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Superfluidity breakdown and multiple roton gaps in spin-orbit-coupled Bose-Einstein condensates in an optical lattice

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We investigate the superfluid phases of a Rashba spin-orbit-coupled Bose-Einstein condensate residing on a two-dimensional square optical lattice in the presence of an effective Zeeman field Ω. At a critical value $\Omega = \Omega_c$, the single-particle spectrum $E_{\mathbf{k}}$ changes from having a set of four degenerate minima to a single minimum at $\mathbf{k} = 0$, corresponding to condensation at finite or zero momentum, respectively. We describe this quantum phase transition and the symmetry breaking of the condensate phases. We use the Bogoliubov theory to treat the superfluid phases and determine the phase diagram, the excitation spectrum, and the sound velocity of the phonon excitations. A dynamically unstable superfluid regime occurring when Ω is close to Ω_c is analytically identified and the behavior of the condensate quantum depletion is discussed. Moreover, we show that there are two types of roton excitations occurring in the $\Omega < \Omega_c$ regime and obtain explicit values for the corresponding energy gaps.

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Introduction. The recent realization of ultracold spin-orbit-coupled (SOC) quantum gases [1] has attracted high interest and has resulted in considerable research efforts both on the theoretical and on the experimental side [2–6], in part due to the possibility to tune the spin-orbit interactions [7] in contrast to solid-state materials. Ultracold quantum gases with spin-orbit coupling manifest novel types of superfluid and magnetic ground states and have also been predicted to host topological excitations like Majorana fermions [8].

The SOC Bose-Einstein condensate (BEC) has intrinsic features that make it different from the standard BEC: the interaction among atoms make a SOC BEC stable since it cannot exist in the free regime [9], the SOC also breaks the Galilean invariance so that the superfluid properties change in different reference frames [10]; for a review see Ref. [11]. Several works have considered different types of SOC in the continuous limit: pure Rashba, mixed and symmetric Rashba-Dresselhaus, in two and three dimensions [12–14]. The exotic properties of the Mott insulating phase arising from the superfluid–Mott-insulator (SF-MI) transition [15,16] were also considered in the case of an optically induced lattice. However, an analytical quantitative description of the SF phase for a SOC BEC in an optically induced lattice is still missing.

In this work, we consider a Bose-Einstein condensate with Rashba SOC residing on a two-dimensional (2D) square optical lattice and prove that the SOC qualitatively affects the features of the superfluid phase. The system's parameters are the Zeeman coupling Ω, the strength of the spin-orbit coupling λ, the hopping t, and the intra- and interspecies interactions U and U'. We discuss the origin and magnitude of these terms in more detail later on. We will in this paper show three main results: (I) With $\lambda \gg t$ the existence of the SF is related to the ratio Ω/U and not to t/U as in the usual Bose-Hubbard models. (II) Ω can trigger a breakdown of SF in a window near the critical value $\Omega_c \equiv 2\lambda^2/t$. In this regime the excitation spectrum assumes complex values, indicating a dynamical instability toward a phase separation [17]. (III) In the regime $\Omega < \Omega_c$, the excitation spectrum has, besides the usual gapless phonon minimum localized at the condensation momentum, three gapped roton minima with different gap energies $\Delta_\perp$ and $\Delta_\parallel$. We provide analytical evidence of all these results.

Bose-Hubbard formulation. It is possible to induce on a dilute atomic boson gas system, through laser-atom interactions, a spin-momentum interaction such that the effective system has two coupled levels. In this sense one may speak of pseudospin-$\frac{1}{2}$ bosons. The confinement on a 2D plane and the periodic potential on it can be experimentally realized through the action of counterpropagating lasers. Our starting point is a two-species Bose-Hubbard–type Hamiltonian [16] $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{\text{int}}$:

$$\mathscr{H}_0 = \sum_{\langle i,j\rangle,\alpha\beta} [-t_\alpha b^\dagger_{i\alpha} b_{j\alpha}\delta_{\alpha\beta} + i\lambda b^\dagger_{i\alpha}\hat{z}\cdot(\boldsymbol{\sigma}\times\boldsymbol{d}_{ij})_{\alpha\beta} b_{j\beta}] + \sum_{i\alpha\beta}[\delta b^\dagger_{i\alpha}(\sigma_y)_{\alpha\beta} b_{i\beta} - \Omega b^\dagger_{i\alpha}(\sigma_z)_{\alpha\beta} b_{i\beta} - \mu b^\dagger_{i\alpha} b_{i\alpha}\delta_{\alpha\beta}],$$

$$\mathscr{H}_{\text{int}} = \sum_{i\alpha} \frac{U}{2} b^\dagger_{i\alpha} b^\dagger_{i\alpha} b_{i\alpha} b_{i\alpha} + \sum_i U' b^\dagger_{iA} b^\dagger_{iB} b_{iA} b_{iB}. \quad (1)$$

Above, i is the lattice-site index, α and β run over the two species A and B that correspond to the pseudospin $\pm\frac{1}{2}$, μ is the chemical potential, t_α is the hopping coefficient, λ is the strength of the spin-orbit coupling, $\hat{z}$ is the unit vector in the z direction, $\boldsymbol{d}_{ij}$ is the nearest neighbor (NN) vector between lattice sites i and j, $\boldsymbol{\sigma}$ is the Pauli matrix vector, δ is the detuning parameter, and Ω is the Zeeman coupling. The square optical lattice is assumed to lie in the xy plane. The interaction part $\mathscr{H}_{\text{int}}$ contains the intra- and interspecies interactions U and U'; we allow these coefficients to be different. We set $\hbar = 1$ in what follows. We diagonalize the noninteracting Hamiltonian $\mathscr{H}_0$ using the quasimomentum basis $\{b_{\mathbf{k}\alpha}, b^\dagger_{\mathbf{k}\alpha}\}$: $b_{i\alpha} = \frac{1}{\sqrt{N_s}}\sum_{\mathbf{k}} b_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\boldsymbol{r}_i}$, where N_s is the total number of sites. We focus on equal hopping coefficients $t \equiv t_A = t_B$ and $\delta = 0$ for the sake of obtaining more tractable analytical expressions that allow for deeper physical insights. The energy bands are $E_{\mathbf{k},\pm} = -2t(\cos k_x + \cos k_y) - \mu \pm [\Omega^2 + 4\lambda^2(\sin^2 k_x + \sin^2 k_y)]^{1/2}$. The spectrum $E_{\mathbf{k},\pm}$ is invariant under parity ($k_x \to -k_x, k_y \to -k_y$) and under permutation of k_x and k_y, ($k_x \to k_y, k_y \to k_x$), so the total symmetry group of $E_{\mathbf{k},\pm}$ is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{S}_2$. The value of Ω strongly affects the shape of $E_{\mathbf{k},-}$. With $\Omega > \Omega_c \equiv 2\lambda^2/t$ it has one minimum at (0,0). With $\Omega < \Omega_c$ it has four degenerate minima at $(\pm k_0, \pm k_0)$:

$$k_0 = \arcsin\sqrt{[1-(\Omega/\Omega_c)^2]/[1+2(t/\lambda)^2]}. \quad (2)$$

At the critical value $\Omega = \Omega_c$, it has one minimum at (0,0) behaving as a fourth order power in momentum. We note that without a lattice structure the minima degeneracy of $E_{\mathbf{k},-}$ is continuous in the (k_x,k_y) plane, whereas it is discrete in our case so that the SF phase is expected to be more robust towards quantum fluctuations. We define the operator basis $\{d_{\mathbf{k},-}, d_{\mathbf{k},+}\}$ that respectively annihilates a boson in the lower band $E_{\mathbf{k},-}$ and in the upper band $E_{\mathbf{k},+}$. These are related to $\{b_{\mathbf{k}A}, b_{\mathbf{k}B}\}$ via the unitary matrix $\mathscr{P}$. We are interested in a low-energy description of the system at $T = 0$ and thus we consider populating only the lowest energy band $E_{\mathbf{k},-}$. This condition is qualitatively satisfied by taking $\Omega > \max\{U,U'\}$. In fact, $2\Omega < (E_{\mathbf{k},+} - E_{\mathbf{k},-}) < 2\Omega + 8\lambda$, and $\max\{U,U'\}$ is an estimate of the disposable energy to scatter from the lower band to the upper band. We define $E_{\mathbf{k}} \equiv E_{\mathbf{k},-}$. With this assumption $d_{\mathbf{k},+} \to 0$ and the operators $b_{\mathbf{k}A}$ and $b_{\mathbf{k}B}$ are directly proportional to $d_{\mathbf{k}} \equiv d_{\mathbf{k},-}$: $b_{\mathbf{k}A} = \alpha_{\mathbf{k}} d_{\mathbf{k}}$ and $b_{\mathbf{k}B} = \beta_{\mathbf{k}} d_{\mathbf{k}}$, where we set $\alpha_{\mathbf{k}} \equiv \mathscr{P}_{1,1}$ and $\beta_{\mathbf{k}} \equiv \mathscr{P}_{2,1}$. The coefficients $\alpha_{\mathbf{k}} \in \mathbb{R}$ and $\beta_{\mathbf{k}} \in \mathbb{C}$ are the probability amplitudes for a particle in the band $E_{\mathbf{k}}$ to be of the A or B type. From the unitarity of $\mathscr{P}$ it follows that $\alpha(\mathbf{k})^2 + |\beta(\mathbf{k})|^2 = 1$;

$$\alpha_{\mathbf{k}} = \sqrt{(1/2)\{1+[1+(2\lambda/\Omega)^2(\sin^2 k_x + \sin^2 k_y)]^{-1/2}\}},$$

$$\beta_{\mathbf{k}} = [(\sin k_y - i \sin k_x)/\sqrt{\sin^2 k_x + \sin^2 k_y}]\sin\theta_{\mathbf{k}}. \quad (3)$$

We define $\cos\theta_{\mathbf{k}} \equiv \alpha_{\mathbf{k}}$ for later purposes. The interaction Hamiltonian as a function of the operators $\{d_{\mathbf{k}}, d_{\mathbf{k}}^\dagger\}$ reads

$$\mathscr{H}_{\text{int}} = \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{p}+\mathbf{p}'} \frac{U}{2N_S}(\alpha_{\mathbf{k}}\alpha_{\mathbf{k}'}\alpha_{\mathbf{p}}\alpha_{\mathbf{p}'} + \beta_{\mathbf{k}}^*\beta_{\mathbf{k}'}^*\beta_{\mathbf{p}}\beta_{\mathbf{p}'}) d_{\mathbf{k}}^\dagger d_{\mathbf{k}'}^\dagger d_{\mathbf{p}} d_{\mathbf{p}'} + \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{p}+\mathbf{p}'} \frac{U'}{N_S}\alpha_{\mathbf{k}}\beta_{\mathbf{k}'}^*\alpha_{\mathbf{p}}\beta_{\mathbf{p}'} d_{\mathbf{k}}^\dagger d_{\mathbf{k}'}^\dagger d_{\mathbf{p}} d_{\mathbf{p}'}. \quad (4)$$

We note that the scattering coefficients in Eq. (4) are invariant under parity. We discard the upper energy band $E_{\mathbf{k},+}$, which corresponds to mapping the original $\{A,B\}$ components into an effective one-component system with momentum-dependent interaction coefficients [Eq. (4)].

In the regime $\Omega < \Omega_c$, the noninteracting energy spectrum $E_{\mathbf{k}}$ has four degenerate minima, which raises the issue of whether the condensation takes place at one or more momenta. As we discuss after the evaluation of the ground-state energy (6), the condensation momentum is unique when $U > U'$: this is the so-called plane-wave phase. Our analysis and results are restricted to this case.

The shape of $E_{\mathbf{k}}$ changes by varying Ω across Ω_c; this determines a quantum phase transition. With $\Omega > \Omega_c$ the condensation momentum is $\mathbf{K}_0 = 0$, the corresponding state preserves the parity symmetry in momentum space; with $\Omega < \Omega_c$ the condensation momentum is $\mathbf{K}_0 \neq 0$; this is a symmetry broken phase because the corresponding condensate state breaks the parity symmetry in momentum space. A natural choice for the order parameter of this QPT is $|\beta_{\mathbf{K}_0}|^2$ that passes from a nonzero value with $\Omega < \Omega_c$ to zero with $\Omega > \Omega_c$, varying continuously.

To treat the condensate phase we apply the Bogoliubov theory which is very well suited to capture the SF properties but not to investigate the SF-MI transition [16], the latter being outside the scope of the present work. Let $\mathbf{K}_0$ denote the condensation momentum which is zero or finite according to the value of Ω. We then have $d_{\mathbf{K}_0}^\dagger d_{\mathbf{K}_0} = N_{\mathbf{K}_0} \gg 1$ and subsequently apply the Bogoliubov approximation $d_{\mathbf{K}_0}^\dagger \sim d_{\mathbf{K}_0} \sim \sqrt{N_{\mathbf{K}_0}}$. We perform a mean-field approximation of Eq. (4) by taking into account the particle-number fluctuations out of the condensate to the first order [18]. The final Hamiltonian is

$$\mathscr{H} = E_0 + {\sum_{\mathbf{k}}}' \left(a_{\mathbf{k}} d_{\mathbf{k}}^\dagger d_{\mathbf{k}} + b_{\mathbf{k}} d_{\mathbf{k}} d_{2\mathbf{K}_0-\mathbf{k}} + b_{\mathbf{k}}^* d_{\mathbf{k}}^\dagger d_{2\mathbf{K}_0-\mathbf{k}}^\dagger\right), \quad (5)$$

where the symbol $'$ indicates that $\mathbf{K}_0$ is excluded from the sum. With $n = (N_A + N_B)/N_S$ we have

$$\begin{aligned}
E_0/N_S &= nE_{\mathbf{K}_0} + n^2\left[(U/2)\left(\alpha_{\mathbf{K}_0}^4 + |\beta_{\mathbf{K}_0}|^4\right) + U'\alpha_{\mathbf{K}_0}^2|\beta_{\mathbf{K}_0}|^2\right],\\
a_{\mathbf{k}} &= E_{\mathbf{k}} - E_{\mathbf{K}_0} + nU\left[2\alpha_{\mathbf{k}}^2\alpha_{\mathbf{K}_0}^2 + 2|\beta_{\mathbf{k}}|^2|\beta_{\mathbf{K}_0}|^2 - \alpha_{\mathbf{K}_0}^4 - |\beta_{\mathbf{K}_0}|^4\right] + nU'\left[\alpha_{\mathbf{k}}^2|\beta_{\mathbf{K}_0}|^2 + \alpha_{\mathbf{K}_0}^2\left(|\beta_{\mathbf{k}}|^2 - 2|\beta_{\mathbf{K}_0}|^2\right)\right.\\
&\quad \left. + 2\alpha_{\mathbf{k}}\alpha_{\mathbf{K}_0}\mathrm{Re}\left(\beta_{\mathbf{k}}\beta_{\mathbf{K}_0}^*\right)\right],\\
b_{\mathbf{k}} &= (n/2)U\left(\alpha_{\mathbf{K}_0}^2\alpha_{\mathbf{k}}\alpha_{2\mathbf{K}_0-\mathbf{k}} + \beta_{\mathbf{K}_0}^{*2}\beta_{\mathbf{k}}\beta_{2\mathbf{K}_0-\mathbf{k}}\right)\\
&\quad + nU'\alpha_{\mathbf{K}_0}\beta_{\mathbf{K}_0}^*\left(\alpha_{\mathbf{k}}\beta_{2\mathbf{K}_0-\mathbf{k}} + \alpha_{2\mathbf{K}_0-\mathbf{k}}\beta_{\mathbf{k}}\right),
\end{aligned} \quad (6)$$

where E_0 is the ground-state energy. Considering $\Omega < \Omega_c$ we can compare E_0 with the ground-state energy obtained by supposing that the condensate state is equally populated by atoms with momenta $\mathbf{K}_0$ and $-\mathbf{K}_0$ (striped phase), this is obtained taking into account in the interaction Hamiltonian (4) values of the momenta $\{\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}'\}$ equal to $\{\pm\mathbf{K}_0, \pm\mathbf{K}_0, \pm\mathbf{K}_0, \pm\mathbf{K}_0\}$, $\{\pm\mathbf{K}_0, \mp\mathbf{K}_0, \pm\mathbf{K}_0, \mp\mathbf{K}_0\}$, or $\{\pm\mathbf{K}_0, \mp\mathbf{K}_0, \mp\mathbf{K}_0, \pm\mathbf{K}_0\}$. With $U > U'$ the favored phase is the plane-wave phase whereas with $U' > U$ the boundary between the two phases is

$$\Omega/\Omega_c = \sqrt{2}t/\sqrt{[(x+1)/(x-1)](\lambda^2+2t^2) - \lambda^2}, \quad (7)$$

with $x = U'/U$ (see Fig. 1). We checked that possible condensate phases that populate all the four minima of $E_{\mathbf{k}}$ always have a higher ground-state energy.

We diagonalize the mean-field Hamiltonian (5) making sure to preserve the boson commutation relations [19], obtaining the excitation spectrum and the final Hamiltonian:

$$\mathscr{E}_{\mathbf{k}} = \frac{1}{2}\left(a_{\mathbf{k}} - a_{2\mathbf{K}_0-\mathbf{k}} + \sqrt{\left(a_{\mathbf{k}} + a_{2\mathbf{K}_0-\mathbf{k}}\right)^2 - 16|b_{\mathbf{k}}|^2}\right), \quad (8)$$

$$\mathscr{H} = E_0 + \frac{1}{2}{\sum_{\mathbf{k}}}'(\mathscr{E}_{\mathbf{k}} - a_{\mathbf{k}}) + {\sum_{\mathbf{k}}}' \mathscr{E}_{\mathbf{k}} C_{\mathbf{k}}^\dagger C_{\mathbf{k}}, \quad (9)$$

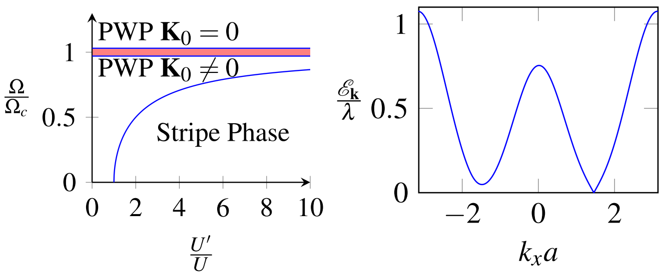

FIG. 1. (Color online) (Left) The red (gray) stripe denotes the instability region. With $\Omega < \Omega_c$ and $U' < U$ the stable phase is the plane-wave phase (PWP) with finite condensation momentum. With $U' > U$ the PWP and the striped phase are competing. With $\Omega > \Omega_c$ the favored phase is the PWP with condensation momentum equal to zero. Increasing U'/U the phase boundary between the striped phase and the PWP tends to 1. (Right) Projection of the excitation spectrum $\mathscr{E}_{\mathbf{k}}$ on $k_y = k_0$ showing the phonon excitation and roton gap. $\tilde{t} = 0.08$, $\tilde{\Omega} = 0.55$, $n = 1$, $\tilde{U} = 0.12$, $\tilde{U}' = 0.11$.

where $C_{\mathbf{k}}$, $C_{\mathbf{k}}^\dagger$ are the bosonic annihilation and creation operators of the excitations $\mathscr{E}_{\mathbf{k}}$. In the noninteracting limit $U = U' = 0$, we have $b_{\mathbf{k}} = 0$ and $a_{\mathbf{k}} = E_{\mathbf{k}} - E_{\mathbf{K}_0}$, so $\mathscr{E}_{\mathbf{k}}$ reduces to $E_{\mathbf{k}} - E_{\mathbf{K}_0}$. We see that $\mathscr{E}_{\mathbf{K_0}} = 0$ so that the excitation spectrum is gapless at the condensation momentum, moreover the square root term of Eq. (8), which is responsible for the phonon excitations, has reflection symmetry across $\mathbf{K}_0$. Equation (8) is the general form of the excitation energies. Before analyzing the features of the excitation spectrum (8) in the regimes $\Omega < \Omega_c$ and $\Omega > \Omega_c$, we determine the effective mass of the particles of our model and also the values of λ, Ω, t, U, and U' that place our system in the SF phase.

Effective masses, superfluidity criterium. The effective masses are the eigenvalues of $\partial^2 E|_{\mathbf{K}_0}$; with $\Omega < \Omega_c$, this matrix is nondiagonal so that the effective masses correspond to motion along a rotated set of orthogonal axis x' and y'. These are

$$m_\pm^* = 2((t^3/\lambda^2)\sin k_0 \tan k_0\{[1 + (\lambda/t)^2] \pm 1\})^{-1}. \quad (10)$$

To give a physical interpretation of Eq. (10), we normalize each quantity choosing λ as the unit of energy and consider the cases $\tilde{t} \equiv t/\lambda \gg 1$, $\tilde{t} \ll 1$. Summarizing:

$$\begin{aligned} &\text{for} \quad \tilde{t} \gg 1: \; m_-^* = \tilde{t}/R, \quad m_+^* = 1/(\tilde{t}R), \\ &\text{for} \quad \tilde{t} \ll 1: \; m_-^* = \tilde{\Omega}/R, \quad m_+^* = \tilde{\Omega}/R, \end{aligned} \quad (11)$$

with $\tilde{\Omega} = \Omega/\lambda$ and $R = [1 - (\Omega/\Omega_c)^2]$. The criterium that we use in order to determine the parameter values ensuring that our system is in a SF phase, and not in a MI phase, comes from the one-component Bose-Hubbard model. There, $m^* \sim 1/t$ and the superfluidity is ensured with $m^* U < 1$ [20]. Considering the same condition $m_\pm^* U < 1$ we see that with $\tilde{t} \ll 1$ the parameter guiding the SF is Ω and not t. Moreover, we see that, with $\Omega \to \Omega_c^-$, the SF is always strongly disfavored. With $\Omega > \Omega_c$, $E_{\mathbf{k}}$ has only one minimum in (0,0) and the effective mass is isotropic $m^* = [2t(1 - \Omega_c/\Omega)]^{-1}$. With $\lambda \to 0$ ($\Omega_c \to 0$), this reduces to the usual result for the standard Bose-Hubbard model $m^* = \frac{1}{2t}$; also in this case with $\Omega \to \Omega_c^+$ SF is disfavored.

The general formula for the sound velocity from Eq. (8) is

$$c_{x,\pm} = \partial_{k_x} a_{\mathbf{k}}\big|_{\mathbf{K}_0} \pm \sqrt{a_{\mathbf{K}_0}\big[\partial_{k_x}^2 (a_{\mathbf{k}} - 2|b_{\mathbf{k}}|)\big|_{\mathbf{K}_0}\big]}. \quad (12)$$

It can be shown that, if $\partial_{k_x}^2 (a_{\mathbf{k}} - 2|b_{\mathbf{k}}|)|_{\mathbf{K}_0} < 0$, $\mathscr{E}_{\mathbf{k}}$ becomes complex around $\mathbf{K}_0$, so looking at the sound velocity is a natural tool to find possible instabilities of $\mathscr{E}_{\mathbf{k}}$.

$\Omega > \Omega_c$, *excitation spectrum, sound velocity, instability.* In this case $E_{\mathbf{k}}$ features only a minimum at $\mathbf{k} = 0$, so that $\mathbf{K}_0 = 0$ and $\beta_{\mathbf{0}} = 0$. Then,

$$\mathscr{E}_{\mathbf{k}} = \sqrt{\big[E_{\mathbf{k}} - E_{\mathbf{K}_0} + nU(2\alpha_{\mathbf{k}}^2 - 1) + nU'(1 - \alpha_{\mathbf{k}}^2)\big]^2 - n^2 U^2 \alpha_{\mathbf{k}}^4}.$$

A phonon excitation appears in the limit $\mathbf{k} \to 0$ with sound velocity $c \equiv c_x = c_y$:

$$c = \sqrt{2nU[t - 2(\lambda^2/\Omega) - n(\lambda/\Omega)^2(U - U')]}. \quad (13)$$

When $\lambda \to 0$, $c \to \sqrt{2nUt}$, which is the one-component Bose-Hubbard result for c. Approaching the critical value $\Omega \to \Omega_c^+$, both $\mathscr{E}_{\mathbf{k}}$ and c become imaginary under the condition $n(U - U')/2\Omega > (\Omega/\Omega_c - 1)$. This is one of our main results. The imaginary eigenvalues are indicative of a dynamical instability for the superfluid phase on an optical lattice when including SOC. A physical interpretation of this instability is related to the real underlying two-component $\{A, B\}$ system that seems to enter a phase-separation regime [17]. This can be understood by considering the left panel of Fig. 2 where we plot the relative population of the atomic species A (pseudospin up) and B (pseudospin down) in the condensate. Due to the Zeeman coupling, Eq. (1), the atoms of the species A are energetically favored respect to the species B in the condensed phase. The two atomic species coexist in the condensate until Ω reaches the value Ω_c at which point the species B is expelled from the BEC.

$\Omega < \Omega_c$, *sound velocities, instability, roton excitation.* In this case $E_{\mathbf{k}}$ has four degenerate minima localized at $(\pm k_0, \pm k_0)$; without loss of generality we assume that the condensation momentum is equal to $\mathbf{K}_0 = (k_0, k_0)$. The excitation spectrum has a cusp at $\mathbf{K}_0$, proving the existence of phonons. The slope differs slightly on the positive and negative direction of the k_x axis, respectively k_y axis; this is associated with the anisotropy of the effective masses. The sound velocity $c_{x,\pm} = c_{y,\pm}$ is given

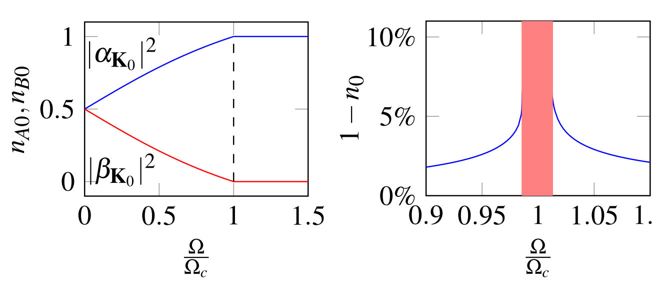

FIG. 2. (Color online) (Left) The relative population of A and B atoms in the condensate phase: $n_{A0} \equiv |\alpha_{\mathbf{K}_0}|^2$, $n_{B0} \equiv |\beta_{\mathbf{K}_0}|^2$. The dashed vertical line corresponds to the value of Ω_c, with $\Omega > \Omega_c$, $n_{A0} = 1$, $n_{B0} = 0$. (Right) Quantum depletion (percentage of the total particle number): the depletion grows approaching the instability region, red (gray), but it is nevertheless small. The parameters of the plot are $\tilde{t} = 1$, $\tilde{\Omega}_c = 2$, $n = 1$, $\tilde{U} = 0.1$, $\tilde{U}' = 0.05$, $N_s = 10^4$.

in the footnote [21], its structure is in agreement with Ref. [14] that considered the continuum case (no optical lattice). From the explicit analytical form of the sound velocity it is possible to determine the values of Ω such that $c_{x,\pm}$ becomes complex which is a sufficient condition for the excitation spectrum to become dynamically unstable. We consider two regimes $\tilde{t} > 1$ and $\tilde{t} < 1$:

$$\begin{aligned} &\tilde{t} > 1 : \tilde{\Omega}_c(1 - \tilde{t}^{-2}/8) < \tilde{\Omega} < \tilde{\Omega}_c, \\ &\tilde{t} < 1 : \tilde{\Omega}_c[1 - n(\tilde{t}/8)(\tilde{U} - \tilde{U}')] < \tilde{\Omega} < \tilde{\Omega}_c. \end{aligned} \tag{14}$$

Thus, just as in the case $\Omega > \Omega_c$, an instability appears when Ω is close to the critical value Ω_c. In addition to the phonon minimum occurring at the condensation momentum, a peculiar feature resulting from the presence of spin-orbit coupling is the presence of additional roton minima. Such objects are absent in multicomponent Bose-Einstein condensates without spin-orbit interactions and may be understood as a consequence of the degenerate nature of the minima in the excitation spectrum $E_{\mathbf{k}}$ without interactions. We find that the roton gaps are *not* degenerate in spite of the single-particle-spectrum minima being degenerate. The excitation spectrum (8) has the usual phonon minimum localized at $\mathbf{K}_0$ whereas we find that the positions of the roton minima are close to the positions of the degenerate minima of the single-particle spectrum as long as one considers weak interaction parameters U and U'. In fact, discarding the second-order terms in U and U', Eq. (8) approximately reduces to $a_{\mathbf{k}}$ far from the condensation momentum $\mathbf{K}_0$. With $\mathbf{K}_0 = (k_0,k_0)$, the positions of the roton excitations are then $(k_0,-k_0)$, $(-k_0,k_0)$, $(-k_0,-k_0)$. The roton gaps $\Delta(\mathbf{k})$ are

$$\begin{aligned} \Delta_\perp &\equiv \Delta(k_0,-k_0) = \Delta(-k_0,k_0) = nU\left(2\alpha_{\mathbf{K}_0}^4 - 2\alpha_{\mathbf{K}_0}^2 + 1\right), \\ \Delta_\parallel &\equiv \Delta(-k_0,-k_0) = nU - n(U+U')2\alpha_{\mathbf{K}_0}^2\left(1-\alpha_{\mathbf{K}_0}^2\right). \end{aligned} \tag{15}$$

All gaps are always positive as long as $U > U'$, which is the regime we are considering (plane-wave phase). As seen, there exist two types of gaps $\Delta_\perp$ and $\Delta_\parallel$: one gap for the roton excitations closest to the condensation momentum ($\Delta_\perp$) and one gap for the roton excitation farthest away from it ($\Delta_\parallel$). The degeneracy of the minima in the noninteracting case is partially lifted when adding interactions U and U'.

Quantum depletion. The BEC depletion at a temperature T is the average relative number of particles not belonging to the BEC: $1 - n_0 = (1/N)\sum_{\mathbf{k}\neq\mathbf{K}_0}\langle d_{\mathbf{k}}^\dagger d_{\mathbf{k}}\rangle$, the operators $d_{\mathbf{k}}$ as in Eq. (6), $n_0 \equiv \langle d_{\mathbf{K}_0}^\dagger d_{\mathbf{K}_0}\rangle/N$. At $T = 0$ only the quantum fluctuations contribute to the depletion. Performing a basis change from $d_{\mathbf{k}}$ to the quasiparticle operators $C_{\mathbf{k}}$ (see, e.g., Sec. 4 in Ref. [19]) it allows us to obtain

$$1 - n_0 = \sum_{\mathbf{k}\neq\mathbf{K}_0}(2N)^{-1} \times \left(\left|a_{\mathbf{k}}+a_{2\mathbf{K}_0-\mathbf{k}}\right| \Big/ \sqrt{\left(a_{\mathbf{k}}+a_{2\mathbf{K}_0-\mathbf{k}}\right)^2 - 16|b_{\mathbf{k}}|^2} - 1\right).$$

Inside the instability region, the above expression of quantum depletion loses its meaning because the sum above becomes complex. In the right panel of Fig. 2, we present a numerical evaluation of the quantum depletion: the depletion increases slightly upon approaching the dynamical unstable region but nevertheless remains small for a system of finite size. We also numerically evaluate the depletion as a function of t/λ with Ω close to the instability region, both on the left and right side, and found that it is always less than 10%. In the thermodynamic limit, the BEC does not exist at the edges of the instability region but the quantum depletion rapidly decreases in the neighborhood of the edges in such a way the instability region is still well defined.

Summary. In summary, we established a phase diagram for the superfluid state of a SOC BEC in the presence of a 2D square optical lattice. We identified an instability regime in a window of values for the Zeeman-coupling Ω near a critical value Ω_c where the excitation energies become complex. We also derived analytical expressions for the roton excitations appearing in the system, and showed that there are two types of inequivalent roton gaps.

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[21]
$$\begin{aligned}
\Delta U &\equiv U - U', \\
\partial_{k_x} a_{\mathbf{k}}\big|_{\mathbf{K}_0} &= -n\Delta U \sin 4\theta_{\mathbf{k}_0} \partial_{k_x}\theta\big|_{\mathbf{k}_0}, \\
c_{x,\pm} &= \partial_{k_x} a_{\mathbf{k}}\big|_{\mathbf{K}_0} \pm \left(nU - \frac{n}{2}\Delta U \sin^2 2\theta_{\mathbf{k}_0}\right)^{\frac{1}{2}} \left[\partial_{k_x}^2 E\big|_{\mathbf{k}_0} - n\Delta U \left(\frac{1}{2}\partial_{k_x}^2\theta\big|_{\mathbf{k}_0} \sin 4\theta_{\mathbf{k}_0} + 2\partial_{k_x}\theta\big|_{\mathbf{k}_0}^2 \cos 4\theta_{\mathbf{k}_0}\right)\right]^{\frac{1}{2}}, \\
\partial_{k_x}\theta_{\mathbf{k}}\big|_{\mathbf{k}_0} &= \frac{t/\Omega}{1 + 2\,(2\lambda/\Omega)^2}\sqrt{\left[1 + \frac{1}{2}\left(\frac{\Omega}{2\lambda}\right)^2\right]\left[1 + 2\left(\frac{t}{\lambda}\right)^2\right]}, \\
\partial_{k_x}^2\theta_{\mathbf{k}}\big|_{\mathbf{k}_0} &= \frac{\Omega t^2}{8\lambda^3}\left(\frac{3}{\sqrt{2}\sin_{k_0}} - \frac{\sqrt{2}}{\sin_{k_0}\cos_{k_0}^2} - \frac{4t^2\sin_{k_0}}{\sqrt{2}\lambda^2\cos_{k_0}^2}\right).
\end{aligned}$$