Chirality asymptotic behavior and non-Markovianity in quantum walks on a line

Margarida Hinarejos,[1] Carlo Di Franco,[2] Alejandro Romanelli,[3] and Armando Pérez[1]
[1]*Departament de Física Teòrica and IFIC, Universitat de València–CSIC Dr. Moliner 50, 46100-Burjassot, Spain*
[2]*Centre for Theoretical Atomic, Molecular and Optical Physics, School of Mathematics and Physics, Queen's University, Belfast BT7 1NN, United Kingdom*
[3]*Instituto de Física, Facultad de Ingeniería, Universidad de la República, C.C. 30, C.P. 11000, Montevideo, Uruguay*
(Received 15 January 2014; published 28 May 2014)

We investigate the time evolution of the chirality reduced density matrix for a discrete-time quantum walk on a one-dimensional lattice. The matrix is obtained by tracing out the spatial degree of freedom. We analyze the standard case, without decoherence, and the situation in which decoherence appears in the form of broken links in the lattice. By examining the trace distance for possible pairs of initial states as a function of time, we conclude that the evolution of the reduced density matrix is non-Markovian, in the sense defined by Breuer, Laine, and Piilo [Phys. Rev. Lett. **103**, 210401 (2009)]. As the level of noise increases, the dynamics approaches a Markovian process. The highest non-Markovianity corresponds to the case without decoherence. The reduced density matrix tends always to a well-defined limit that we calculate, but only in the decoherence-free case is this limit nontrivial.

DOI: 10.1103/PhysRevA.89.052330 PACS number(s): 03.67.Ac, 03.65.Yz

I. INTRODUCTION

The Markov approximation is a valuable and powerful tool for studying the dynamics of an open system interacting with its environment. The random walk is an example of a classical Markovian process that has found applications in many fields. In quantum mechanics, important physical scenarios leading to decoherence can be analyzed by means of simple Markovian models. For instance, in quantum optics, the time evolution of an open system characterized by a nonunitary behavior can be described by a master equation written generally in the form of a Lindblad equation [1].

In quantum information theory, the discrete-time quantum walk (QW) on a line has been studied as a natural generalization of the classical random walk [2]. It has been shown in detail how the unitary quantum-mechanical evolution of the QW can be separated into Markovian and interference terms [3,4]. The Markovian terms responsible for the diffusion obey a master equation, while the interference terms are needed in order to preserve the unitary character of the evolution. The approach provides an intuitive framework which becomes useful for analyzing the behavior of quantum systems in which decoherence plays a central role. In other words, it shows in a transparent form that the primary effect of decoherence here is to make the interference terms negligible in the evolution equation.

It is clearly important to find a way to evaluate how non-Markovian a quantum system is. Reference [5] has proposed a general measure for the degree of non-Markovian behavior in open quantum systems. It is based on the trace distance, which quantifies the distinguishability of quantum states, and it can be interpreted in terms of the information flow between the open system and its environment. The measure takes nonzero values whenever there is a flow of information from the environment back to the open system, and it has already been used in different contexts [6].

On the other hand, the asymptotic behavior of the QW has been recently investigated focusing on the chirality reduced density matrix, obtained when the position degree of freedom is traced out [7–10]. This matrix has a long-time limit that depends on the initial conditions. One finds thus the following situation: the dynamical evolution of the QW is a unitary process, however the asymptotic behavior of the reduced density matrix has some properties that are shared by some diffusive Markovian processes. This allows to amalgamate concepts such as thermodynamic equilibrium with the idea of a system that follows a unitary evolution. References [8,10] have developed a thermodynamic theory to describe the QW equilibrium between the position and chirality degrees of freedom; it is possible to introduce the concept of temperature for an isolated quantum system that evolves in a composite Hilbert space (i.e., the tensor product of several subspaces). Additionally, Ref. [8] has shown that the transient behavior toward thermodynamic equilibrium is described by a master equation with a time-dependent population rate.

In this paper, we study the asymptotic QW behavior with and without decoherence, and we exploit the measure proposed in Ref. [5] to evaluate its non-Markovianity. We show that, without decoherence, the reduced density matrix dynamics has a clear time dependence and a well-defined limit that can be calculated in terms of the initial conditions. This corresponds, when comparing the evolution of two different initial states, to a reduced asymptotic trace distance. The introduction of decoherence translates, as far as the long-time limit is concerned, into a trivial result, since all states evolve toward the maximally decohered state (proportional to the identity matrix).

The evolution during the first time steps of the QW features an interesting phenomenon, i.e., the presence of oscillations in the trace distance between pairs of states, which is interpreted as a signature of a non-Markovian process. Such oscillations occur both with and without decoherence, even though they become more and more attenuated as the level of noise increases. In agreement with our observations for the asymptotic limit, the trace distance tends to zero when decoherence affects the system.

This paper is organized as follows. In Sec. II we introduce the basic features of the QW, and we obtain the asymptotic

 ©2014 American Physical Society

limit for the reduced density matrix in the chiral space. In Sec. III we recast the QW in the form of a map equation for the generalized chiral distribution (GCD), i.e., the diagonal terms of the reduced density matrix. The asymptotic limit under the effect of decoherence is addressed in Sec. IV. In Sec. V we discuss the short-time behavior, where non-Markovian effects clearly manifest as oscillations of the trace distance between pairs of states. Section VI summarizes our main results.

II. ASYMPTOTIC REDUCED DENSITY MATRIX FOR THE QW

The standard QW corresponds to the discrete (both in time and in space) evolution of a one-dimensional quantum system (the walker) in a direction that depends on an additional degree of freedom, the chirality, with two possible states: "left" $|L\rangle$ or "right" $|R\rangle$. The global Hilbert space of the system is the tensor product $H_s \otimes H_c$. H_s is the Hilbert space associated with the motion on the line, and it is spanned by the basis $\{|x\rangle : x \in \mathbb{Z}\}$. H_c is the chirality (or coin) Hilbert space, defined as a two-dimensional space that can correspond, for example, to a spin-1/2 particle, or to a two-level energy system. Let us call T_- (T_+) the operators in H_s that move the walker one site to the left (right), and $|L\rangle\langle L|$ and $|R\rangle\langle R|$ the chirality projector operators in H_c. We consider the unitary transformation

$$U(\theta) = \{T_- \otimes |L\rangle\langle L| + T_+ \otimes |R\rangle\langle R|\} \circ \{I \otimes K(\theta)\}, \quad (1)$$

where $K(\theta) = \sigma_z e^{-i\theta\sigma_y}$, $\theta \in [0,\pi/2]$ is a parameter defining the bias of the coin toss, I is the identity operator in H_s, and σ_y and σ_z are Pauli matrices acting on H_c. The effect of the unitary operator $U(\theta)$ on the state of the system in one time step τ is $|\Psi(t+\tau)\rangle = U(\theta)|\Psi(t)\rangle$. The state vector can be expressed as the spinor

$$|\Psi(t)\rangle = \sum_{x=-\infty}^{\infty} \begin{bmatrix} a_x(t) \\ b_x(t) \end{bmatrix} |x\rangle, \quad (2)$$

where the upper (lower) component is associated with the left (right) chirality. The unitary evolution implied by Eq. (1) can be written as the map

$$a_x(t+\tau) = a_{x+1}(t)\cos\theta + b_{x+1}(t)\sin\theta, \quad (3)$$

$$b_x(t+\tau) = a_{x-1}(t)\sin\theta - b_{x-1}(t)\cos\theta. \quad (4)$$

In this paper, we select $\theta = \frac{\pi}{4}$ to obtain an unbiased coin (Hadamard coin).

The density matrix of the quantum system is

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|. \quad (5)$$

To study the QW time dependence on the initial conditions, we take the initial state of the walker as sharply localized at the origin with arbitrary chirality, thus

$$|\Psi(0)\rangle = |0\rangle \otimes |\Phi_0\rangle. \quad (6)$$

Here

$$|\Phi_0\rangle = \begin{pmatrix} \cos\frac{\gamma}{2} \\ e^{i\varphi}\sin\frac{\gamma}{2} \end{pmatrix}, \quad (7)$$

with $\gamma \in [0,\pi]$ and $\varphi \in [0,2\pi]$ defining a point on the unit three-dimensional Bloch sphere. In this case, the initial density matrix is

$$\rho(0) = |0\rangle\langle 0| \otimes |\Phi_0\rangle\langle\Phi_0|, \quad (8)$$

where

$$|\Phi_0\rangle\langle\Phi_0| = \begin{pmatrix} \cos^2\frac{\gamma}{2} & \frac{e^{-i\varphi}}{2}\sin\gamma \\ \frac{e^{i\varphi}}{2}\sin\gamma & \sin^2\frac{\gamma}{2} \end{pmatrix}. \quad (9)$$

To use the affine map approach [11,12], Eq. (9) can be transformed to express the two-by-two matrix as a four-dimensional column vector, obtaining

$$|\Phi_0\rangle\langle\Phi_0| = r_0 I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \cos\varphi\sin\gamma \\ -\sin\varphi\sin\gamma \\ \cos\gamma \end{pmatrix}, \quad (10)$$

with σ_i ($i = 1,2,3$) the Pauli matrices, and

$$r_i = \tfrac{1}{2}\mathrm{tr}(|\Phi_0\rangle\langle\Phi_0|\sigma_i). \quad (11)$$

The reduced density operator is defined as

$$\rho_c(t) = \mathrm{tr}_s[\rho(t)]) = \sum_{x=-\infty}^{\infty} \langle x|\rho(t)|x\rangle, \quad (12)$$

where the partial trace is taken over the positions. Following the method introduced in Ref. [11] and generalized in Ref. [12], Eq. (12) can be transformed into

$$\rho_c(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} L_k^t |\Phi_0\rangle\langle\Phi_0|, \quad (13)$$

with L_k the superoperator defined as

$$L_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sin 2k & \cos 2k \\ 0 & 0 & -\cos 2k & \sin 2k \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (14)$$

To obtain the eigenvalues of L_k, it is necessary to find the eigenvalues of the following associated matrix:

$$M_k = \begin{pmatrix} 0 & \sin 2k & \cos 2k \\ 0 & -\cos 2k & \sin 2k \\ 1 & 0 & 0 \end{pmatrix}. \quad (15)$$

The eigenvalues of Eq. (15) are

$$\lambda_1 = 1, \ \lambda_2 = e^{i(\alpha+\pi)}, \ \lambda_3 = e^{-i(\alpha+\pi)}, \quad (16)$$

where

$$\cos\alpha = \tfrac{1}{2}(1+\cos 2k) = (\cos k)^2. \quad (17)$$

The corresponding eigenvectors are

$$\vec{v_1} = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \frac{\sqrt{2}\cos k}{\sqrt{3+\cos 2k}}\begin{pmatrix} 1 \\ (1-\cos 2k)/\sin 2k \\ 1 \end{pmatrix}, \quad (18)$$

$$\vec{v_2} = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \frac{1}{N_2}\begin{pmatrix} e^{i(\alpha+\pi)} \\ -(e^{i(\alpha+\pi)} - 2\cos 2k)/(2\sin 2k) \\ 1 \end{pmatrix}, \quad (19)$$

$$\vec{v_3} = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \frac{1}{N_3}\begin{pmatrix} e^{-i(\alpha+\pi)} \\ -(e^{-i(\alpha+\pi)} - 2\cos 2k)/(2\sin 2k) \\ 1 \end{pmatrix}, \quad (20)$$

with N_2 and N_3 normalization factors. It is now straightforward to obtain $(L_k)^t$ using the diagonal expression for L_k, that is,

$$L_k = B\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{it(\alpha+\pi)} & 0 \\ 0 & 0 & 0 & e^{-it(\alpha+\pi)} \end{pmatrix} B^\dagger. \quad (21)$$

Here, B is the eigenvector matrix,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{11} & v_{12} & v_{13} \\ 0 & v_{21} & v_{22} & v_{23} \\ 0 & v_{31} & v_{32} & v_{33} \end{pmatrix}, \quad (22)$$

and $B^\dagger$ is its conjugate transpose. Substituting Eq. (22) into Eq. (21) and exploiting the stationary phase theorem to neglect the oscillatory terms $e^{\pm it(\alpha+\pi)}$ when time goes to infinity, one finds the following asymptotic equation:

$$(L_k)^t \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & |v_{11}|^2 & v_{11}v_{21}^* & v_{11}v_{31}^* \\ 0 & v_{21}v_{11}^* & |v_{21}|^2 & v_{21}v_{31}^* \\ 0 & v_{31}v_{11}^* & v_{31}v_{21}^* & |v_{31}|^2 \end{pmatrix}. \quad (23)$$

The reduced density matrix in the asymptotic regime, $\tilde{\rho}_c$, can be calculated using Eq. (13) as

$$\tilde{\rho}_c \equiv \lim_{t\to\infty} \rho_c(t) = \lim_{t\to\infty} \int_{-\pi}^{\pi} \frac{dk}{2\pi} L_k^t |\Phi_0\rangle\langle\Phi_0|. \quad (24)$$

To work out this expression, it is necessary to solve the following integrals:

$$\int_{-\pi}^{\pi} \frac{|v_{11}|^2}{2\pi} dk = 1 - \frac{1}{\sqrt{2}}, \quad (25)$$

$$\int_{-\pi}^{\pi} \frac{|v_{21}|^2}{2\pi} dk = \sqrt{2} - 1, \quad (26)$$

$$\int_{-\pi}^{\pi} \frac{|v_{31}|^2}{2\pi} dk = 1 - \frac{1}{\sqrt{2}}, \quad (27)$$

$$\int_{-\pi}^{\pi} \frac{v_{11}v_{21}^*}{2\pi} dk = \int_{-\pi}^{\pi} \frac{v_{11}v_{31}^*}{2\pi} dk = \int_{-\pi}^{\pi} \frac{v_{21}v_{31}^*}{2\pi} dk = 0. \quad (28)$$

Therefore, we obtain analytically the QW reduced density matrix in the asymptotic regime,

$$\tilde{\rho}_c = \begin{pmatrix} r_0 \\ \left(1 - \frac{1}{\sqrt{2}}\right)(r_1 + r_3) \\ (\sqrt{2} - 1)r_2 \\ \left(1 - \frac{1}{\sqrt{2}}\right)(r_1 + r_3) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \left(1 - \frac{1}{\sqrt{2}}\right)(\cos\varphi \sin\gamma + \cos\gamma) \\ (\sqrt{2} - 1)\sin\varphi \sin\gamma \\ \left(1 - \frac{1}{\sqrt{2}}\right)(\cos\varphi \sin\gamma + \cos\gamma) \end{pmatrix}. \quad (29)$$

Going back to the 2×2 matrix formalism, the reduced density matrix in the asymptotic regime can finally be written as

$$\tilde{\rho}_c = \begin{pmatrix} \Pi_L & Q_0 \\ Q_0^* & \Pi_R \end{pmatrix}, \quad (30)$$

where

$$\Pi_L = \frac{1}{2}\left[1 + \left(1 - \frac{1}{\sqrt{2}}\right)(\cos\varphi \sin\gamma + \cos\gamma)\right],$$
$$\Pi_R = \frac{1}{2}\left[1 - \left(1 - \frac{1}{\sqrt{2}}\right)(\cos\varphi \sin\gamma + \cos\gamma)\right],$$
$$Q_0 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)[(\cos\varphi \sin\gamma + \cos\gamma) - i\sqrt{2}\sin\varphi \sin\gamma]. \quad (31)$$

III. QW MAP EQUATION

The aim of this section is to study the populations of the reduced density matrix for the standard (decoherence-free) QW. Using Eqs. (2), (5), and (12), this matrix is expressed as

$$\rho_c(t) = \begin{pmatrix} P_L(t) & Q(t) \\ Q^*(t) & P_R(t) \end{pmatrix}, \quad (32)$$

where

$$P_L(t) = \sum_{k=-\infty}^{\infty} |a_k(t)|^2, \quad (33)$$

$$P_R(t) = \sum_{k=-\infty}^{\infty} |b_k(t)|^2, \quad (34)$$

$$Q(t) \equiv \sum_{k=-\infty}^{\infty} a_k(t) b_k^*(t). \quad (35)$$

The global chirality distribution (GCD) is defined as the distribution

$$\begin{bmatrix} P_L(t) \\ P_R(t) \end{bmatrix}, \quad (36)$$

with $P_R(t) + P_L(t) = 1$.

It is shown in Ref. [7] that the GCD satisfies the following map:

$$\begin{bmatrix} P_L(t+1) \\ P_R(t+1) \end{bmatrix} = \begin{pmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{pmatrix} \begin{bmatrix} P_L(t) \\ P_R(t) \end{bmatrix} + \mathrm{Re}\,[Q(t)] \sin 2\theta \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (37)$$

We observe that, if the "interference term"

$$\mathrm{Re}[Q(t)] \sin 2\theta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (38)$$

in Eq. (37) is neglected, the time evolution of the GCD could be described by a classical Markovian process. The two-dimensional matrix

$$\begin{pmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{pmatrix} \quad (39)$$

can be interpreted as the corresponding transition probability matrix for a Markov chain, since it satisfies the necessary requirements: all its elements are positive, and the sum over the elements of any column or row is equal to 1. However, $Q(t)$ [together with $P_L(t)$ and $P_R(t)$] is a time-dependent function. This implies that the map defined by Eq. (37) does not correspond to a classical Markovian process. It is important to stress that, here, we are just analyzing the GCD (i.e., the left and right populations of the chiral degree of freedom) in terms of classical Markovian behavior. A study of quantum Markovianity requires, on the other hand, that we consider the evolution of the complete matrix $\rho_c(t)$. This will be performed in Sec. V.

In spite of the time dependence manifested by Eq. (37), the GCD does possess a long-time limiting value, as obtained in the previous section. Equation (37) can be used to derive a consistency condition relating Π_L, Π_R, and Q_0, by taking the limit $t \to \infty$. One then finds

$$\begin{bmatrix} \Pi_L \\ \Pi_R \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + 2\,\mathrm{Re}(Q_0)/\tan\theta \\ 1 - 2\,\mathrm{Re}(Q_0)/\tan\theta \end{bmatrix}. \quad (40)$$

When $\theta = \pi/4$, Eq. (40) agrees with the expressions given by Eq. (31). This interesting result shows that the long-time probability to find the system with left or right chirality only depends on the asymptotic interference term. Although the dynamical evolution of the QW is unitary, the evolution of its GCD has an asymptotic limit, a feature that is characteristic of a diffusive behavior. The situation is even more surprising if we compare our case with the case of the QW on finite graphs [2], where it is shown that there is no convergence to a stationary distribution. To quantify how much the asymptotic limit keeps track of the initial state, we use the trace distance

$$D(\rho_1 - \rho_2) = \tfrac{1}{2}\mathrm{tr}|\rho_1 - \rho_2|,$$

which gives us a measure for the distinguishability of two quantum states. Here, $|\rho| = \sqrt{\rho^\dagger \rho}$. We calculate this quantity for two reduced density matrices (in the chiral space) that correspond to two different initial states of Eq. (9). Following the notation defined in Eq. (32), we write

$$\rho_1(t) - \rho_2(t) = \begin{pmatrix} P_{1L}(t) - P_{2L}(t) & Q_1(t) - Q_2(t) \\ Q_1^*(t) - Q_2^*(t) & P_{1R}(t) - P_{2R}(t) \end{pmatrix}. \quad (41)$$

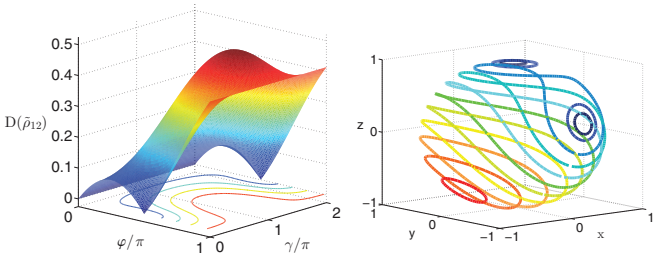

FIG. 1. (Color online) Left panel: Asymptotic trace distance as a function of the angles γ and φ, representing the initial conditions of ρ_2. The initial conditions of ρ_1 are given by $\gamma = 0$. Right panel: The contour levels corresponding to the left panel are mapped to the Bloch sphere, using the same color convention.

We now evaluate the trace distance between the asymptotic reduced density matrices for the QW without decoherence, while in the next section we extend the investigation to the scenario that takes into account decoherence introduced by broken links. Considering two different initial conditions, the difference between their asymptotic reduced density matrices is

$$\tilde{\rho}_{12} = \begin{pmatrix} \Pi_{1L} - \Pi_{2L} & Q_{10} - Q_{20} \\ Q_{10}^* - Q_{20}^* & \Pi_{1R} - \Pi_{2R} \end{pmatrix}. \quad (42)$$

Therefore, the distance between the asymptotic reduced density matrices is defined as

$$D(\tilde{\rho}_{12}) = \tfrac{1}{2}\mathrm{tr}|\tilde{\rho}_{12}|. \quad (43)$$

After some algebra, taking into account Eq. (40) with $\theta = \pi/4$, the asymptotic trace distance can be expressed, in terms of the initial conditions, as

$$D(\tilde{\rho}_{12}) = \sqrt{2[\mathrm{Re}(Q_{10} - Q_{20})]^2 + [\mathrm{Im}(Q_{10} - Q_{20})]^2}. \quad (44)$$

Here, $\mathrm{Re}(Q_0)$ [$\mathrm{Im}(Q_0)$] is the real (imaginary) part of Q_0, and Q_0 is given by Eq. (31). To study the dependence on the initial conditions, we consider the evolution of pairs of independent states under the QW map. We fix the initial conditions for the first state and study Eq. (44) by considering different points on the Bloch sphere as the initial conditions for the second state.

Figures 1 and 2 show our results in two nonequivalent scenarios. As can be seen from these figures, the asymptotic trace distance has a nontrivial behavior as a function of the second state, once the first one is fixed. The left panel gives an idea of how much the trace distance is reduced (the minimum reduction being of the order of 1/2 in the case represented in Fig. 1, whereas lower values are reached for the parameters

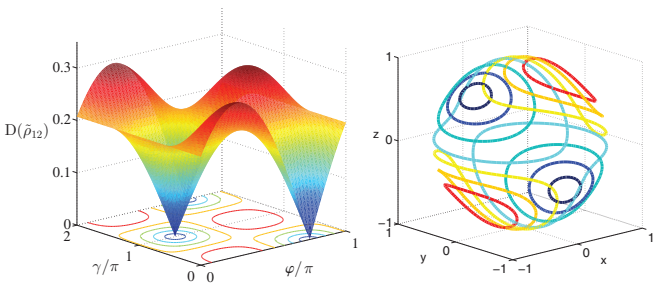

FIG. 2. (Color online) Same as Fig. 1, but now the initial conditions for ρ_1 are given by $\gamma = \pi/4$ and $\varphi = \pi$.

that correspond to Fig. 2). The contour levels can be mapped to the points of the Bloch sphere associated with the second state (right panel), thus providing a closer relationship to physical states. As we see by comparing the two figures, changing the first state does not translate into a simple rotation of the Bloch sphere representation, the reason being that the coin operator does not commute with arbitrary rotations.

IV. ASYMPTOTIC DENSITY MATRIX WITH DECOHERENCE

We now study the dynamics of the reduced density matrix for the QW under the effect of decoherence. We exploit the model, known as *broken links*, that was proposed for the first time in Ref. [13] and analyzed in the frame of the previous section in Ref. [12]. It induces decoherence in both degrees of freedom, i.e., coin and position. Similar results to those presented here can also be found for other decoherence models.

At each time step t, the state of the links in the line is defined. Each link has a probability p of breaking in a given time step. Clearly, for $p = 0$, the ideal decoherence-free QW is recovered. During the movement stage, if the walker is in a site with both the links on the right and left broken (this happens with probability p^2), the walker does not move. With probability $(1-p)^2$, both links are not broken and, in this case, the evolution normally occurs. With probability $p(1-p)$, only one link is broken and the walker is *forced* to move to the other direction. Notice, however, that the limit $p \longrightarrow 1$ implies that the walker is forced to stay at the initial position, since the links with neighboring sites are broken with probability 1 (or close to 1), and only the coin operator acts. This limit is no longer connected with the QW, and, for this reason, we restrict ourselves to small values of p.

Reference [12] obtains the superoperator L_k that determines the dynamical evolution of the QW with broken links,

$$L_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e & f+p^2 \\ 0 & 0 & p^2-f & e \\ 0 & 1-2p & -2g & -2h \end{pmatrix}, \tag{45}$$

where

$$\begin{aligned} e &= (1-p)^2 \sin 2k, \\ f &= (1-p)^2 \cos 2k, \\ g &= p(1-p)\sin k, \\ h &= p(1-p)\cos k. \end{aligned} \tag{46}$$

The dynamics of the reduced density matrix is again described by Eq. (24), but now L_k is given by Eq. (45). Redefining M_k as

$$M_k = \begin{pmatrix} 0 & e & f+p^2 \\ 0 & p^2-f & e \\ 1-2p & -2g & -2h \end{pmatrix}, \tag{47}$$

its eigenvalues $\{\lambda_i : i = 1,2,3\}$ satisfy $|\lambda_i| < 1$ for $0 < p < 1$. If A is the matrix constructed from the eigenvectors of M_k, and Λ is the diagonal matrix with the corresponding eigenvalues as elements, it is straightforward to prove that

$$\lim_{t\to\infty} M_k^t = \lim_{t\to\infty} (A \Lambda^t A^\dagger) = 0. \tag{48}$$

In this case, Eq. (24) gives us

$$\tilde{\rho}_c = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{49}$$

In other words, going back to the formalism of 2×2 matrices, the reduced density matrix in the asymptotic regime is simply

$$\tilde{\rho}_c = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{50}$$

regardless of the initial state. Thus, in the presence of noise, the trace distance between any two different initial states approaches zero, i.e.,

$$\lim_{t\to\infty} D(\rho_1(t) - \rho_2(t)) = 0. \tag{51}$$

V. SHORT-TIME BEHAVIOR

So far we have investigated the properties of the reduced density matrix in the long-time regime. We have obtained a well-defined limit for both the decoherence-free scenario and the case with decoherence. We now discuss the situation in which one considers not the asymptotic limit but a finite number of steps in the QW. Our study, as before, is focused on the time evolution of $D(\rho_1 - \rho_2)$.

The measure of quantum non-Markovianity given by Ref. [5] is based on the rate of change [14] of the trace distance,

$$\sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t) - \rho_2(t)). \tag{52}$$

Figure 3 shows the time evolution of the trace distance, both for the whole density matrices and for the corresponding reduced density matrices. We have taken, as initial conditions, the pair of states giving the maximum value of the measure (see below). If one starts from a different pair of states, the curves look qualitatively similar, although the overall scale is smaller.

We have considered various values of the decoherence parameter p, the case $p = 0$ corresponding to the absence of

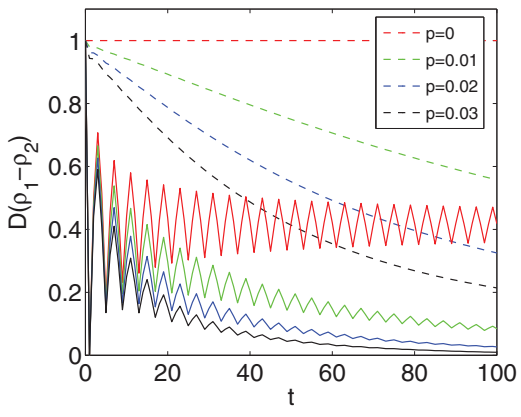

FIG. 3. (Color online) Trace distance, as a function of the number of time steps, between the whole density matrices (dashed lines) and between the corresponding reduced density matrices (solid lines). The initial state $\rho_1(0)$ is defined by Eq. (9) with $\gamma = 0$, while $\rho_2(0)$ with $\gamma = \pi$. Different values of the decoherence parameter p have been considered.

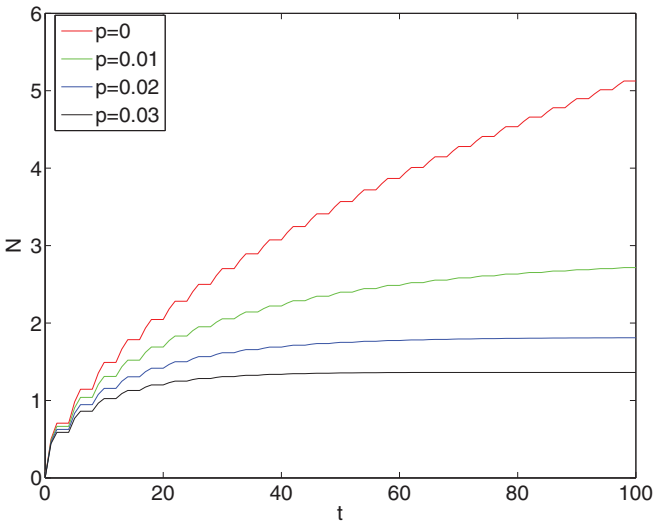

FIG. 4. (Color online) Contribution to the non-Markovianity measure, as a function of the number of time steps, evaluated for the pair of initial states $\rho_1(0)$ and $\rho_2(0)$ that maximizes the integral in Eq. (53). Different values of the decoherence parameter p have been considered.

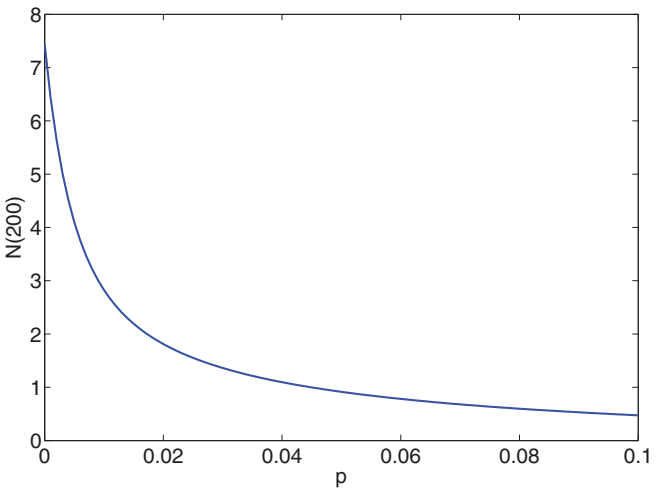

FIG. 5. (Color online) Contribution to the non-Markovianity measure, as a function of the decoherence parameter, calculated for the pair of initial states $\rho_1(0)$ and $\rho_2(0)$ that maximizes the integral in Eq. (53), and for the time interval $[0,200]$.

decoherence. Without decoherence, the QW evolves unitarily, so the trace distance between two total states is preserved. If $p > 0$, the evolution for the total state is clearly Markovian, as indicated by a monotonous decrease in the trace distance (this happens of course for any possible pair of initial states). The reduced density matrices, however, show a completely different behavior. Analyzing first the case $p = 0$, we observe the presence of oscillations. In other words, the trace distance increases during some time intervals, giving a positive value of σ in Eq. (52). As discussed in Ref. [5], this feature is a clear signature of a quantum non-Markovian process. We notice that the amplitude of these oscillations decreases with t. For values $p > 0$, we also observe the presence of such oscillations. In fact, the curves look similar during the first time steps. However, as t increases, the oscillations are more strongly damped than in the decoherence-free case. This effect is even more pronounced for larger values of p. In addition to these features, we also notice that the trace distance goes asymptotically to zero, consistently with our results in Sec. IV.

To obtain a quantitative idea about the degree of the non-Markovianity observed in the previous plots, the authors in Ref. [5] suggest, as a figure of merit, the accumulated area of the trace distance variation for those time intervals where the trace distance is increasing, which amounts to calculating

$$N_{\max} = \max_{\rho_1,\rho_2} \int_{\sigma>0} \sigma(\tau, \rho_{1,2}(0)) d\tau. \tag{53}$$

The maximization is performed over all the possible pairs of initial states $\rho_1(0)$ and $\rho_2(0)$. We have checked numerically that the pair of states maximizing Eq. (53) is the same both with and without decoherence, and corresponds to the north and south poles of the Bloch sphere. We have verified, by performing several numerical simulations, that this does not depend either on the total number of time steps considered or on the value of p. We have therefore plotted in Fig. 4 the value

$$N(t) = \int_{\sigma>0;\tau\in[0,t]} \sigma(\tau, \rho_{1,2}(0)) d\tau, \tag{54}$$

evaluated for this pair of initial states. $N(t)$ can be seen as the contribution to the non-Markovianity measure in the time window $[0,t]$. Even if in the time window allowed by our computational power it is not possible to evaluate (if any) the asymptotic value of $N(t)$ for $t \to \infty$ (i.e., the non-Markovianity measure $N_{\max}$) in the decoherence-free case, the results reported in Fig. 4 give already a very precise picture of how the decoherence affects the degree of non-Markovianity of the coin evolution. The non-Markovianity is stronger as the magnitude of decoherence decreases, with the largest value of its measure corresponding to the decoherence-free case. This feature is clearly shown in Fig. 5, where we have plotted the contribution to the non-Markovianity measure $N_{\max}$ calculated for the time interval $[0,200]$ as a function of p, for the north-south pair of states.

VI. CONCLUSIONS

In this work, we have analyzed both the short-time behavior and the asymptotic limit of the reduced (or chiral) density matrix for the discrete-time QW on a one-dimensional lattice. We have found that this reduced system shows clear features that can be associated with a non-Markovian evolution. First, we have considered the case in which the QW proceeds without decoherence. The chiral density matrix possesses a well-defined asymptotic limit in time. This allows us to calculate the limiting value of the trace distance for pairs of different initial states.

We have studied the effect of decoherence, modeled as the random presence of broken links on the lattice. The case with decoherence possesses a trivial asymptotic limit, since all states converge to (one-half of) the identity, so that the trace distance between pairs of them always tends to zero.

The short-time behavior of the reduced system features quite interesting results. One observes the presence of oscillations in the trace distance for reduced matrices that correspond to two different initial states, a phenomenon that clearly indicates a non-Markovian time evolution. These oscillations appear even when the system does not suffer from decoherence, and they are damped as the number of time steps increases, thus allowing for a convergence of the trace distance in accordance with our previous observations. As the level of noise becomes larger, the amplitude of the oscillations is also reduced for a given number of time steps. In addition, the trace distance approaches zero asymptotically, as already predicted from our long-time analysis. The contribution to the non-Markovianity measure reported in Eq. (53), as a function of the number of time steps, then tends to a value that decreases as the level of decoherence increases.

To conclude, we have found and characterized a non-Markovian behavior for a relatively simple and yet nontrivial system as the coin in a QW on a line. The results that we have presented for the particular model of decoherence chosen here can also be found for other models, such as the one investigated in Ref. [9]. They provide a step forward in our understanding of phenomena such as the transition from unitary to diffusive processes and of the thermalization of quantum systems, and they clearly deserve further attention.

ACKNOWLEDGMENTS

This work has been supported by the Spanish Ministerio de Educación e Innovación, MICIN-FEDER project FPA2011-23897, and "Generalitat Valenciana" Grant No. PROMETEO/2009/128. C.D.F. acknowledges the support from the UK EPSRC, Grant No. EP/G004579/1 under the "New directions for EPSRC research leaders" initiative. A.R. acknowledges the support from PEDECIBA; ANII, FCE-2-211-1-6281, Uruguay. C.D.F. is thankful to A. Pérez and the Universitat de València for the kind hospitality.

[1] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[2] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).

[3] A. Romanelli *et al.*, Phys. Lett. A **313**, 325 (2003).

[4] A. Romanelli *et al.*, Physica A **338**, 395 (2004).

[5] H. P. Breuer, E. M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).

[6] T. J. G. Apollaro, C. Di Franco, F. Plastina, and M. Paternostro, Phys. Rev. A **83**, 032103 (2011); P. Rebentrost and A. Aspuru-Guzik, J. Chem. Phys. **134**, 101103 (2011); P. Haikka, S. McEndoo, G. De Chiara, G. M. Palma, and S. Maniscalco, Phys. Rev. A **84**, 031602(R) (2011); S. Lorenzo, F. Plastina, and M. Paternostro, *ibid.* **87**, 022317 (2013); J.-S. Tang *et al.*, Europhys. Lett. **97**, 10002 (2012); A. Sindona, J. Goold, N. Lo Gullo, S. Lorenzo, and F. Plastina, Phys. Rev. Lett. **111**, 165303 (2013); B.-H. Liu *et al.*, Nat. Phys. **7**, 931 (2011); B.-H. Liu, D.-Y. Cao, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, Sci. Rep. **3**, 1781 (2013); T. J. G. Apollaro *et al.*, arXiv:1311.2045.

[7] A. Romanelli, Phys. Rev. A **81**, 062349 (2010).

[8] A. Romanelli, Phys. Rev. A **85**, 012319 (2012).

[9] A. Pérez and A. Romanelli, J. Computat. Theor. Nanosci. **10**, 1 (2013).

[10] A. Romanelli and G. Segundo, Physica A **393**, 646 (2014).

[11] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 032304 (2003).

[12] M. Annabestani, S. J. Akhtarshenas, and M. R. Abolhassani, Phys. Rev. A **81**, 032321 (2010).

[13] A. Romanelli *et al.*, Physica A **347**, 137 (2005).

[14] For the QW considered here, t takes only discrete values $t \in \mathbb{N}$, therefore the time derivatives and integrals in time have to be understood as finite differences and sums.