Intelligent states for a number-operator–annihilation-operator uncertainty relation

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(Received 11 September 2013; published 11 June 2014)

Recently a new uncertainty relation was found as an alternative to a number-phase uncertainty relation for a harmonic oscillator. In this paper we determine numerically, via the discrete-variable-representation method known from quantum chemistry, the exact states that saturate this new uncertainty relation. We analyze the physical properties of the states and compare them to the intelligent states of the Pegg-Barnett uncertainty relation. We find that for a given set of expectation values of the physical parameters, which are the particle number and the two quadratures, the two kinds of intelligent states are equivalent. The intelligent states are the eigenstates corresponding to the lowest eigenvalue of a Hermitian operator, which, when interpreted as a Hamiltonian of a physical sytem, describes a nonlinear driven harmonic oscillator, for example, a Duffing oscillator for a certain parameter range. Hence, our results can be interpreted as the determination of the ground state of such physical systems in an explicit analytic form as well. As the Pegg-Barnett intelligent states we use are expressed in terms of a coherent-state superposition facilitating experimental synthesis, we relate the states determined here to experimentally feasible ones.

DOI: 10.1103/PhysRevA.89.062108 PACS number(s): 03.65.Fd, 42.50.Dv

I. INTRODUCTION

Defining a phase operator for a quantum harmonic oscillator which is satisfactory from both physical and mathematical points of view is still an intriguing and not completely solved problem of quantum mechanics. As is well known, it is not possible to introduce a phase operator conjugate to the excitation number operator which satisfies all the requirements of Hermiticity, experimental measurability, calculability, and soundness of definition in an infinite-dimensional separable Hilbert space simultaneously. Due to this fact and the fundamental relevance of phase, especially in quantum optics, several proposals exist in the literature for the definition of a suitable phase operator conjugate to the number operator (such as, e.g., the Susskind-Glogower [1], the Pegg-Barnett [2] or the Garrison-Wong [3] operator), albeit all of the operators fail to satisfy some of the aforementioned properties. The relevance of the problem is also justified by the fact that the quantum harmonic oscillator is experimentally feasible in many physical systems including light, vibrational degrees of freedom of ions, etc.

The existence of a pair of canonically conjugate observables implies an uncertainty relation that quantifies their complementarity, which is maybe the most fundamental feature of quantum mechanics. In the case of the number-phase pair also, the corresponding number-phase uncertainty relations have been constructed for the introduced operators [1,2,4–8]. Entropic uncertainty relations have also been introduced for describing number-phase uncertainty [9–13].

The states saturating a number-phase uncertainty relation are called number-phase intelligent states. Intelligent states for the Susskind-Glogower [1] and for the Pegg-Barnett [2] number-phase uncertainty relations have been found previously [14,15]. Uncertainty relations and the respective intelligent states constitute the starting point for the discussion of squeezing phenomena. Finding the intelligent states for an uncertainty relation is crucial in order to characterize quantum states prepared in experiments with a reduced uncertainty of one of the given physical quantities [16,17]. States of harmonic oscillator systems with reduced number uncertainty are called amplitude-squeezed states. Amplitude squeezing of light and its generation have been widely studied in the literature because of the potential application in quantum metrology, quantum information processing, and optical communication [18–22].

Recently an uncertainty relation involving the number operator N and annihilation operator a was proposed as a well-behaved alternative to known number-phase relations [23]. This expression reads

$$\left[(\Delta N)^2+\frac{1}{4}\right]\left[(\Delta a)^2+\frac{1}{2}\right]\geqslant\frac{\langle N\rangle}{4}+\frac{1}{8}, \qquad (1)$$

and we shall refer to it as the N-a uncertainty relation in what follows. The physical relevance of this uncertainty relation is that it makes it possible to define amplitude squeezing without reference to a phase operator, hence avoiding the problem of its introduction in a rather elegant manner. The quantities found in this uncertainty relation can be measured easily with relatively low experimental requirements [24,25]. We note that a family of uncertainty relations with N and a has already been used for the detection of quantum entanglement [26,27].

In this article we determine the number-phase intelligent states saturating the N-a uncertainty relation. It was already conjectured in Ref. [23] that they coincide with those of the Pegg-Barnett case; however, it was not proven there. Nevertheless, it was derived that that these states are the

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eigenstates of the operator

$$O(\lambda_N,\lambda_X,\lambda_Y) = \lambda_N^{(k)} a^\dagger a + (a^\dagger a)^2 + \left(\frac{\lambda_X^{(k)} + i\lambda_Y^{(k)}}{\sqrt{2}}\right)a + \left(\frac{\lambda_X^{(k)} - i\lambda_Y^{(k)}}{\sqrt{2}}\right)a^\dagger \quad (2)$$

with the smallest eigenvalue. In Eq. (2), the Lagrange multipliers λ_X, λ_Y, and λ_N are there to set the appropriate expectation values of the quadratures $\langle X\rangle$ and $\langle Y\rangle$ and that of the particle number $\langle N\rangle$ in the resulting intelligent state. This is the starting point of our present considerations. We determine the N-a intelligent states, as the respective eigenstates of the operator in Eq. (2), by a method which is known in quantum chemistry, namely, the discrete-variable-representation method (see Refs. [28–33] and references therein).

The solution to the eigenvalue problem described above appears to be useful also in a completely different context. The Hermitian operator O given in Eq. (2) describes a nonlinear driven harmonic oscillator when it is considered as the Hamiltonian of a physical system. As noted in Ref. [23] it also appears as a system Hamiltonian in self-consistent calculations for the Bose-Hubbard model based on the Gutzwiller ansatz [34–38]. The operator O, as a Hamiltonian, also describes a Duffing oscillator in a frame rotating with the driving frequency when the detuning of the driving and the nonlinearity for typical values of the position are small [39]. This system can be realized in circuit QED experiments based on Josephson junctions in superconducting circuits [40,41]. Hence, the eigenstate of the operator in Eq. (2) for the smallest eigenvalue describes the ground state of a physically interesting quantum system characterized by the operator O as a Hamiltonian.

We compare the numerically obtained ideal intelligent states to the intelligent states of the Pegg-Barnett number-phase uncertainty relation defined by a one-dimensional coherent-state superposition [15]

$$|\alpha_0,u,\delta\rangle = c\int \exp\left(-\frac{1}{2}u^2\phi^2 - i\delta\phi\right)|\alpha_0\exp(i\phi)\rangle d\phi. \quad (3)$$

In addition to the calculational convenience, the one-dimensional coherent-state representation has a further important benefit: in various physical systems it is relatively easy to prepare experimentally discrete coherent-state superpositions derived from the continuous superposition that generate the original state [42–50]. Hence, the expression in (3) warrants the practical feasibility of the Pegg-Barnett states. The states defined in Eq. (3) interpolate between coherent states and Fock states and describe amplitude-squeezed states [51] between these two limits for a given parameter set. As the states in Eq. (3) have three parameters u, δ, and α_0 which affect the expectation values of the particle number and quadratures, the proper means of a comparison of the states in (3) and the ones found numerically and described in this paper is to compare the difference in the two sides of the uncertainty relation in Eq. (1), for a fixed set of expectation values of the particle number and the two quadratures. Were this difference equal for the two kinds of states, we could conclude that the states in (3) are intelligent states for Eq. (1) as well. In what follows we shall show that this is indeed the case, within numerical precision.

The paper is organized as follows. In Sec. II we present the numerical method for solving the eigenvalue equation for the operator of Eq. (2) and analyze the physical properties of the determined ideal intelligent states for the number-operator–annihilation-operator uncertainty relation. In Sec. III we compare these ideal states to the intelligent states for the Pegg-Barnett uncertainty relation. Finally, Sec. IV summarizes the results.

II. IDEAL INTELLIGENT STATES

The analytical solution of the eigenvalue problem of the operator O (and the Hamiltonians of the same form) is not known. Therefore we apply a numerical method for obtaining its ground state. A numerically exact approximation to the ground state of the operator O can be obtained by expanding the ground state in terms of a truncated, orthonormal basis. In addition to determining the ground state, the ground-state expectation values of the quadrature operators $X = (a + a^\dagger)/\sqrt{2}$, $Y = (a - a^\dagger)/i\sqrt{2}$, and that of the number operator N and other quantities appearing in Eq. (1) are also required. Both the ground state and the ground-state expectation values of operators can be calculated conveniently by employing the discrete-variable-representation (DVR) method [28–33].

When calculating the expectation values an interesting property of the DVR method is exploited. Namely, the α_ith element of the ith eigenvector corresponding to the numerically converged ith eigenvalue obtained in a DVR calculation is proportional to the value of the ith eigenfunction taken at the αth DVR grid point. The proportionality factor is just the square root of the αth quadrature weight. This property makes the calculation of expectation values of operators over converged eigenstates very simple once the DVR of these operators have been calculated. The DVR of the operators X, Y, and N can be constructed either numerically, by employing the transformation method [52–54], or analytically [55,56]. The form of O suggests that harmonic oscillator eigenstates may be a suitable basis. Therefore the Gauss-Hermite DVR is employed in our calculations. The calculations proceed as follows.

Given a truncated basis formed by the first N harmonic oscillator eigenstates $\chi_n(X)$, $n = 0,1,\ldots,N-1$, Gauss-Hermite DVR basis functions $\phi_\alpha(X)$, $\alpha = 1,\ldots,N$, are defined as

$$\phi_\alpha(X) = \sum_n \chi_n(X)T_{n\alpha}, \quad (4)$$

where $T_{n\alpha}$ are elements of the matrix $\boldsymbol{T}$ diagonalizing the matrix $\boldsymbol{X}$ of the operator X formed in the truncated harmonic oscillator basis. It has been shown [53,54] that, with X_α denoting the αth eigenvalue of $\boldsymbol{X}$, $T_{n\alpha} = w_\alpha^{1/2}\chi_n(X_\alpha)$ and X_α and w_α are just the points and weights, respectively, of the N-point Gauss-Hermite quadrature.

Since the operator O can be expressed in terms of the operators N, X, and Y as

$$O = \lambda_N N + N^2 + \lambda_X X - \lambda_Y Y, \quad (5)$$

and $Y = -i(d/dX)$, one can set up its matrix representation $\boldsymbol{O}$ readily in the Gauss-Hermite DVR basis. The $\beta\alpha$th element reads as

$$O_{\beta\alpha} = \lambda_N N_{\beta\alpha} + (N^2)_{\beta\alpha} + \lambda_X X_\alpha\delta_{\beta\alpha} - \lambda_Y Y_{\beta\alpha}, \quad (6)$$

where $\delta_{\beta\alpha}$ is the Kronecker delta symbol, and $A_{\beta\alpha}$ denotes the $\beta\alpha$ element of an operator A in the DVR basis. The diagonalization of $\boldsymbol{O}$ gives approximate eigenvalues and eigenfunctions of O. To make sure that the ground state has converged at least within ten significant digits, the first 400 harmonic oscillator eigenstates were used as a truncated basis along with the corresponding 400-point Gauss-Hermite quadrature points (and weights) as the DVR grid. Convergence has been checked by increasing the size of the DVR basis to 500.

Truncation of the basis set amounts to approximating the eigenfunctions by finite linear combinations of the basis functions. In particular, the ith eigenfunction is approximated as

$$\Psi_i(X) \approx \sum_\alpha c_{\alpha i}\phi_\alpha(X). \tag{7}$$

One of the points of employing the DVR even for such a relatively simple operator as O is that for converged eigenfunctions, i.e., when there is equality in Eq. (7),

$$c_{\alpha i} = w_\alpha^{1/2}\Psi_i(X_\alpha) \tag{8}$$

holds. Then, once $\boldsymbol{O}$ has been diagonalized one can immediately plot the eigenfunctions and the calculation of the expectation value of an operator $A(X)$ reduces to the simple Gaussian quadrature, i.e.,

$$\begin{aligned}\langle A\rangle &= \langle\Psi_i|A|\Psi_i\rangle \\ &= \sum_\beta\sum_\alpha w_\beta^{1/2}\Psi_i^*(X_\beta)A_{\beta\alpha}w_\alpha^{1/2}\Psi_i(X_\alpha) \\ &\approx \sum_\alpha w_\alpha\Psi_i^*(X_\alpha)A(X_\alpha)\Psi_i(X_\alpha).\end{aligned} \tag{9}$$

Using the procedure described above we have determined the wave functions of the ideal intelligent states for the parameter ranges $\lambda_N,\lambda_X,\lambda_Y = [-100,100]$ with average density $\Delta\lambda_i = 4$. Then we calculated all relevant physical quantities, i.e., mean values of the particle number $\langle N\rangle$ and the quadrature operators $\langle X\rangle$ and $\langle Y\rangle$, and the uncertainties ΔN and Δa.

Figure 1 shows the mean value of the particle number as a function of the parameters λ_X and λ_Y for $\lambda_N = -100$.

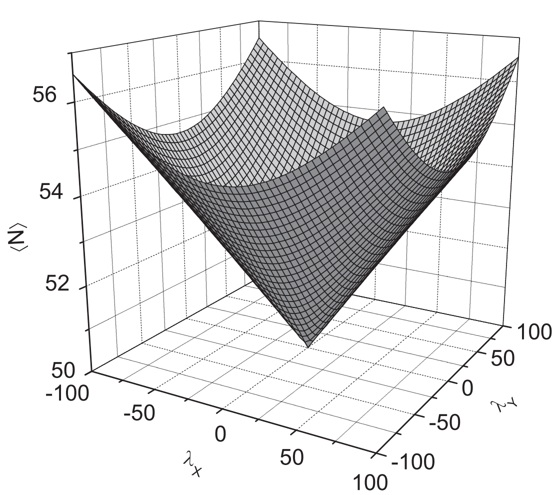

FIG. 1. The mean value of the number operator $\langle N\rangle$ as a function of the parameters λ_X and λ_Y for $\lambda_N = -100$.

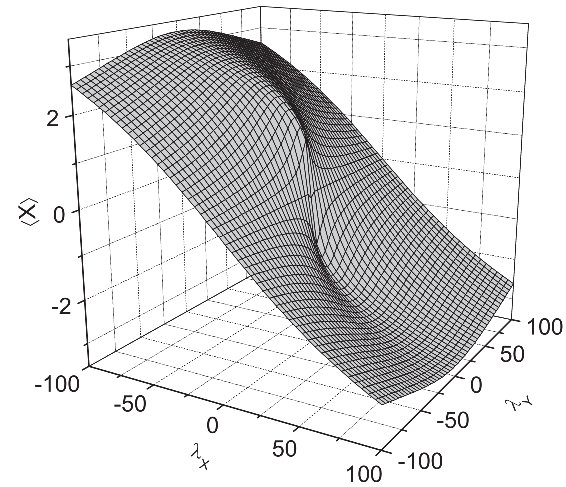

FIG. 2. The mean value of the quadrature operator $\langle X\rangle$ as a function of the parameters λ_X and λ_Y for $\lambda_N = -1$.

The symmetry in λ_X and λ_Y follows from the corresponding symmetry of Eq. (2). The mean value of the quadrature operator $\langle X\rangle$ as a function of the parameters λ_X and λ_Y is presented in Fig. 2 for $\lambda_N = -1$.

Figures 3 and 4 show the mean values of the quadrature operators $\langle X\rangle$ and $\langle Y\rangle$, respectively, as functions of λ_N and λ_Y for $\lambda_X = -1$. Due to the mutual correspondence of the parameters λ_X and λ_Y from now on we can choose $\lambda_Y = 0$ without loss of generality. In Fig. 4 one can see that the mean value of the quadrature operator Y equals to zero in this case.

After calculating all physical quantities as functions of the parameters $\lambda_X,\lambda_Y,\lambda_N$ we checked that the dependence is monovalent. This means that one point in the three-dimensional parameter space yields only one point in the space of the physical quantities $\langle N\rangle$, $\langle X\rangle$, and $\langle Y\rangle$. As a consequence we can examine the properties of the intelligent states as a function of the physical quantities.

At this point we should note that there is an interesting property of the uncertainty relation (1), namely, it does not

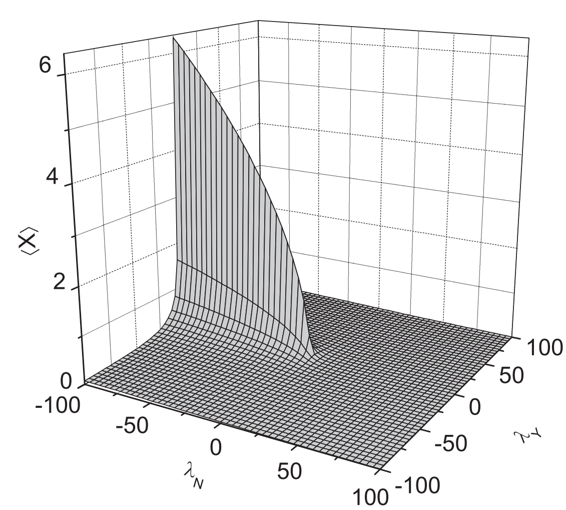

FIG. 3. The mean value of the quadrature operator $\langle X\rangle$ as a function of the parameters λ_N and λ_Y for $\lambda_X = -1$.

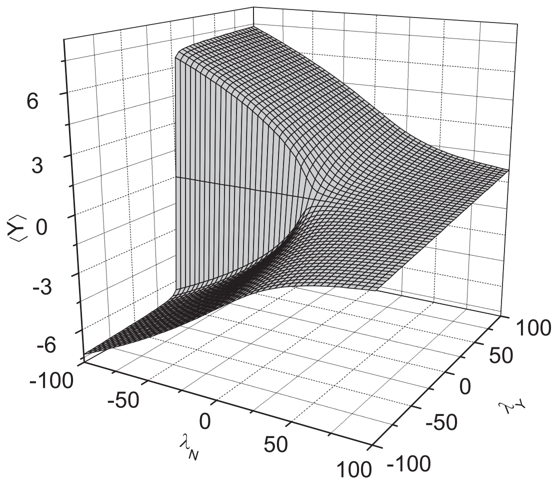

FIG. 4. The mean value of the quadrature operator $\langle Y\rangle$ as a function of the parameters λ_N and λ_Y for $\lambda_X = -1$.

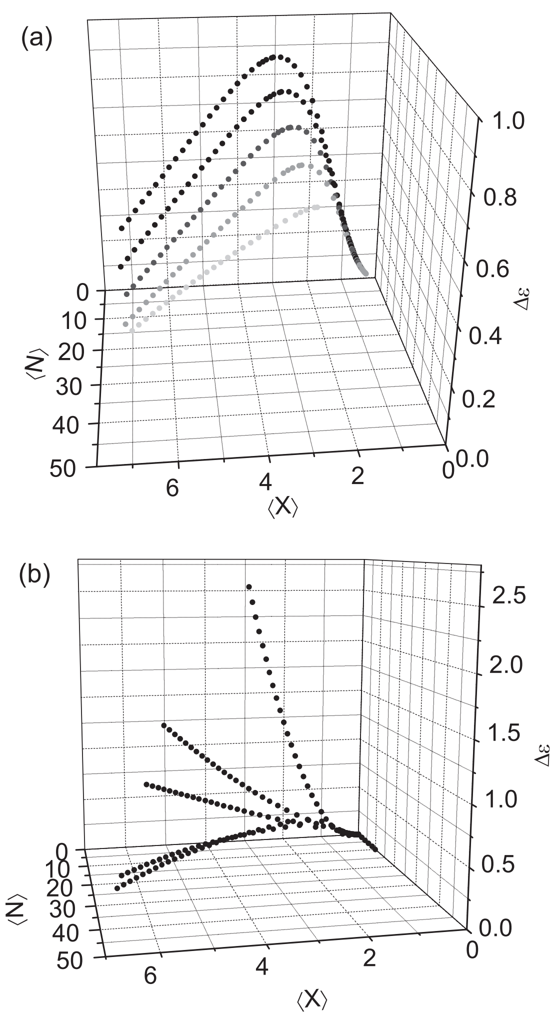

FIG. 5. The difference $\Delta\varepsilon$ between the left-hand side and the right-hand side of Eq. (1) as a function of the mean value of the number operator $\langle N\rangle$ and the quadrature operator $\langle X\rangle$ for (a) $\lambda_X = [-20, -40, -60, -80, -100]$ from bottom to top and (b) $\lambda_X = [0, -0.1, -0.5, -1, -6, -10]$ from right to left.

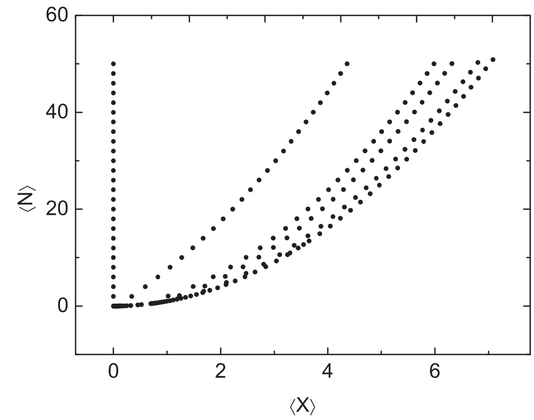

FIG. 6. The mean value of the number operator $\langle N\rangle$ as a function of the mean value of the quadrature operator $\langle X\rangle$ for $\lambda_X = [0, -0.1, -0.5, -1, -6, -100]$ from left to right.

lead to equality even for an obviously number-phase intelligent state, e.g., for a coherent state with a large amplitude. Moreover, having predefined expectation values of the particle number $\langle N\rangle$, and the two quadratures $\langle X\rangle$ and $\langle Y\rangle$, there might even not exist a state for which (1) holds as an equality. Hence, by saturation of the inequality, the minimization of the difference of the left- and right-hand sides of the inequality is meant. Also, an *intelligent state* is one which minimizes this difference. We note that the same definition holds for the case of the uncertainty relation involving the particle number and the Pegg-Barnett phase operator.

In Fig. 5 we show the difference between the left-hand side and the right-hand side of Eq. (1), $\Delta\varepsilon$, as a function of the mean values $\langle X\rangle$ and $\langle N\rangle$ for various values of the parameter λ_X. We note that the straight line in Fig. 5(b) at $\lambda_X = 0$, $\langle X\rangle = 0$ corresponds to Fock states. For these states obviously $\Delta\varepsilon = 0$. One can see that the intelligent states of the uncertainty relation Eq. (1) generally do not yield equality corresponding to $\Delta\varepsilon = 0$. As shown in Ref. [23], even for coherent states $|\alpha_0\rangle$, which are obviously intelligent states for this uncertainty relation, $\Delta\varepsilon = \sqrt{\alpha_0}/4$.

Figure 6 shows the mean value of the number operator $\langle N\rangle$ as a function of the mean value of the quadrature operator $\langle X\rangle$ for various values of the parameter λ_X corresponding to the projection of the functions $\Delta\varepsilon$ of Fig. 5(b) on the parameter plane of $\langle N\rangle$ and $\langle X\rangle$.

In the next section we will show that the same range of the expectation values $\langle N\rangle$ and $\langle X\rangle$ can be achieved by adjusting the parameters u, δ, and α_0 in the intelligent states of the Pegg-Barnett uncertainty relation defined in Eq. (3).

III. COMPARISON TO THE INTELLIGENT STATES FOR THE PEGG-BARNETT UNCERTAINTY RELATION

In Ref. [15] number-phase intelligent states were determined for the number-phase uncertainty relation defined through the Pegg-Barnett Hermitian phase operator. These states have been defined as continuous coherent-state superpositions on a circle according to Eq. (3) and they have the following expansion in the number

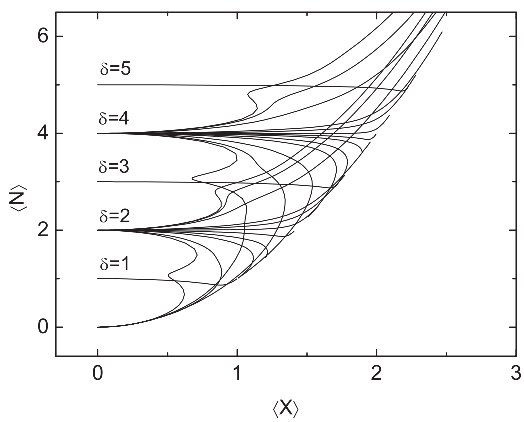

FIG. 7. The mean value of the number operator $\langle N\rangle$ as a function of the mean value of the quadrature operator $\langle X\rangle$ for $\delta = 1,2,3,4,5$ and a series of α_0 parameter values. (For $\delta = 1,3,5$ only the case $\alpha_0 = \sqrt{\delta}$ is plotted while for $\delta = 4$, $\alpha_0 = [0.1,0.5,1,1.5,1.7,1.9,2,2.1,2.3,2.5,5,10,20]$, and for $\delta = 2$, $\alpha_0 = [0.1,0.5,1,1.2,\sqrt{2},1.6,1.8,2,5,10,15]$ are plotted. In these latter cases the curve belonging to a lower α_0 runs always below the one with a greater α_0.)

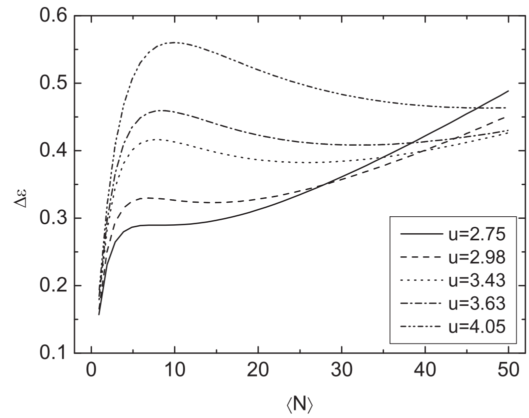

FIG. 8. The difference $\Delta\varepsilon$ between the left-hand side and the right-hand side of Eq. (1), as a function of the mean value of the number operator $\langle N\rangle$ for various values of u.

states:

$$|\alpha_0,u,\delta\rangle = c\sum_{n=0}^{\infty}\exp\left(-\frac{\alpha_0^2}{2}\right)\frac{\alpha_0^n}{\sqrt{n!}}\frac{\sqrt{2\pi}}{u} \times \exp\left(-\frac{(\delta-n)^2}{2u^2}\right)|n\rangle. \quad (10)$$

These states lead to the coherent state $|\alpha_0\rangle$ in the limit $u\to\infty$ and yield the number state $|\delta\rangle$ when δ is a non-negative integer in the limit $u \to 0$. We note that the state of Eq. (10) corresponds to the most illustratively parametrized version of the mathematically obtained intelligent state for the Pegg-Barnett uncertainty relation. For further calculations we choose α_0 to be real, which corresponds to the choice of $\lambda_Y = 0$ in the case of the ideal intelligent states.

In Fig. 7 we present the mean value of the particle number $\langle N\rangle$ as a function of the mean value of the quadrature operator $\langle X\rangle$ for some values of the parameter δ and various values of α_0. Comparing this figure to Fig. 6, it is easy to conclude that the ideal intelligent states and the ones defined by Eq. (3) cover the same domain of the $\langle X\rangle$-$\langle N\rangle$ plane. The area of the possible values of $\langle X\rangle$ and $\langle N\rangle$ is bounded by a parabolic curve corresponding to the values of these quantities in coherent states. When $\langle X\rangle = 0$ we have the number states. Note that some of the curves corresponding to the different parameters in Fig. 7 eventually intersect. Indeed, different parameter sets can produce states with the same expectation values. Moreover, we have checked that in such cases not only the expectation values but all the coefficients in the number-state expansion of the states are the same within numerical precision, hence, different parameter sets may yield the same state.

Figure 8 shows the difference $\Delta\varepsilon$ between the left-hand side and the right-hand side of Eq. (1) as a function of the mean value of the number operator $\langle N\rangle$ for various values of u. For comparison we present the same function for the ideal intelligent state for various values of λ_X (Fig. 9). From the similarity of the curves it is clear that the parameter λ_X plays the same role for the ideal intelligent states as does the parameter u for the states $|\alpha_0,u,\delta\rangle$. Of course, the points of a curve in Fig. 8 correspond to different values of the parameters δ and α_0 (here $\delta = \alpha_0^2$), and the points of a curve in Fig. 9 correspond to different values of the parameter λ_N.

In the end, we have verified numerically that for the states $|\alpha_0,u,\delta\rangle$ it is easy to reach with high precision all discrete values of $\langle N\rangle$ and $\langle X\rangle$ calculated numerically for the ideal intelligent states for the considered parameter range of λ_X, λ_Y, and λ_N by adjusting the values of u, δ, and α_0. In this way the states $|\alpha_0,u,\delta\rangle$ can be compared to the ideal intelligent state. We have found that the difference $\Delta\varepsilon$ between the two sides of the uncertainty relation of Eq. (1) for a fixed set of expectation values of the particle number and the two quadratures is equal for the two kind of states within numerical precision. In Table I we present the mean values of the particle number $\langle N\rangle$ and the quadrature $\langle X\rangle$ and the precision of the equivalence of the difference $\Delta_{\Delta\varepsilon}$ between the two sides of the uncertainty relation of Eq. (1) for five different corresponding parameter sets of the two kind of states. It means that the ideal intelligent states of the number-operator–annihilation-operator

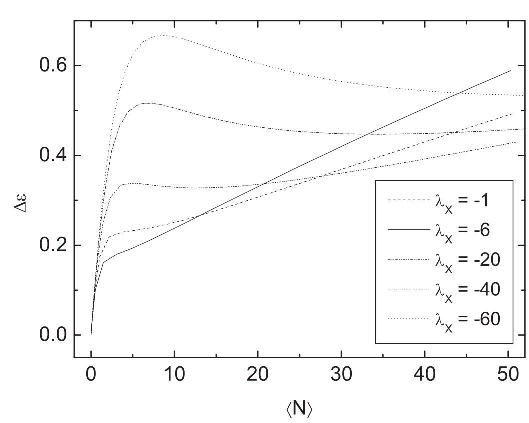

FIG. 9. The difference $\Delta\varepsilon$ between the left-hand side and the right-hand side of Eq. (1) as a function of the mean value of the number operator $\langle N\rangle$ for various values of λ_X.

TABLE I. The mean values of the particle number $\langle N\rangle$ and the quadrature $\langle X\rangle$ and the precision of the equivalence of the difference $\Delta_{\Delta\varepsilon}$ between the two sides of the uncertainty relation of Eq. (1) for five different corresponding parameter sets of the two kinds of states.

λ_X	λ_N	u	δ	α_0	$\langle N\rangle$	$\langle X\rangle$	$\Delta_{\Delta\varepsilon}$
–20	4	1.789	3	1.48	2.32	2.14	11×10^{-4}
–6	2	2.31	1	0.99	0.89	1.32	2×10^{-4}
–10	–4	2.396	4	1.877	3.666	2.644	3×10^{-4}
–20	–28	3.02	16	3.88	15.70	5.49	16×10^{-4}
–0.5	–12	0.94	6	2.534	6.02	2.64	14×10^{-4}

relation coincide with the intelligent states of the Pegg-Barnett uncertainty relation when the mean values of the physical quantities $\langle N\rangle$, $\langle X\rangle$, and $\langle Y\rangle$ are equal.

As a straightforward consequence of this result, the coherent superposition states defined in Eq. (3) approximate with high precision the ground-state wave function of nonlinear driven harmonic oscillators described by the Hamiltonian of the form of Eq. (2). This bears additional experimental relevance, as we have already discussed in the Introduction.

IV. CONCLUSION

We have determined numerically the exact states that saturate the number-operator–annihilation-operator uncertainty relation that has been introduced recently in Ref. [23] as an alternative approach to the problem of the number-phase uncertainty relation. We proved that these ideal intelligent states coincide with the intelligent states of the Pegg-Barnett number-phase uncertainty relation. This result justifies the correctness of the alternative uncertainty relation introduced in Ref. [23] and establishes its relation to the the Pegg-Barnett formalism which, due to its ease in calculation, is applied in most of the quantum optics literature. Moreover, as Pegg-Barnett intelligent states are expressed in the form of a one-dimensional coherent-state representation, we have related the newly determined states to experimentally feasible ones.

We have also recognized that the operator in the eigenvalue equation yielding the ideal intelligent state coincides with the Hamiltonian of nonlinear driven harmonic oscillator systems realized recently in circuit QED experiments. Therefore our result implies that the ground state of such system is the intelligent state of the Pegg-Barnett and the number-operator–annihilation-operator uncertainty relations.

ACKNOWLEDGMENT

We thank the National Research Fund of Hungary OTKA (Contract No. K83858) for support.

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