_{j=1}^{n} A_j \otimes B_j$ where the $d_A \times d_A$ matrices $A_1,\dots,A_n$ are linearly independent, and the $d_B \times d_B$ matrices $B_1,\dots,B_n$ are linearly independent. Such decomposition will be called Schmidt expansion in this paper, but note that the same term may sometimes imply the additional requirement that A_j (and B_j) are orthogonal under the Hilbert-Schmidt inner product. We say that U is a *controlled unitary gate*, if U is locally equivalent to $\sum_{j=1}^{d_A} |j\rangle\langle j| \otimes U_j$ or $\sum_{j=1}^{d_B} V_j \otimes |j\rangle\langle j|$. To be specific, U is a controlled unitary from the A or B side, respectively. Clearly the matrices U_j,V_j are unitary. We further say that system A (or B) controls with n terms if $U = \sum_{i=1}^{n} P_i \otimes U_i$ (or $\sum_{i=1}^{n} U_i \otimes P_i$), where the $U_1,\dots,U_n$ are unitaries and the P_i are orthogonal projectors, i.e., $P_i P_j = \delta_{ij} P_i$.

It is known that any multipartite (i.e., nonlocal) unitary gate of Schmidt rank two is a controlled unitary [10]. However, there are bipartite unitaries of Schmidt rank four, e.g., the two-qubit SWAP gate [10], that are not controlled unitaries. It is then an interesting question to characterize the bipartite unitaries of Schmidt rank three. Formally, we investigate the following conjecture in the next section.

Conjecture 1. Any bipartite unitary operator of Schmidt rank three is a controlled unitary operator.

To approach this conjecture, we generalize the concept of controlled unitary gate; see an example in the next paragraph. We split the space into a direct sum: $\mathcal{H}_A = \oplus_{i=1}^{m} \mathcal{H}_i$, $m > 1$, $\mathrm{Dim}\,\mathcal{H}_i = m_i$, and $\mathcal{H}_i \perp \mathcal{H}_j$ for distinct $i,j = 1,\dots,m$. We say that U is a *block-controlled unitary (BCU) gate controlled from the A side*, if U is locally equivalent to $\sum_{i=1}^{m} \sum_{j,k=1}^{m_i} |u_{ij}\rangle\langle u_{ik}| \otimes V_{ijk}$, where $|u_{ij}\rangle \in \mathcal{H}_i$ for $j,k = 1,\dots,m_i$, and $m > 1$. Note that the V_{ijk} are not necessarily unitary. For simplicity we denote the decomposition as $\oplus_A V_i$ where $V_i = \sum_{j,k=1}^{m_i} |u_{ij}\rangle\langle u_{ik}| \otimes U_{ijk}$, and we denote $|V_i|_A = m_i$. We have $UU^\dagger = \sum_{i=1}^{m} P_i \otimes I_B = I$, where P_i is the projector on the space $\mathcal{H}_i$. So the BCU from the A side can be understood as the direct sum of nonlocal unitaries on the spaces $\mathcal{H}_i \otimes \mathcal{H}_B$, $i = 1,\dots,m$. In particular, if $m_i = 1$ for all i, then U degenerates to a controlled unitary gate from the A side. So a BCU has more general properties than those a controlled unitary gate has. One may similarly define the BCU gate controlled from the B side.

Although the controlled unitary gate is a BCU gate, the converse is wrong. An example is the following qutrit-qubit unitary gate: $U = \frac{1}{2}(I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) + |3\rangle\langle 3| \otimes I_2$, where $\sigma_x,\sigma_y,\sigma_z$ are the standard Pauli operators acting on the two-dimensional subspace of $\mathcal{H}_A$ spanned by $|1\rangle,|2\rangle$. They can be regarded as qutrit operators by letting $\sigma_i|3\rangle = 0$ for $i = x,y,z$. By definition U is a BCU gate from A side with $m = 2$, $\mathcal{H}_1 = \mathrm{span}\{|1\rangle,|2\rangle\}$, and $\mathcal{H}_2 = \mathrm{span}\{|3\rangle\}$. If U is a controlled unitary from A or B side, then it has Schmidt rank at most three or two. It is a contradiction with the fact that U has Schmidt rank four. So U is not a controlled unitary.

Since a nonlocal unitary and controlled unitary may be not locally equivalent, one may ask when they are locally equivalent. The question has been addressed in [10, Lemma 2], which says the A_j in the Schmidt expansion of U have simultaneous singular value decomposition, i.e., they are simultaneously diagonal under the same two local unitaries before and after them. However, the lemma is not very operational in practice. Below we present an operational criterion based on [17, Corollary 5] and [10, Lemma 2]. Note that [18] also cited some related work of the authors of [17], and made implicit use of the same lemma below, in studying the entanglement cost of more general types of bipartite unitaries.

Lemma 2. Let $U = \sum_{j=1}^{r} A_j \otimes B_j$ be a nonlocal unitary of Schmidt rank r. Then U is a controlled unitary from the A side if and only if $A_i A_j^\dagger$, $i,j = 1,\dots,r$ are all normal and commute with each other, and $A_i^\dagger A_j$, $i,j = 1,\dots,r$ are all normal and commute with each other.

Proof. The assertion immediately follows from [17, Corollary 5] and [10, Lemma 2]. ■

Note that the matrices A_j (B_j) are not necessarily orthogonal to each other.

III. PROVING CONJECTURE 1 FOR $d_A = 2,3$

Conjecture 1 trivially holds for $d_A = 1$. In this section we show that Conjecture 1 holds when one of the systems A,B is a qubit or a qutrit. The first case is demonstrated by the following theorem.

Theorem 3. Any bipartite unitary on $2 \times d_B$ of Schmidt rank three is locally equivalent to a controlled unitary controlled from the B side.

Proof. Let U be a bipartite unitary on $2 \times d_B$ of Schmidt rank three. Suppose U has an operator Schmidt expansion $U = \sum_{j=1}^{3} E_j \otimes F_j$. Using the orthogonality under the Hilbert-Schmidt inner product, there is a 2×2 matrix E_4 orthogonal to E_1,E_2,E_3. Let the non-negative real numbers a, b be the singular values of E_4. Up to local unitaries we may assume $E_4 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$. Since U has Schmidt rank 3, U is locally equivalent to the unitary $U_1 = \sum_{j=1}^3 A_j \otimes B_j$, where $A_1 = |0\rangle\langle 1|$, $A_2 = |1\rangle\langle 0|$, and $A_3 = b|0\rangle\langle 0| - a|1\rangle\langle 1|$.

Let $\mathcal{H}_B = \oplus_{i=1}^k V_i$ be an orthogonal decomposition and the diagonal matrix P_i the projector on the subspace V_i, $\forall i$. So $P_i P_j = \delta_{ij} P_i$ and $\sum_{i=1}^k P_i = I_B$. Up to local unitaries on $\mathcal{H}_B$ we may assume the orthogonal decomposition $B_3 = \sum_{i=1}^k c_i P_i$, where the diagonal entries $c_i > c_j \geqslant 0$ for all $i < j$. Since U_1 is unitary, we have

$$b^2 B_3^\dagger B_3 + B_2^\dagger B_2 = B_1^\dagger B_1 + a^2 B_3^\dagger B_3 = I_B, \tag{1}$$

$$b^2 B_3 B_3^\dagger + B_1 B_1^\dagger = B_2 B_2^\dagger + a^2 B_3 B_3^\dagger = I_B. \tag{2}$$

Taking the trace in Eqs. (1) and (2), we have $a = b > 0$. Since U_1 is unitary, we have

$$B_3^\dagger B_1 = B_2^\dagger B_3, \quad B_1 B_3^\dagger = B_3 B_2^\dagger. \tag{3}$$

Since $B_3 = B_3^\dagger = \sum_{i=1}^k c_i P_i$, it follows from (3) that $B_3^2 B_1 = B_1 B_3^2 = B_3 B_2^\dagger B_3$. So B_1 commutes with B_3^2. One can similarly show that B_2 commutes with B_3^2. Since $c_i > c_j \geqslant 0$ for all $i < j$, we have $B_1 = \oplus_{i=1}^k G_i$ and $B_2 = \oplus_{i=1}^k H_i$, where the square blocks G_i, H_i act on the space V_i, $\forall i$. By (3) we have $G_1 = H_1^\dagger, \ldots, G_{k-1} = H_{k-1}^\dagger$ and $c_k G_k = c_k H_k^\dagger$. It follows from (1) and (2) that

$$B_1^\dagger B_1 = B_1 B_1^\dagger = B_2^\dagger B_2 = B_2 B_2^\dagger = I_B - a^2 B_3 B_3^\dagger. \tag{4}$$

So the matrices $G_1, \ldots, G_{k-1}$ and $H_1, \ldots, H_{k-1}$ are normal. If $c_k > 0$, then $B_1 = B_2^\dagger$ and G_k is also normal by (4). So B_1 is normal, and B_1, B_2, B_3 are simultaneously diagonalizable under unitary similarity transform. So U_1 is locally equivalent to a controlled unitary controlled from the B side. Since U, U_1 are locally equivalent, the assertion follows. On the other hand, if $c_k = 0$, by (4) we have $G_k G_k^\dagger = H_k H_k^\dagger = P_k$, i.e., both G_k, H_k are unitary. So B_1, B_2, B_3 are simultaneously locally equivalent to diagonal matrices, and the assertion follows. This completes the proof. ∎

No controlled unitary on $2 \times d_B$ of Schmidt rank three can be controlled from the A side; otherwise, the Schmidt rank would be decreased. Below we construct a controlled unitary on 3×3 of Schmidt rank three which is not controlled from the A side. Let $U = \sum_{i=1}^3 V_i \otimes |i\rangle\langle i|$, where $V_i = U_i \oplus |3\rangle\langle 3|$, $i = 1,2,3$ and the U_i are linearly independent unitaries acting on the space span$\{|1\rangle, |2\rangle\}$. One can verify that U is a controlled unitary of Schmidt rank three controlled from the B side. If it is also controlled from the A side, then the three-dimensional subspace H spanned by the V_i is also spanned by three matrices of rank one. This is a contradiction with the fact that there is no matrix of rank one in H. So U is not controlled from the A side. Next let $U' = \sum_{i=1}^3 V_i \otimes P_i$ be a controlled unitary on $3 \times d_B$ and B control with three terms. Using a similar argument above, we can show that U' is not controlled from the A side.

It is known that [10, Theorem 6] shows two facts. Any bipartite unitary U of Schmidt rank two (i) is controlled from both A and B sides, and (ii) has at least one of the two systems A, B controlling with two terms. Can these two statements be generalized to unitaries of Schmidt rank three? The bipartite unitary in Theorem 3 and U, U' in the last paragraph have Schmidt rank three and violate statement (i). Next we show that statement (ii) cannot be generalized to that one side controls with three terms. Consider the controlled unitary

$$V = I_2 \otimes |1\rangle\langle 1| + \sigma_x \otimes |2\rangle\langle 2| + \sigma_z \otimes |3\rangle\langle 3| + \frac{\sigma_x + \sigma_z}{\sqrt{2}} \otimes |4\rangle\langle 4| \tag{5}$$

of Schmidt rank three on a 2×4 system. Evidently, the A side cannot control with three terms. If the B side controls with three terms, then V is locally equivalent to $V' = \sum_{i=1}^3 U_i \otimes P_i$, where P_i are pairwise orthogonal projectors. In any expansion of the Schmidt-rank-three unitary V of the form $V = \sum_{j=1}^3 A_j \otimes B_j$, the subspace span$\{B_j\}$ is well defined in the sense that it is determined solely by V and is independent of the form of the expansion (as long as the expansion has only three terms), so for this particular V this subspace is the three-dimensional subspace S_1 spanned by the matrices $|1\rangle\langle 1|, |2\rangle\langle 2| + \frac{1}{\sqrt{2}}|4\rangle\langle 4|, |3\rangle\langle 3| + \frac{1}{\sqrt{2}}|4\rangle\langle 4|$, because we can choose A_j to be I_2, σ_x, and σ_z. The corresponding subspace for V' is span$\{P_i\}$, which contains two linearly independent matrices of rank one. As V and V' are locally equivalent, the subspace S_1 also contains two linearly independent matrices of rank one. This is impossible and hence the B side cannot control with three terms. Therefore, the statement (ii) cannot be directly generalized to the case of Schmidt rank three.

It is known [19, proposition 3] that a two-qubit unitary cannot have Schmidt rank exactly three. Theorem 3 implies an alternative proof for this fact. If a two-qubit unitary has Schmidt rank three, then from Theorem 3 it must be controlled from the B side which is two-dimensional; hence the unitary has Schmidt rank at most two, a contradiction with the assumption that it has Schmidt rank three.

In the paragraph below proposition 3 in [19], it was conjectured that a unitary operator on $d_A \times d_B$ system of Schmidt rank k exists if and only if k divides $d_A d_B$. In the same paragraph there was an alternative conjecture. They are both false as the V in Eq. (5) is a counterexample.

To investigate Conjecture 1 with a qutrit system, we present two preliminary lemmas.

Lemma 4. Assertion (i) implies assertion (ii): (i) any bipartite unitary on $d_A \times d_B$ system of Schmidt rank three is locally equivalent to a controlled unitary; (ii) any bipartite BCU (see the definition in the paragraph following Conjecture 1) from the A side on $(d_A + 1) \times d_B$ system of Schmidt rank three is locally equivalent to a controlled unitary.

Proof. Let U be a bipartite BCU from the A side on $(d_A + 1) \times d_B$ of Schmidt rank three. We may assume $U = U_1 \oplus_A U_2$. Since U has Schmidt rank three, U_1, U_2 have Schmidt rank at most three. Since both $|U_1|_A, |U_2|_A \leqslant d_A$, it follows from (i) and [10] that both of them are equivalent to controlled unitaries. If both of them are controlled from the A side, then (ii) holds. If one of them is controlled from only the B side, then it has Schmidt rank three [10]. Suppose it is U_1; then it has Schmidt expansion $U_1 = \sum_{j=1}^3 A_j \otimes B_j$, where A_j and B_j are operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. As U_1 is controlled from the B side, from [10, Lemma 2], B_j can be simultaneously diagonalized under local unitaries. Since U also has Schmidt rank three, it can be expanded in three terms with the same

B_j on the B side. Again using [10, Lemma 2], U is also a controlled unitary from the B side. Thus $(i) \to (ii)$ holds. ■

An open problem is whether the converse is true, i.e., $(ii) \to (i)$.

Lemma 5. Any bipartite operator on $\mathcal{H}$ of Schmidt rank at most d_A is locally equivalent to another operator $\sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes U_{ij}$ such that $U_{ij} = 0$ for some pair of subscripts (i,j).

Proof. Let $V = \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes V_{ij}$ be an arbitrary bipartite operator of Schmidt rank at most d_A. Using the row and column operations, we only need to show that we can always realize $V_{11} = 0$. First this is evidently true when the blocks $V_{i,1}, \ldots, V_{i,d_A}$ are linearly dependent for some i. Next suppose they are linearly independent for $i = 1, \ldots, d_A$. Since the Schmidt rank of V is at most d_A, it becomes exactly d_A now. So $V_{1,1}, \ldots, V_{1,d_A}$ are in the d_A-dimensional subspace spanned by $V_{2,1}, \ldots, V_{2,d_A}$. There is a unit vector (x,y) such that the following d_A matrix pencils $xV_{1,1} + yV_{2,1}, \ldots, xV_{1,d_A} + yV_{2,d_A}$ are linearly dependent. Let W be a unitary matrix of the first row $(x,y,0,\ldots,0)$. Then the d_A top blocks of size $d_B \times d_B$ in $(W \otimes I)V$ are exactly the above matrix pencils, so they are linearly dependent. Now the claim follows from the first case. This completes the proof. ■

The assertion of this lemma can be easily generalized to the case in which the bipartite operator V is replaced by an isometry mapping the space $\mathbf{C}^n \otimes \mathbf{C}^q$ to $\mathbf{C}^m \otimes \mathbf{C}^p$, i.e., $V = \sum_{i=1}^{m} \sum_{j=1}^{n} |i\rangle\langle j| \otimes V_{ij}$, where V_{ij} is of size $p \times q$. Now we are in a position to prove Conjecture 1 with $d_A = 3$. We shall denote $A \propto B$ for two linearly dependent matrices A, B.

Theorem 6. Any bipartite unitary on $3 \times d_B$ of Schmidt rank three is locally equivalent to a controlled unitary.

Proof. Let U be a bipartite unitary on $3 \times d_B$ of Schmidt rank three. For $d_B = 2$ the assertion follows from Theorem 3. The overall proof strategy is by induction over d_B. The inductive assumption is that the assertion holds for $d_B = 2, \ldots, k-1$. Then, for $d_B = k$, we prove by considering the cases that U is a BCU and not a BCU, respectively.

We claim that, under the inductive assumption above, the assertion for $d_B = k$ holds for BCUs. If U is a BCU controlled from the A side, then the claim follows from Lemma 4 and Theorem 3. Let U be a BCU controlled from the B side. We have $U = U_1 \oplus_B U_2$, where the unitaries U_i act on $\mathcal{H}_A \otimes \mathcal{H}_i$, $i = 1,2$, and $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_B$. Since U has Schmidt rank three, U_1, U_2 have Schmidt rank at most three. It follows from $U = U_1 \oplus_B U_2$ that both dimensions $|U_1|_B, |U_2|_B < k$. So the induction hypothesis and [10] (for Schmidt rank three and two, respectively) imply that U_1, U_2 are controlled unitaries. If both U_1, U_2 are controlled from the B side, then the claim follows. Suppose one of them, say U_1, is controlled from the A side only. So U_1 has Schmidt rank three [10]. Since U also has Schmidt rank three, it is a controlled unitary from the A side (by the same argument as in the proof of Lemma 4), so the claim follows. From now on we assume that U is not a BCU.

By Lemma 5 we may assume the bipartite unitary $U = \sum_{i,j=1}^{3} |i\rangle\langle j| \otimes U_{ij}$, with U_{ij} of size $d_B \times d_B$ and $U_{13} = 0$. Since U is unitary, the rows of submatrix (U_{11}, U_{12}) are orthogonal to the rows of another submatrix $\sum_{i=2}^{3} \sum_{j=1}^{2} |i\rangle\langle j| \otimes U_{ij}$. Since the former has rank d_B, the latter has rank at most d_B. So the space spanned by the rows of two matrices (U_{21}, U_{22}) and (U_{31}, U_{32}) has dimension at most d_B. It implies that there is a unit vector (x,y) such that the matrix pencil $x(U_{21}, U_{22}) + y(U_{31}, U_{32})$ has rank at most $d_B - 1$. Let V_1 be a 3×3 unitary with the bottom row $(0,x,y)$, and $U' = (V_1 \otimes I_B)U$. The bottom left $2d_B \times 2d_B$ submatrix of U' consisting of four $d_B \times d_B$ blocks is exactly the above-mentioned matrix pencil. So U' is locally equivalent to $W = \sum_{i,j=1}^{3} |i\rangle\langle j| \otimes W_{ij}$, where the bottom row of W is $(0, \ldots, 0, 1)$. To obtain this form we have used $W = (I_A \otimes S_B)U'(I_A \otimes T_B)$, where the local unitaries S_B and T_B are for obtaining the first $2d_B$ and last d_B elements of the bottom row, respectively. As W is unitary, W_{33} is block diagonal with a $(d_B - 1) \times (d_B - 1)$ block and a 1×1 block. Since W is not a BCU, the three blocks W_{31}, W_{32}, W_{33} are linearly dependent (otherwise, all the nine blocks W_{ij} would be spanned by these three blocks and hence are block diagonal, so U is a BCU controlled from the B side). This implies $W_{31} \propto W_{32}$. So W is locally equivalent to $W' = \sum_{i,j=1}^{3} |i\rangle\langle j| \otimes W'_{ij}$ with $W'_{13} = W'_{31} = 0$. Since W' is unitary, the ranks of W'_{23} and W'_{32} are equal. The first big row $(W'_{11}, W'_{12}, W'_{13})$ being orthogonal to the last big row implies the sum of ranks of W'_{12} and W'_{32} is not greater than d_B. Similarly, the sum of ranks of W'_{21} and W'_{23} is not greater than d_B. Thus under local unitaries on $\mathcal{H}_B$, W' is equivalent to $V = \begin{pmatrix} v_1 & v_2 & 0 \\ v_3 & v_4 & v_5 \\ 0 & v_6 & v_7 \end{pmatrix}$, where the $d_B \times d_B$ blocks v_i have the expression $v_i = \begin{pmatrix} v_{i1} & v_{i2} \\ v_{i3} & v_{i4} \end{pmatrix}$, $i = 1, \ldots, 7$, and $v_{22} = v_{24} = v_{33} = v_{34} = v_{42} = v_{43} = v_{51} = v_{52} = v_{61} = v_{63} = 0$, where the equations $v_{42} = v_{43} = 0$ are natural consequences of the other equations, and the choices of the local unitaries on $\mathcal{H}_B$ before and after W' are such that the other equations hold. The blocks v_{i1} are of size $(d_B - r) \times (d_B - r)$, v_{i2} of size $(d_B - r) \times r$, v_{i3} of size $r \times (d_B - r)$, and v_{i4} of size $r \times r$. Since W is not a BCU, none of $v_1, v_2, v_3, v_5, v_6, v_7$ is zero. So $r \in [1, d_B - 1]$, and v_2, v_6 are linearly independent. Let H be the space spanned by v_2, v_3, v_5, v_6. Since the Schmidt rank of V is three, we have $\mathrm{Dim}\, H = 2$ or 3.

Suppose $\mathrm{Dim}\, H = 2$, so $v_3, v_5 \in H = \mathrm{span}\{v_2, v_6\}$. We have two cases (1) $v_{23} = 0$, $v_{64} \neq 0$, and (2) $v_{23} \neq 0$, $v_{64} = 0$. In case (1), we have $v_{32} = v_{53} = v_{62} = 0$, $v_{21} \propto v_{31}$, and $v_{54} \propto v_{64}$. If $v_4 \in H$, then v_{21} and v_{64} are both invertible. Since V is unitary, we have $v_{12} = v_{13} = v_{72} = v_{73} = 0$. Then V becomes a BCU which gives us a contradiction with the assumption. So $v_4 \notin H$. The space spanned by the v_i is spanned by v_3, v_4, v_5. Again V becomes a BCU and we have a contradiction. In case (2), we have $v_{31} = v_{21} = v_{54} = 0$, $v_{32} \propto v_{62}$, and $v_{23} \propto v_{53}$. If $v_4 \in H$, then $v_4 = 0$. Since V is unitary, we have $r = d_B - r$ and v_{23}, v_{32} are unitary, and the only nonzero blocks in v_1 and v_7 are v_{11} and v_{74}, respectively. So V has Schmidt rank four and we have a contradiction. So $v_4 \notin H$. Since V has Schmidt rank three, we have $v_{i3} \propto v_{23}$ and $v_{i2} \propto v_{32}$ $(i = 1, \ldots, 7$, same below). Since V is unitary, we have $v_{12} = v_{13} = v_{72} = v_{73} = 0$. Since V has Schmidt rank three, the three blocks v_1, v_4, v_7 are pairwise linearly dependent. So V is locally equivalent to a matrix S the same as V, except that v_{11}, v_{14} are replaced by scalar matrices, and hence v_{71}, v_{74} are also scalar. It follows from $VV^\dagger = I$ that $r = d_B - r$. As $v_{23} \neq 0, v_{12} = v_{21} = 0$, and the rows of V are normalized, we have that the singular values of v_{11} are larger than those of v_{14}. As the blocks v_1 and v_7 are proportional, the singular values of v_{71} are larger than those of v_{74}, but this contradicts with that $v_{62} \neq 0, v_{64} = v_{73} = 0$ and that the rows of V are normalized. Thus this case does not exist. So the case $\text{Dim}\, H = 2$ is excluded.

Let $\text{Dim}\, H = 3$. Up to local unitaries we may assume that H is spanned by v_2, v_3, v_6. Since V has Schmidt rank three, we have $v_1, v_4, v_5, v_7 \in H$. Hence $v_{i3} \propto v_{23}$ and $v_{i4} \propto v_{64}$. We now prove $v_{23} \neq 0$ and $v_{64} \neq 0$. Suppose $v_{23} = 0$, then $v_{53} = v_{73} = 0$, so that the submatrix formed by $v_{44}, v_{54}, v_{64}, v_{74}$ has rows orthogonal and normalized, so it is unitary and hence its columns are normalized; therefore, $v_{62} = v_{72} = 0$. Similarly from $v_{13} = v_{23} = 0$, we get that $v_{12} = v_{32} = 0$. So U is a BCU controlled from the B side, a contradiction. So we have proved $v_{23} \neq 0$. Now suppose $v_{64} = 0$; we have $v_{44} = v_{54} = v_{74} = 0$, so v_{53} and v_{73} are the only possibly nonzero blocks on their respective rows, so they are nonzero, and $v_{i3} \propto v_{23}$ implies $v_{53} \propto v_{73}$, so the corresponding rows are not orthogonal, violating that V is unitary. This proves $v_{64} \neq 0$.

Next, we prove two statements: $v_{32} \propto v_{62}$ and $v_{21} \propto v_{31}$. Suppose v_{32}, v_{62} are linearly independent. It follows from $v_4, v_5 \in H$ that $v_4 = 0$, $v_{21} = v_{54} = 0$, and $v_{23} \propto v_{53}$. Since V is unitary, by looking at the rows and columns that v_{23} and v_{53} are in, we have $r = d_B - r$, and hence $v_{13} = v_{14} = v_{71} = v_{73} = 0$. By $v_1 \in H$ and $v_{32} \neq 0$, $v_{64} \neq 0$, we have $v_1 \propto v_3$, and as $v_{21} = v_{41} = 0$, the rows containing v_{11}, v_{12} are proportional to the rows containing v_{31}, v_{32}, a contradiction with that V is unitary. Hence $v_{32} \propto v_{62}$. Next suppose v_{21}, v_{31} are linearly independent. Since $v_5 \in H$, we have $v_{53} = v_{62} = 0$. And because V is unitary, the submatrix formed by $v_{44}, v_{54}, v_{64}, v_{74}$ is unitary; hence $v_{72} = v_{73} = 0$. Since $v_7 \in H$, we have $v_{32} = 0$. From $v_1 \in H$, we have $v_{12} = 0$; hence v_{14} is unitary, which contradicts $v_{23} \neq 0$. So $v_{21} \propto v_{31}$.

From the results in the previous paragraph and that $v_i \in H$, we have $v_{i1} \propto v_{21}$ and $v_{i2} \propto v_{32}$. Therefore, we may assume $v_{i1} = a_i A$, $v_{i2} = b_i B$, $v_{i3} = c_i C$, and $v_{i4} = d_i D$ with nonzero blocks A, B, C, D for $i = 1, \ldots, 7$. Since we have proved $v_{23} \neq 0$, $v_{64} \neq 0$, we have $c_2 \neq 0$, $d_6 \neq 0$. Since V is unitary, we have

$$(|a_1|^2 + |a_2|^2)AA^\dagger + |b_1|^2 BB^\dagger = I_{d_B - r}, \qquad (6)$$

$$(|a_3|^2 + |a_4|^2)AA^\dagger + |b_3|^2 BB^\dagger = I_{d_B - r}, \qquad (7)$$

$$(a_1 a_3^* + a_2 a_4^*)AA^\dagger + b_1 b_3^* BB^\dagger = 0, \qquad (8)$$

$$(|c_1|^2 + |c_2|^2)CC^\dagger + |d_1|^2 DD^\dagger = I_r, \qquad (9)$$

$$|c_5|^2 CC^\dagger + (|d_4|^2 + |d_5|^2)DD^\dagger = I_r, \qquad (10)$$

$$|c_7|^2 CC^\dagger + (|d_6|^2 + |d_7|^2)DD^\dagger = I_r, \qquad (11)$$

$$c_5 c_7^* CC^\dagger + (d_4 d_6^* + d_5 d_7^*)DD^\dagger = 0. \qquad (12)$$

Since V is unitary and $c_2 \neq 0$, at least one of b_1, b_3 is nonzero. If one of them is zero, then (6) and (7) imply that A is proportional to a unitary. If both b_1, b_3 are nonzero, then (6) and (8) imply that A is proportional to a unitary. So we have proved A is always proportional to a unitary. It follows from (6) and (7) that $BB^\dagger$ is proportional to $I_{d_B - r}$. Next, if one of c_5, c_7 is zero then (10) and (11) imply that D is proportional to a unitary. If both c_5, c_7 are nonzero, then (11) and (12) imply that D is proportional to a unitary. So we have proved D is always proportional to a unitary. It follows from $c_2 \neq 0$ and (9) that $CC^\dagger$ is proportional to I_r. So V is locally equivalent to the following matrix:

$$V' = \begin{pmatrix} a_1 I_{d_B - r} & b_1 B & a_2 I_{d_B - r} & 0 & 0 & 0 \\ c_1 C & d_1 I_r & c_2 C & 0 & 0 & 0 \\ a_3 I_{d_B - r} & b_3 B & a_4 I_{d_B - r} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 I_r & c_5 C & d_5 I_r \\ 0 & 0 & 0 & b_6 B & a_7 I_{d_B - r} & b_7 B \\ 0 & 0 & 0 & d_6 I_r & c_7 C & d_7 I_r \end{pmatrix}, \qquad (13)$$

where we still use the complex numbers a_i, b_i, c_i, d_i and blocks B, C, since there is no confusion. By adjusting the coefficients for the B blocks, we may assume that $BB^\dagger = I_{d_B - r}$, and similarly we may assume $CC^\dagger = I_r$. Since V' is unitary, we have

$$(|a_1|^2 + |a_2|^2)I_{d_B - r} + |b_1|^2 BB^\dagger = I_{d_B - r}, \qquad (14)$$

$$(|a_3|^2 + |a_4|^2)I_{d_B - r} + |b_3|^2 BB^\dagger = I_{d_B - r}, \qquad (15)$$

$$|d_1|^2 I_r + (|b_1|^2 + |b_3|^2)B^\dagger B = I_r. \qquad (16)$$

Recall that one of b_1, b_3 is nonzero. As $B^\dagger B$ and $BB^\dagger$ have the same rank, from the three equations above we have $d_B - r = r$. Hence B and C are square matrices and are unitaries. Next we apply local unitaries to V' to turn the B into I_r, while preserving the other identity blocks in V'. The transform is given by $V'' = [I_A \otimes (I_r \oplus R)]V'[I_A \otimes (I_r \oplus R^\dagger)]$, where I_r and R are $r \times r$ unitaries acting on subspaces of $\mathcal{H}_B$, and $R = B$. So when V'' is expressed in the form of Eq. (13), the B becomes I_r, while all the coefficients are unchanged. The C becomes a unitary matrix $C' = RC$ after the transform, and thus diagonalizable under a unitary similarity transform. So the C' blocks in V'' can be diagonalized with the following overall transform $X = [I_A \otimes (S \oplus S)]V''[I_A \otimes (S^\dagger \oplus S^\dagger)]$, where S is a $r \times r$ unitary, and it can be verified that other blocks in X are still I_r with coefficients. So V' is locally equivalent to a matrix X, which is still of the form (13) but B and C are replaced by diagonal matrices. So X is a BCU from the B side, and we have a contradiction. This completes the proof. ■

Let U be a bipartite unitary on $d_A \times d_B$ of Schmidt rank three and $d_A = 2,3$. It follows from Theorems 3 and 6 that U is a controlled unitary. We can further decide the side from which U is controlled by Lemma 2. To find out the explicit decomposition of U into a controlled unitary, we refer to an efficient algorithm constructed in [17] and references therein. The algorithm is proposed for finding the finest simultaneous singular value decomposition for simultaneous block diagonalization of square matrices under unitary similarity.

IV. CHARACTERIZATION OF NONLOCAL UNITARY OPERATORS

In this section we propose a few applications of our results on general nonlocal unitary operators. First we characterize the equivalence of nonlocal unitaries and relate them to the controlled unitaries. In Theorem 7 we show that the SL-equivalent multipartite unitary operators are indeed locally equivalent. Using it and Theorem 6 we can simplify the problem of deciding the SL equivalence of two bipartite unitaries of Schmidt rank three with $d_A = 2,3$. Using Theorem 7 we provide a sufficient condition by which a bipartite unitary is locally equivalent to a controlled unitary in Corollary 8. Next we propose an upper bound on the quantum resources implementing bipartite unitaries of Schmidt rank three with $d_A = 2,3$; see Lemma 9. We also show that this upper bound is saturated for some bipartite unitary. Third we apply our results to a special case of Conjecture 10 on the ranks of multipartite quantum states. This conjecture is to construct inequalities analogous to those in terms of von Neumann entropy such as the strong subadditivity [20].

A. Equivalence of nonlocal unitary operators

We start by presenting the following observation on the *SL* equivalence of general nonlocal unitary operators.

Theorem 7. Suppose U and V are multipartite unitaries acting on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$, and they satisfy $U = (S_1 \otimes S_2 \otimes \cdots \otimes S_p)V(T_1 \otimes T_2 \otimes \cdots \otimes T_p)$ for invertible operators S_i and T_i acting on $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_p$, respectively. Then $U = (Q_1 \otimes Q_2 \otimes \cdots \otimes Q_p)V(R_1 \otimes R_2 \otimes \cdots \otimes R_p)$, where Q_i and R_i are unitaries acting on $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_p$, respectively. In particular, when S_i and T_i are identity operators on any party i, we can choose Q_i and R_i to be identity operators.

Proof. Suppose S_i and T_i have singular value decompositions of the form $S_i = E_i C_i F_i$, $T_i = G_i D_i H_i$, where E_i, F_i, G_i, and H_i are unitaries, and C_i and D_i are real diagonal matrices with the diagonal elements sorted in nonincreasing order. The diagonal elements of C_i and D_i are the singular values of S_i and T_i, respectively. Since S_i and T_i are invertible, all the diagonal elements of C_i and D_i are positive.

Let $U' = (E_1^\dagger \otimes E_2^\dagger \otimes \cdots \otimes E_p^\dagger)U(H_1^\dagger \otimes H_2^\dagger \otimes \cdots \otimes H_p^\dagger)$, and let $V' = (F_1 \otimes F_2 \otimes \cdots \otimes F_p)V(G_1 \otimes G_2 \otimes \cdots \otimes G_p)$; then U' and V' are unitaries and satisfy

$$U' = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'(D_A \otimes D_2 \otimes \cdots \otimes D_p). \quad (17)$$

Using $U'U'^\dagger = I$, where I is the identity operator on the entire space, we have

$$I = U'U'^\dagger = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'\big(D_A^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2\big)V'^\dagger(C_1 \otimes C_2 \otimes \cdots \otimes C_p) \quad (18)$$

and, using $V'^\dagger = V'^{-1}$, we get

$$V' = \big(C_1^2 \otimes C_2^2 \otimes \cdots \otimes C_p^2\big)V'\big(D_1^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2\big). \quad (19)$$

And since C_i and D_i are diagonal, $\tilde{C} := C_1^2 \otimes C_2^2 \otimes \cdots \otimes C_p^2$ and $\tilde{D} := D_1^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2$ are diagonal. Consider any nonzero element in the matrix V', and let us suppose it is on row j and column k of V'. Then Eq. (19) implies $\tilde{C}_{jj}\tilde{D}_{kk} = 1$, where $\tilde{C}_{jj}$ means the jth diagonal element of $\tilde{C}$, and $\tilde{D}_{kk}$ is similarly defined. And since $\tilde{C}$ and $\tilde{D}$ only contain positive elements on their diagonals, we have $\sqrt{\tilde{C}_{jj}}\sqrt{\tilde{D}_{kk}} = 1$. This holds for any 2-tuple (j,k) satisfying that the element on row j and column k of V' is nonzero, and since $C_1 \otimes C_2 \otimes \cdots \otimes C_p$ and $D_1 \otimes D_2 \otimes \cdots \otimes D_p$ are diagonal, this implies

$$V' = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'(D_1 \otimes D_2 \otimes \cdots \otimes D_p). \quad (20)$$

Together with Eq. (17), we get $U' = V'$; hence

$$U = (E_1F_1 \otimes E_2F_2 \otimes \cdots \otimes E_pF_p)V(G_1H_1 \otimes G_2H_2 \otimes \cdots \otimes G_pH_p), \quad (21)$$

where E_iF_i and G_iH_i are unitaries by construction. From the proof above we see that when S_i and T_i are identity operators on any party i, we can choose E_i, F_i, G_i, and H_i to be identity operators. This completes the proof of Theorem 7. ∎

The theorem implies that two SL-equivalent multipartite unitary operators are indeed locally equivalent to each other. Such two unitaries can be viewed as the same nonlocal resource in quantum information processing tasks. In contrast, two stochastic LOCC (SLOCC)-equivalent pure states may be not locally equivalent, and generally they can only probabilistically simulate each other in quantum information processing tasks. For example, the three-qubit W state $|\,\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ [21] and W-like state $|\,\mathrm{W}'\rangle = \frac{1}{2}|001\rangle + \frac{1}{2}|010\rangle + \frac{1}{\sqrt{2}}|100\rangle$ are SLOCC equivalent but not locally equivalent, as the bipartition of them gives rise to a nonmaximally entangled state and a maximally entangled state, respectively.

It is known that the classification of multipartite states under LOCC and SLOCC are different, because they are realized with probability one and less than one, respectively. So the former is more coarse grained than the latter. For example, the three-qubit pure states have infinitely many orbits under LOCC [22], while there are only two kinds of fully entangled states under SLOCC, namely the GHZ and W states [21]. In contrast, Theorem 7 implies that the classification of multipartite unitary operations under local unitaries and SL are essentially the same; the latter does not give any additional advantage the former does not have. There are other ways of classifying nonlocal unitaries, such as the LO, LOCC, and SLOCC equivalences discussed in [6], which implicitly assume the use of ancillas.

Based on the previous results we can simplify the decision of SL equivalence of two bipartite unitaries U,V of Schmidt rank three and $d_A = 2,3$. In practice, this is motivated by the simulation of one of them by the other, and the implementation of them. Using Theorem 7 we only need to study the equivalence under local unitaries. It follows from Theorem 6 that both U,V are controlled unitaries. They are not locally equivalent if they are not controlled from the same side, which can be decided by the algorithm in [17]. Nevertheless, deciding the equivalence of two controlled unitary controlled from the same side remains unknown.

Below we characterize the controlled unitaries using Theorem 7.

Corollary 8. If a unitary U on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is SL equivalent to

$$V = \sum_{j=1}^{m} R_j \otimes V_j, \tag{22}$$

where R_j are operators on $\mathcal{H}_A$ satisfying

$$P_j R_j P_j = R_j, \quad \forall j, \tag{23}$$

with $\{P_j\}$ being a set of mutually orthogonal projectors on $\mathcal{H}_A$, and V_j are arbitrary operators on $\mathcal{H}_B$, then U is equivalent under local unitaries to the block diagonal form

$$U' = \sum_{j=1}^{m} P_j \otimes V_j', \tag{24}$$

where V_j' are unitary operators on $\mathcal{H}_B$.

In particular, if a unitary U on $\mathcal{H}$ is SL equivalent to $\sum_{j=1}^{d_A} |j\rangle\langle j| \otimes U_j$ for nonzero matrices U_j, then U is a controlled unitary gate controlled from the A side.

Proof. Note that the general case is reducible to the particular case by first doing singular value decompositions of R_j, and at the end noting that the final local unitaries V_j' on $\mathcal{H}_B$ corresponding to the same R_j are the same. Hence we only need to prove the particular case in the last paragraph of the assertion.

By hypothesis, U is locally equivalent to a unitary $W = \sum_{j=1}^{d_A} |\alpha_j\rangle\langle\beta_j| \otimes W_j$. The states $|\alpha_1\rangle,\dots,|\alpha_{d_A}\rangle \in \mathcal{H}_A$ are linearly independent, and the states $|\beta_1\rangle,\dots,|\beta_{d_A}\rangle \in \mathcal{H}_A$ are normalized (by absorbing constant factors into the corresponding $|\alpha_j\rangle$) and are also linearly independent. Let $|\gamma\rangle \perp P$, and $P = I_A - |\gamma\rangle\langle\gamma|$ the projector on the hyperplane of $\mathcal{H}_A$ spanned by $|\alpha_2\rangle,\dots,|\alpha_{d_A}\rangle$. Since W is unitary, we have $\langle\gamma|_A W W^\dagger |\gamma\rangle_A = |\langle\gamma|\alpha_1\rangle|^2 W_1 W_1^\dagger = I_B$. So the matrix W_1 is proportional to a unitary matrix. We may replace $|\alpha_2\rangle,\dots,|\alpha_{d_A}\rangle$ by any $d_A - 1$ states of $|\alpha_1\rangle,\dots,|\alpha_{d_A}\rangle$ in the above argument, and similarly obtain that the W_i's are proportional to unitary matrices, $i = 2,\dots,d_A$. So U is SL equivalent to a controlled unitary from the A side. The assertion then follows from Theorem 7. This completes the proof. ■

An explanation of Corollary 8 is as follows: if the effect of a unitary is to stochastically implement a controlled type operation of the form in Eq. (22), then the unitary must be a controlled unitary.

B. Entanglement cost of implementing a bipartite unitary

Computing the entanglement cost of implementing a nonlocal unitary is an important question in quantum information [6]. For this purpose a few protocols have been constructed. For example, one can use teleportation [23] twice to implement a nonlocal unitary by using LOCC and two maximally entangled states $|\Psi_{d_A}\rangle$ ($d_A \leqslant d_B$), which contains $2\log_2 d_A$ ebits [8]: Alice teleports her input system to Bob, and Bob does the unitary locally, and teleports back the part of the output system belonging to Alice to her. In Ref. [8], another protocol has been proposed to implement any bipartite controlled unitary controlled from the A side by LOCC and the maximally entangled state $|\Psi_{d_A}\rangle$. Using these protocols, and Theorems 3 and 6, we have the following.

Lemma 9. Let $d_A = 2,3 \leqslant d_B$. Any bipartite unitary of Schmidt rank three can be implemented by using LOCC and the maximally entangled state $|\Psi_k\rangle$, where $k = \min\{d_A^2, d_B\}$.

From this lemma, $\log_2 d_B$ ebits is an upper bound of the amount of entanglement needed to implement a bipartite unitary of Schmidt rank three. In the following we show that this upper bound can be saturated for some unitary with $d_A = 2$, $d_B = 3$. Let $U = I_2 \otimes |1\rangle\langle 1| + \sigma_x \otimes |2\rangle\langle 2| + \sigma_y \otimes |3\rangle\langle 3|$ be on the space $\mathcal{H}_A \otimes \mathcal{H}_B$, A' be the ancilla qubit system, and the bipartite space $\mathcal{K} = \mathcal{H}_{AA'} \otimes \mathcal{H}_B$. Let the product state $|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \otimes \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \in \mathcal{K}$. Then $U|\psi\rangle \in \mathcal{K}$ is a uniformly entangled state of Schmidt rank three. That is, U creates $\log_2 3$ ebits and therefore implementing U must cost at least so much entanglement [7]. On the other hand, Lemma 9 implies that U can be implemented using $\log_2 3$ ebits and LOCC.

We leave as an open question whether there is a Schmidt-rank-three unitary on 2×4 system that needs more than $\log_2 3$ ebits to implement using LOCC. Similar questions can be asked about Schmidt-rank-three unitaries on $3 \times d_B$ systems with $d_B \geqslant 4$. This is a question about the lower bound of the entanglement cost of unitaries, and there are a few results in the literature: Soeda *et al.* [24] proved that one ebit of entanglement is needed for implementing a two-qubit controlled unitary by LOCC when the resource is a bipartite entangled state with Schmidt number two. It is proved in Stahlke *et al.* [25] that if the Schmidt rank of the resource state is equal to the Schmidt rank of the bipartite unitary, and the unitary can be implemented by the state using LOCC or separable operations, then the resource state must be uniformly entangled, i.e., with equal nonzero Schmidt coefficients, and higher Schmidt rank resource states may require less entanglement to implement the same unitary. From these results we see that there are two branches to consider: using a resource state of Schmidt rank equal to that of the unitary or a state of higher Schmidt rank.

C. Conjecture for the ranks of quantum states

The following conjecture is proposed in [20]. In the following T denotes the matrix transpose.

Conjecture 10. Let $R_1,\dots,R_K$ be $m_1 \times n_1$ complex matrices, and let $S_1,\dots,S_K$ be $m_2 \times n_2$ complex matrices. Then

$$\operatorname{rank}\left(\sum_{i=1}^{K} R_i \otimes S_i^T\right) \leqslant K \times \operatorname{rank}\left(\sum_{i=1}^{K} R_i \otimes S_i\right). \tag{25}$$

Note that $\operatorname{rank}(\sum_{i=1}^{K} R_i^T \otimes S_i) = \operatorname{rank}(\sum_{i=1}^{K} R_i \otimes S_i^T)$ holds generally. The motivation of this conjecture is to construct basic inequalities in terms of ranks of multipartite quantum states, and some of them have been constructed in [20]. They are analogous to the inequalities in terms of von Neumann entropy such as the strong subadditivity. Using the basic inequalities one can constrain the relation of the ranks of different marginals and quantify the multipartite entanglement dimensionality.

The conjecture with $K = 1$ is trivial, as the transpose does not change the rank of a matrix. Next, Conjecture 10 with $K = 2$ has been proved in [20]. However, the conjecture with $K \geqslant 3$ is still an open problem and is considered to be highly nontrivial in matrix theory. Nevertheless, the results in the last section shed some light on the conjecture with $K = 3$. Let $U = \sum_{i=1}^{3} R_i \otimes S_i$ be a $3 \times d_B$ unitary matrix. Let $U^\Gamma = \sum_{i=1}^{3} R_i \otimes S_i^T$ be the partial transpose of U [26] with the B side transposed. If U is of Schmidt rank three, Theorems 3 and 6 imply that U is locally equivalent to a controlled unitary; if the Schmidt rank of U is less than three, U is also locally equivalent to a controlled unitary, according to [10]. The controlled unitary could be controlled from either side, and in either case we have $\operatorname{rank} U^\Gamma = \operatorname{rank} U$. Hence $\operatorname{rank} U^\Gamma \leqslant 3 \times \operatorname{rank} U$, which is Conjecture 10 with $K = 3$. Evidently, if Theorem 6 can be generalized to any $d_A > 3$, Conjecture 10 would hold for all Schmidt-rank-three unitaries $U = \sum_{i=1}^{3} R_i \otimes S_i$.

V. CONCLUSIONS

We have shown that the nonlocal unitary operator of Schmidt rank three on the $d_A \times d_B$ system is locally equivalent to a controlled unitary when $d_A \leqslant 3$. Using this result we have shown that LOCC and the $r \times r$ maximally entangled state of $r = \min\{d_A^2, d_B\}$ are sufficient to implement such operators. We also have shown that SL-equivalent nonlocal unitary operators are indeed locally equivalent. In addition we have verified a special case of Conjecture 10 on the ranks of multipartite quantum states, when the argument in the bracket of (25) is a bipartite unitary of Schmidt rank three and $d_A \leqslant 3$.

Unfortunately, we are not able to prove Conjecture 1 when $d_A > 3$, as the proof of Theorem 6 cannot be easily generalized. We believe that the generalization of this theorem will prove Conjecture 1 and verify more cases of Conjecture 10. Otherwise, the first counterexample to Conjecture 1 might exist when $d_A = d_B = 4$. The next interesting question is to find generalizations of Lemma 4. Finally, apart from the Schmidt rank, is there another physical quantity which describes the local equivalence between a nonlocal unitary and a controlled unitary? It remains to investigate the connection between nonlocal and controlled unitaries of arbitrary Schmidt rank.

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