Group-representation approach to 1 → N universal quantum cloning machines

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In this work, we revisit the problem of finding an admissible region of fidelities obtained after the application of an arbitrary $1 \to N$ universal quantum cloner which has been recently solved in A. Kay *et al.* [Quant. Inf. Comput **13**, 880 (2013)] from the side of cloning machines. Using group-theory formalism, we show that the allowed region for fidelities can be alternatively expressed in terms of overlaps of pure states with recently found irreducible representations of the commutant $U \otimes U \otimes \cdots \otimes U \otimes U^*$, which gives the characterization of the allowed region where states being cloned are a figure of merit. Additionally, it is sufficient to take pure states with real coefficients only, which makes calculations simpler. To obtain the allowed region, we make a convex hull of possible ranges of fidelities related to a given irrep. Subsequently, two cases, $1 \to 2$ and $1 \to 3$ cloners, are studied for different dimensions of states as illustrative examples.

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I. INTRODUCTION

A basic feature of entanglement is that contrary to classical correlations, it is monogamous. For example, if there is maximal entanglement between two parties, then no other party can be entangled with those parties. More generally, if A is entangled with B and C, then the entanglement must be considerably weaker. This phenomenon gives rise to the fact that quantum information cannot be copied, in contrast to information from the "classical world." In other words, one is not able to copy perfectly an arbitrary quantum state. In terms of monogamy, if one wants to prepare some number of copies of the initially unknown quantum state, the fidelities of cloning cannot all be equal to 1; there is a trade-off. This basic feature is known as the "no-cloning theorem" and was recognized by Wootters and Żurek [1] and Dieks [2].

On the other hand, copying is possible, but the quality of the copy can be very bad sometimes. That is why the goal of finding the ultimate bounds for the quality of copying is an important task. A big effort has been made to solve it, starting from the work of Hillery and Bužek [3]. In general, the subject was studied intensively, both for symmetric (all fidelities are equal) universal quantum cloning machines (UQCM) [4–8] and asymmetric (unequal fidelities) UQCM [8–16]. See also [17,18] for reviews. Nevertheless, for a long time there was a "gap" in studies of quantum cloning: there were no general results on an admissible region of fidelities for universal asymmetric $1 \to N$ quantum cloning machines. The problem has been solved just recently in a series of papers [14,15] from the point of view of cloning machines. In [19] the problem for qubits has been revisited using a group-representation approach, namely, Schur-Weyl duality, where the authors characterized the problem from the side of a cloned state and obtained that regions for fidelities can be obtained from plain and basic calculations of overlaps of pure quantum states with irreps of a symmetric group $S(n)$.

In this paper, we shall consider a $1 \to N$ quantum cloning machine for qudits. Our task it to obtain an admissible region of fidelities after an application of that UQCM. In [19], it has been shown that it is possible to solve the problem for qubits using Schur-Weyl duality. Unfortunately, it works only for that dimension of states, and there is no way to extend it to higher dimensions by using that dualism. Motivated by this, we turn our attention to a recently developed systematic method, decomposition of a partially transposed permutation operator into its irreducible components [20,21], which allows us to omit severe restrictions for the dimensions of states that has appeared previously. However, some modifications are necessary first so that the method suits our problem of cloning machines.

This work is organized as follows. In Sec. II, we formulate our main problem: which values of fidelities are allowed after applying a $1 \to N$ quantum cloning machine for qudits. First, we reformulate the cloning problem in terms of entanglement sharing and recall that cloning fidelity can be connected with a singlet fraction value. Then, we point out that the strategy used in [19] to solve a $1 \to N$ UQCM for qubits is insufficient when one deals with higher dimensions of states d $(d > 2)$ since using Schur-Weyl duality, one is not able to find a maximally entangled state that is invariant under $U \otimes U$ transformations; the only thing that is known is the invariance under $U^* \otimes U$ transformations. That is why the commutant structure of $U^* \otimes U \otimes \cdots \otimes U$ is needed instead of that known from Schur-Weyl duality: $U \otimes U \otimes \cdots \otimes U$. In Sec. II B, mathematical tools from [20] that are necessary to solve the problem are very briefly mentioned, namely, examples of irreducible representations that are needed in our case study problems: $1 \to 2$ and $1 \to 3$ UQCM. Then, we proceed in Sec. II C to show how to connect the method of calculations of the admissible region of fidelities from [19] with mathematical tools from the previous section. It allows us to present in Sec. II D the regions (focusing mainly on our examples of $1 \to 2$ and $1 \to 3$ machines) that are allowed in the problem of $1 \to N$ cloning. Finally, we compare our results in Sec. II E with those obtained in [6], where results for symmetric cloning has been presented, and those from [14,15], where the same problem as ours has been solved but cloning machines were figures of merit. We obtain matching of results in both cases.

II. FORMULATION AND SOLUTION TO THE PROBLEM

A. Background of the problem

Suppose that one has a universal cloning machine that produces clones with cloning fidelities f_{1k}, where $k \in 2,3\dots n$ and the general admissible region of fidelities is the figure of merit. The question that one can ask is the following: Which values of cloning fidelities $(f_{12},f_{13}\dots f_{1n})$ are allowed for a (qudit) universal cloning machine? Keeping in mind that quantum cloning can be recast in a picture where one wants to share entanglement between some number of parties (see, for example, [19,22]), we can equivalently state our problems in this formalism, where one evaluates singlet fractions F_{1i} between the initial state and one of the copies. This allows us to restate our question as follows: Which values of n-tuples of singlet fractions $(F_{12},F_{13}\dots F_{1n})$ are allowed for an arbitrary state of a maximally mixed first subsystem?

Remark 1. Since these two quantities, cloning fidelities and singlet fractions, are connected [22], in the next section we will adapt the term "fidelities" for the latter.

Let us now consider in more detail the relation between cloning fidelities f and the fidelities (singlet fractions) F.

Suppose that we are given the maximally entangled qudit state

$$|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle, \tag{1}$$

and we apply the $1\to N$ cloning machine $\mathcal{CM}$ to the second subsystem of $|\psi^+\rangle$ when the first is untouched.[1] As a result we obtain an $(N+1)$-partite mixed state that possesses all the information about the cloning map $\widetilde{\Lambda}$. The state is of the form

$$\rho_{1\dots n} = (\mathbb{1}\otimes\widetilde{\Lambda})(|\psi^+\rangle\langle\psi^+|), \tag{2}$$

where $n=N+1$, so that the index $i=1$ is related to an initial state and $i=2,\dots N+1$ are related to clones. The fidelities of clones are strictly related to the fidelities of reduced states ρ_{1k} with the maximally entangled state [22]:

$$f_i = \frac{F_i d+1}{d+1}. \tag{3}$$

Here $f_i = \langle\langle\psi_{\text{in}}|\rho^i_{\text{out}}|\psi_{\text{in}}\rangle\rangle$ is the fidelity of ith clone, where $\langle\cdots\rangle$ is the uniform average over an input state ψ_{in}, and $F_i = \langle\psi_+|\rho_{1,i}|\psi_+\rangle$.

An allowed region for quantum cloning can then be calculated by evaluating singlet fractions F_{1i} between the initial state and one of the copies, denoted by

$$F_{1i} = \langle\psi^+_{1i}|\,\mathrm{Tr}_{\overline{1i}}(\rho_{1\dots n})|\psi^+_{1i}\rangle \quad\text{or}\quad F_{1i} = \langle\psi^-_{1i}|\,\mathrm{Tr}_{\overline{1i}}(\widetilde{\rho}_{1\dots n})|\psi^-_{1i}\rangle, \tag{4}$$

where $1<i\leqslant n$, $\mathrm{Tr}_{\overline{1i}}$ means the partial trace over all systems except $1i$, and $|\psi^-_{1i}\rangle$ and $\widetilde{\rho}_{1\dots n}$ are defined below.

Let us show here why we have been able to use Schur-Weyl duality and the commutant structure of $U^{\otimes n}$ for qubit cloning machines [19] and explain why it does not work for higher dimensions of states ($d>2$). For qudits, in principle, the vector $|\psi^-_{1i}\rangle = U\otimes\mathbb{1}|\psi^+_{1\tilde{1}}\rangle$, $|\psi^-\rangle$ needs to be obtained after an application of U. For qubits, one can use Bell states $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$ and can show that the vector $|\psi^-\rangle$ is obtained after the action of the Pauli matrix $-i\sigma_y$ on $|\psi^+\rangle$. Using that, we can write

$$|\psi^-_{1\tilde{1}}\rangle = U\otimes\mathbb{1}|\psi^+_{1\tilde{1}}\rangle, \tag{5}$$

where $U=-i\sigma_y$. State $\widetilde{\rho}_{1234}$ from Eq. (4) is obtained after the following transformation:

$$\begin{aligned}\widetilde{\rho}_{1\dots n} &= (\mathbb{1}\otimes\widetilde{\Lambda})|\psi^-_{1\tilde{1}}\rangle\langle\psi^-_{1\tilde{1}}| \\ &= (U\otimes\mathbb{1})[(\mathbb{1}\otimes\widetilde{\Lambda})|\psi^+_{1\tilde{1}}\rangle\langle\psi^+_{1\tilde{1}}|](U\otimes\mathbb{1})^\dagger.\end{aligned} \tag{6}$$

The n-partite states $\widetilde{\rho}_{1\dots n}$, with the constraint $\widetilde{\rho}_1 = \mathbb{1}/2$, are in one-to-one correspondence with cloning machines.

However, now the problem is formulated in terms of singlet fractions with states $|\psi^-\rangle$ rather than $|\psi^+\rangle$. The former states are invariant under the $U\otimes U$ transformation for any U. Therefore to obtain the region of fidelities with $|\psi^-\rangle$ states it is enough to consider states $\rho_{1\dots n}$ that are invariant under $U^{\otimes n}$ transformations. There exists a well-known formalism that allows us to deal with states possessing such symmetry, called Schur-Weyl duality, that combines representation theory for a unitary group with that of a group of permutations. We have successfully applied this formalism in [19]. However, in dimensions $d>2$ there is no maximally entangled state, which would be $U\otimes U$ invariant. Therefore the Schur-Weyl formalism cannot be used.

Instead, it is known that the state $|\psi^+\rangle$ is $U^*\otimes U$ invariant [23]; hence we should consider $U^*\otimes U^{\otimes n-1}$ invariant states. The formalism related to this kind of symmetry is not as well developed as the previous one, and there are quite basic differences between the two. In particular, while the representation of $U^{\otimes n}$ is dual to the representation of another group, the symmetric group, that is not the case for $U^*\otimes U^{\otimes n-1}$, which is dual to the representation of an algebra that does not satisfy group axioms, an instance of the so-called Brauer algebra. While some general results concerning this type of algebra have been given in the literature (see, for example, [24–26]), it has not been described in depth, in contrast to the Schur-Weyl theory. In particular, the explicit form of matrix elements of representations of the algebra have been provided recently in [20,21]. In the following we solve the cloning problem applying these new tools.

B. Mathematical tools

As stated before, to solve our problem, the knowledge of irreducible representations of a $U^*\otimes U\otimes\cdots\otimes U$ case is necessary. In recent papers [20,21] this problem has been addressed, so we can use the formalism presented there.[2]

In Refs. [20,21], the authors presented irreducible representations of partially transposed permutation operators $V^{t_n}(\sigma)$, where $\sigma\in S(n)$ and t_n denotes partial transposition over the last subsystem. In our approach, we need similar results for irreps when partial transposition is taken over the first

[1] The $1\to N$ cloning machine is described by a completely positive, trace-preserving map $\widetilde{\Lambda}$.

[2] See also Sec. A1 for a short review of this topic.

subsystem, i.e., we need irreps of $V^{t_1}(1k)$, where $1 \leqslant k \leqslant n$ for $U^* \otimes U \otimes \cdots \otimes U$ instead of $U \otimes \cdots \otimes U \otimes U^*$. That is why, first, some work needs to be done to adapt the results so they suit our problem. One can see that to obtain the correct results, we have to take irreps for permutations in the form (in), where $1 \leqslant i \leqslant n-1$, i.e., we have the following mapping:

$$(12) \mapsto (1n),\ (13) \mapsto (2n)\ \ldots\ (1n) \mapsto (n-1n). \tag{7}$$

In the next sections, for simplicity, we introduce the notation that $t_n \equiv\ '$. Now we are ready to present all irreps that are essential for our paper (case study examples). Of course our method works efficiently for an arbitrary number of particles n and dimensions of Hilbert space d, but here we present them only for $n = 3{,}4$ because for these cases we are able to represent our results graphically.

(i) In the case when $n = 3$, in algebra $\mathcal{M}$ we have only one irrep labeled by trivial partition $\alpha = (1)$.

$$\mathbb{V}'_\alpha(13) = \frac{1}{2}\begin{pmatrix} d+1 & -\sqrt{d^2-1} \\ -\sqrt{d^2-1} & d-1 \end{pmatrix},\quad \mathbb{V}'_\alpha(23) = \frac{1}{2}\begin{pmatrix} d+1 & \sqrt{d^2-1} \\ \sqrt{d^2-1} & d-1 \end{pmatrix}. \tag{8}$$

(ii) In the case when $n = 4$, in algebra $\mathcal{M}$ we have two irreps labeled by partitions $\alpha_1 = (2)$ and $\alpha_2 = (1,1)$. For partition α_1 we deal with 3×3 matrices for any $d \geqslant 1$:

$$\mathbb{V}'_{\alpha_1}(14) = \frac{1}{3}D^{\alpha_1}\begin{pmatrix} \frac{1}{6} & \frac{-1}{2\sqrt{3}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{2\sqrt{3}} & \frac{1}{2} & \frac{-1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{3} \end{pmatrix} D^{\alpha_1},$$
$$\mathbb{V}'_{\alpha_1}(24) = \frac{1}{3}D^{\alpha_1}\begin{pmatrix} \frac{1}{6} & \frac{1}{2\sqrt{3}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3} \end{pmatrix} D^{\alpha_1}, \tag{9}$$
$$\mathbb{V}'_{\alpha_1}(34) = \frac{1}{3}D^{\alpha_1}\begin{pmatrix} \frac{2}{3} & 0 & \frac{-2}{3\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{-2}{3\sqrt{2}} & 0 & \frac{1}{3} \end{pmatrix} D^{\alpha_1},$$

where

$$D^{\alpha_1} = \begin{pmatrix} \sqrt{d-1} & 0 & 0 \\ 0 & \sqrt{d-1} & 0 \\ 0 & 0 & \sqrt{d+2} \end{pmatrix} \tag{10}$$

and $\varepsilon^2 = 1$. For partition α_2 the situation is more complicated. The dimension of irrep α_2 depends on the dimension of the local Hilbert space d. Namely, for any $d \geqslant 3$ we have

$$\mathbb{V}'_{\alpha_2}(14) = \frac{1}{3}D^{\alpha_2}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{2\sqrt{3}} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix} D^{\alpha_2},$$
$$\mathbb{V}'_{\alpha_2}(24) = \frac{1}{3}D^{\alpha_2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix} D^{\alpha_2},$$
$$\mathbb{V}'_{\alpha_2}(34) = \frac{1}{3}D^{\alpha_2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{-\sqrt{2}}{3} \\ 0 & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} D^{\alpha_2}, \tag{11}$$

where

$$D^{\alpha_2} = \begin{pmatrix} \sqrt{d+1} & 0 & 0 \\ 0 & \sqrt{d+1} & 0 \\ 0 & 0 & \sqrt{d-2} \end{pmatrix}. \tag{12}$$

For every $d < 3$ (in our case only $d = 2$ is interesting) we deal with 2×2 matrices:

$$\mathbb{V}'_{\alpha_2}(14) = 3\begin{pmatrix} \frac{1}{2} & \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} & \frac{1}{6} \end{pmatrix},\quad \mathbb{V}'_{\alpha_2}(24) = 3\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{6} \end{pmatrix},\quad \mathbb{V}'_{\alpha_2}(34) = 3\begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}. \tag{13}$$

The full knowledge of irreps of $V'(\sigma_{ab})$, where $\sigma_{ab} \in S(n)$ (see Notation 1 in Sec. A1), allows us to decompose these operators and density operators $\rho_{1\ldots n}$, which are $U^* \otimes U \otimes \cdots \otimes U$ invariant, into block diagonal form:

$$V'(\sigma_{ab}) = \bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \mathbb{V}'_\alpha(\sigma_{ab}),\quad \rho_{1\ldots n} = \bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \widetilde{\rho}^\alpha, \tag{14}$$

where the direct sum runs over all inequivalent irreps α, $r(\alpha)$ denotes the dimension of irrep α, and $\widetilde{\rho}^\alpha$ is a representation of operator $\rho_{1\ldots n}$ on irrep α. In the next paragraph we present how to use the decomposition from formula (14) and the explicit matrix form of irreps of $V'(\sigma_{ab})$ to calculate fidelities.

C. Method of calculations

Since, in principle, calculation techniques are similar to those from [19], in most cases, proofs are skipped, and unless specified otherwise, we refer to the above-mentioned work for them.

In this section we provide a general formula for an allowed region of N-tuples of fidelities in terms of overlaps of pure states with irreducible representations from the previous section. This is contained in Theorem 1.

Lemma 1. Fidelity F_{1k} as defined in (4) is of the form

$$F_{1k} = \sum_\alpha F^\alpha_{1k}, \tag{15}$$

where

$$F^\alpha_{1k} = \frac{1}{d}\mathrm{Tr}[\rho^\alpha \mathbb{V}'_\alpha(k-1n)], \tag{16}$$

the index $(k-1n)$ means a permutation that swaps $k-1$ and n, and ρ^α's are arbitrary normalized states on partition α.

Again, from Refs. [20,21] we know that the algebra of partially transposed permutation operators $\mathcal{A}'_n(d)$ splits into the sum of two ideals, i.e., we have $\mathcal{A}'_n(d) = \mathcal{M} \oplus \mathcal{N}$. In Lemma 1 we derived formulas for fidelities for elements in ideal $\mathcal{M}$; now we give similar formulas for elements in ideal $\mathcal{N}$. Physically, this means that we are looking for fidelities between a maximally entangled state and some product state between the input state and clones.

Fact 1. Fidelity $F_{1k}^{\mathcal{N}}$ between state $|\psi_{1k}\rangle$ and a product state $\rho_{1k} = \frac{1}{d}\operatorname{Tr}_{\overline{1k}}(\mathbb{1}_1 \otimes \rho_{2\ldots n})$ is equal to $1/d$.[3]

Now we are in a position to formulate the main theorem of this section:

Theorem 1. The set $\mathcal{F}$ of admissible vectors of fidelities $\{F_{12}\ldots F_{1n}\}$ is of the form

$$\mathcal{F} = \operatorname{conv}\left(\bigcup_{\alpha} \mathcal{F}^{\alpha}\right), \tag{17}$$

where conv stands for a convex hull, the union runs over all irreps, and

$$\mathcal{F}^{\alpha} = \left\{\left(F_{12}^{\alpha}\ldots F_{1n}^{\alpha}\right) : |\psi\rangle \in \mathcal{C}^{d_\alpha}\right\}, \tag{18}$$

where F_{1k}^{α} are of the form $F_{1k}^{\alpha} = \frac{1}{d}\langle\psi|\mathbb{V}'_{\alpha}(k-1n)|\psi\rangle$ and where $|\psi\rangle$ is a pure state.

Let us note that to determine the allowed region of fidelities, it is enough to consider only vectors of real coefficients.

Lemma 2. To generate a convex hull of the allowed region of fidelities, it is sufficient to consider pure states of real coefficients only.

D. Main result

In this section we present our results for two particular cases, $1 \to 2$ and $1 \to 3$ universal quantum cloners.

Let us start with noting that to obtain a general answer to our question from Sec. II, we need to have a mixture of all fidelities connected with our irreps: $\sum_{\alpha} p_{\alpha} F_{1N}^{\alpha}$. This implies that a convex hull is needed. In Figs. 1 and 2 we show plots for $N = 2,3$ and different dimensions d before taking the convex hull, so one can see a contribution from each irrep. Then, we take one particular case, namely, $1 \to N$ UQCM and $d = 3$, and present the convex hull for it that reproduces the allowed region for fidelities (Fig. 3). All plots are obtained using *Mathematica* software.

Remark 2. Because of the properties of the cloning map $\widetilde{\Lambda}$ (see Sec. II) all possible convex mixtures of the partitions produce a correct quantum cloner, i.e., a trace-preserving completely positive map.

E. Comparison with other methods

First of all, let us notice that our method gives the correct results (according to Werner's formula [6]) in the case of symmetric cloning (see [19] for a possible technique for checking that). What is more, the regions of fidelities obtained for $d = 2$ (qubits) match those obtained using Schur-Weyl duality [19]. Last but not least, our method seems to correctly reproduce results obtained in [15], where the solution to the $1 \to N$ universal asymmetric qudit cloning problem for which the exact trade-off in the fidelities of the clones for every N and d has been derived. The authors obtained their result using various tools, such as the Choi-Jamiołkowski isomorphism [27,28] and some variance of the Lieb-Mattis theorem [29,30]. The crucial part of their proof is the observation that the cloning problem can be mapped to some Heisenberg Hamiltonian on a star. Comparing their technique with ours, one can observe that they solve the problem from the side of the cloning map $\widetilde{\Lambda}$, whereas we attack it from the side of the n-party quantum state [see Eqs. (2) and (6)].

[3]By $\operatorname{Tr}_{\overline{1k}}$ we denote the partial trace over all subsystems except the first and kth.

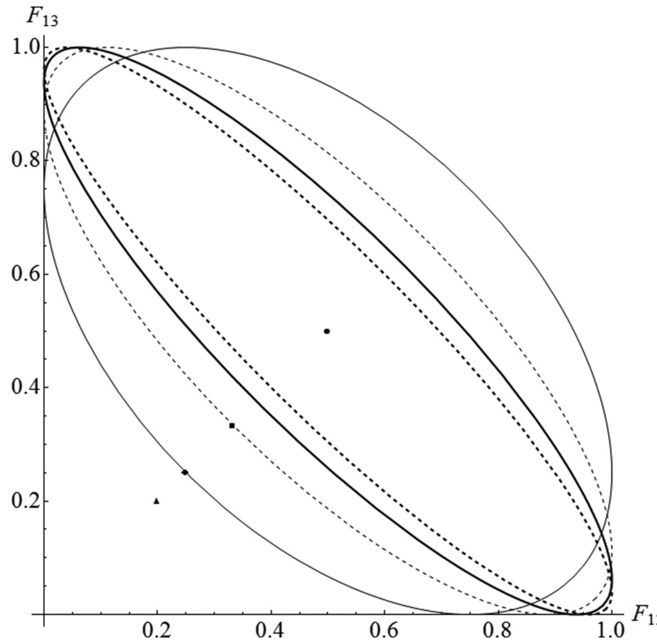

FIG. 1. Allowed regions of fidelities for $1 \to 2$ UQCM. Views for various dimensions d of the Hilbert space are presented: thin gray line and black point, $d = 2$; thin dashed grey line and square, $d = 3$; thick line and diamond, $d = 4$; thick dashed line and triangle, $d = 5$. One can see that for $d \to \infty$ the ellipse is squeezed to the line $F_{13} = -F_{12} + 1$ and coordinates of the point obtained from part $\mathcal{N}$ go to zero.

III. CONCLUSIONS

We have shown that using a more general version of Schur-Weyl duality, the action of the universal $1 \to N$ quantum cloning machine can be described, allowing us to obtain the admissible general region for fidelities. Contrary to other known methods, in our method, quantum states are figures of merit. The method exploits decomposition of (usually big) Hilbert space into blocks of smaller dimensions which, of course, are easier to deal with. Fidelity expressions are then quite easy to obtain; one only needs to know representations of all possible irreps for a given case. Another advantage is that one can consider real pure states in each of the blocks only when generating convex hulls to obtain an allowed region for fidelities.

Let us now briefly discuss the results. First of all, suppose that we choose some point that lays outside of the allowed convex hull. Then a quantum state that would correspond to that point does not exist. On the other hand, whenever we choose points from the convex hull (from inside or from the

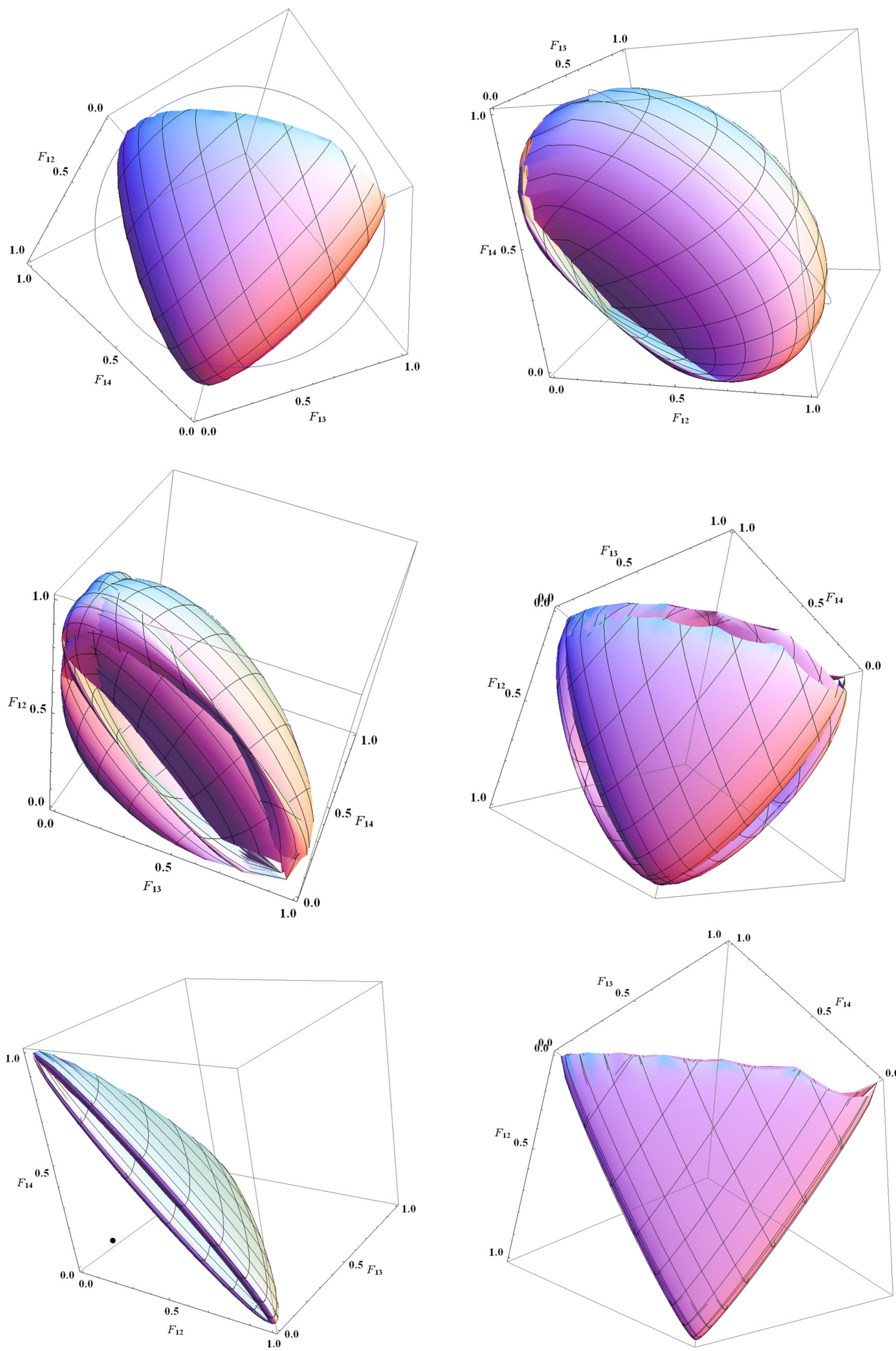

FIG. 2. (Color online) Allowed regions of fidelities for $1 \to 3$ UQCM. Views for various dimensions d of the Hilbert space and all allowed irreps are presented. From the top: $d = 2$, $d = 3$, and $d = 10$. One can see that for $d = 2$ we match results from [19], and this is the only case where irreps from $\mathcal{M}$ are two-dimensional (in this case we have an ellipse). For higher dimensions $d \to \infty$ all regions obtained from part $\mathcal{M}$ are squeezed, and coordinates of points from part $\mathcal{N}$ go to zero.

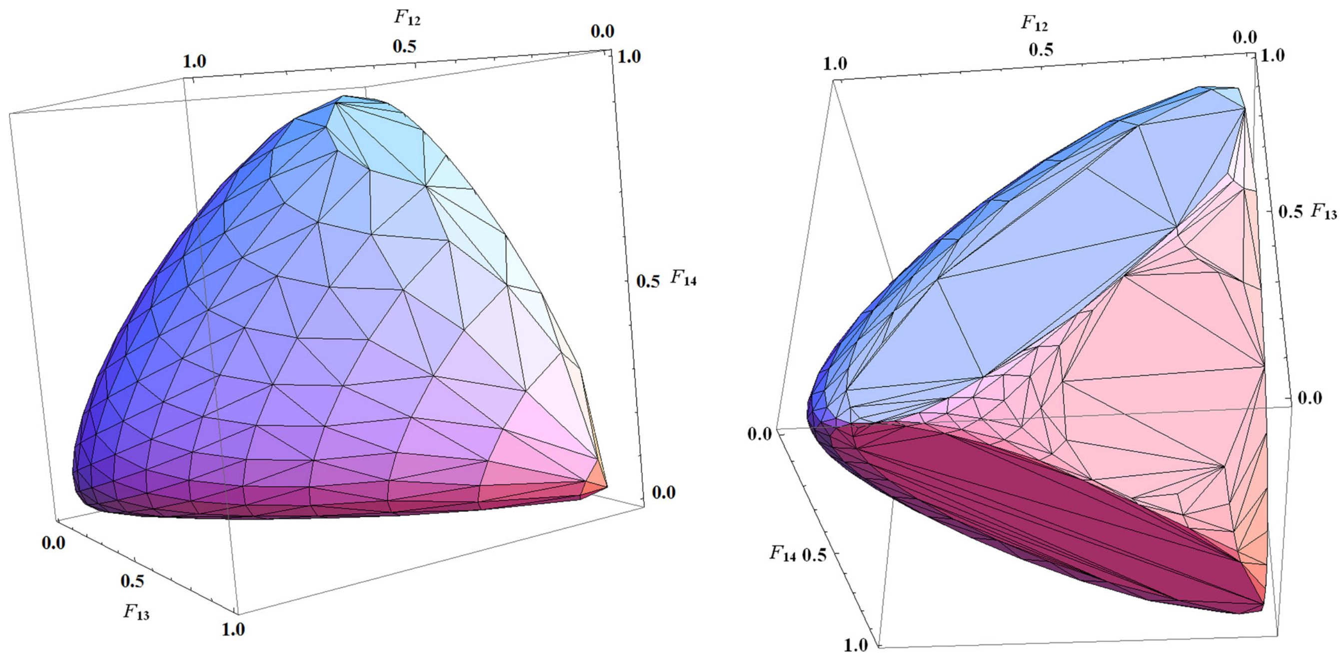

FIG. 3. (Color online) Convex hull for $1 \rightarrow 3$ of UQCM and $d = 3$.

edge), we are able to derive a family of quantum states for which fidelities are fixed and have values determined by the chosen point. Apart from the above-mentioned reconstruction of states from the convex hull, we can try to find, for example, all allowed quantum states which satisfy some required condition for relations between fidelities F_{1k}. For example, for $1 \rightarrow 3$ universal cloning machines we can demand the following constraint:

$$F_{12} + F_{13} = 2F_{14}, \tag{19}$$

where we take maximization over F_{12}. Such a reconstruction was presented in our previous paper regarding the admissible region of fidelities for the qubit case [19]. Finally, having these states, we can reconstruct a cloning machine which returns clones with fidelities f_i, corresponding to fidelities F_{1i} given by the chosen point.

We also have an interesting interpretation of the bottommost part of our plots as optimal anticlones. First of all, one can notice that our convex hulls are invariant with respect to rotations around the straight line $F_{12} = F_{13} = F_{14}$ by the angle $\beta = 2\pi/3$ in the case $1 \rightarrow 3$ UQCM, and they are symmetric with respect to the straight line $F_{12} = F_{13}$ in the case $1 \rightarrow 2$ UQCM. The bottommost point is determined by the intersection between the symmetry line and convex hull, and it corresponds to a minimum value of fidelities which are equal in these cases.

In the future, it would be interesting to obtain optimal clones starting from our method. Numerically, it is not that hard; one just needs to add a cut to the general region to end with the optimal region of fidelities. Analytically, the answer does not seem to be so trivial, but we still hope that the employed group-theoretic techniques are interesting and may provide some new insight into the inner structure of the optimal universal asymmetric quantum cloners.

Finally, let us note that to solve a $M \rightarrow N$ ($M < N$, $M + N = n$) cloning problem, one needs to possess a knowledge of the commutant structure of a $U^{\otimes N} \otimes (U^*)^{\otimes M}$ transformation, where one has M conjugate elements U^* and N elements U [20,21].

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APPENDIX

1. Algebra of partially transposed permutation operators

Here we present a short summary of Ref. [21] which is crucial for the construction of our results. For the reader's convenience we keep here the original notation. It appears that the structure irreducible representations of the algebra $\mathcal{A}_n'(d)$ are closely related to the structure of the representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$ of the group $S(n-1)$ induced by irreducible representations φ^α of the group $S(n-2)$ and the properties of irreducible representations of $\mathcal{A}_n'(d)$ depend strongly on the relation between d and n. Before presenting the main ideas of this appendix we have to describe briefly some objects appearing in the structure of the algebra $\mathcal{A}_n'(d)$, in particular the properties of the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$. The irreducible representations of the group $S(n-2)$ are characterized by the partitions $\alpha=(\alpha_1,\dots,\alpha_k)$ of $n-2$, which describe also the corresponding Young diagram $Y(\alpha)$. The representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$ is completely and simply reducible, i.e., we have the following proposition [31].

Proposition 1.

$$\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)=\bigoplus_\nu \psi^\nu, \tag{A1}$$

where the sum is over all partitions $\nu=(\nu_1,\dots,\nu_k)$ of $n-1$, such that their Young diagrams $Y(\nu)$ are obtained from $Y(\alpha)$ by adding, in a proper way, one box.

Definition 1 [32]. Let $\varphi: H\to M(n,\mathbb{C})$ be a matrix representation of a subgroup H of the group G. Then the matrix form of the induced representation $\pi=\operatorname{ind}_H^G(\varphi)$ of a group G induced by an irrep φ of the subgroup $H\subset G$ has the following block matrix form:

$$\forall\, g\in G \quad \pi_{ai}^{bj}(g)=\widehat{\varphi}_{ij}\left(g_a^{-1}gg_b\right),$$

where g_a, $a=1,\dots,[G:H]$ are representatives of the left cosets G/H and

$$\widehat{\varphi}_{ij}\left(g_a^{-1}gg_b\right)=\begin{cases}\varphi_{ij}\left(g_a^{-1}gg_b\right) & \text{if} \quad g_a^{-1}gg_b\in H,\\ 0 & \text{if} \quad g_a^{-1}gg_b\notin H.\end{cases}$$

Before we discuss the main considerations for the appendix let us introduce some notation.

Notation 1. Any permutation $\sigma\in S(n)$ defines, in a natural and unique way, two natural numbers $a,b\in\{1,2,\dots,n\}$,

$$n=\sigma(a),\quad b=\sigma(n).$$

Thus we may characterize any permutation by these two numbers in the following way:

$$\sigma\equiv\sigma_{(a,b)}\equiv\sigma_{ab}.$$

Note that, in general, a,b may be different except in the case when one of them is equal to n because in this case we have

$$a=n\Leftrightarrow b=n.$$

When $a=n=b$, then $\sigma(n)=n$, and we will use the abbreviation $\sigma=\sigma_{(n,n)}\equiv\sigma_n\in S(n-1)\subset S(n)$.

From Proposition 1 and Definition 1 it follows that the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$ may be described in two bases. The first one is the basis of the matrix form of the induced representation of the form

$$\left\{e_i^a(\alpha): a=1,\dots,n-1,\quad i=1,\dots,\dim\varphi^\alpha\right\}, \tag{A2}$$

where the index $a=1,\dots,n-1$ describes the cosets $S(n-1)/S(n-2)$ and the index $i=1,\dots,\dim\varphi^\alpha$ is the index of a matrix form of φ^α. The second one is the basis of the reduced form of $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$, which is of the form

$$\left\{f_{j_\nu}^\nu:\psi^\nu\in\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha),\quad j_\nu=1,\dots,\dim\psi^\nu\right\}. \tag{A3}$$

The next important objects are the following matrices.

Definition 2. For any irreducible representation φ^α of the group $S(n-2)$ we define the block matrix

$$Q_{n-1}^d(\alpha)\equiv Q(\alpha)=\left(d^{\delta_{ab}}\varphi_{ij}^\alpha[(an-1)(ab)(bn-1)]\right)=\left(Q_{ij}^{ab}(\alpha)\right), \tag{A4}$$

where $a,b=1,\dots,n-1$, $i,j=1,\dots,\dim\varphi^\alpha$, and the blocks of the matrix $Q(\alpha)$ are labeled by indices (a,b), whereas the elements of the blocks are labeled by the indices of the irreducible representation $\varphi^\alpha=(\varphi_{ij}^\alpha)$ of the group $S(n-2)$ and $Q(\alpha)\in M((n-1)w^\alpha,\mathbb{C})$.

The matrices $Q(\alpha)$ are Hermitian, and their structure and properties are described in the [21], where it has been shown that the eigenvalues λ_ν of the matrix $Q(\alpha)$ are labeled by the irreducible representations $\psi^\nu\in\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$ and the multiplicity of λ_ν is equal to $\dim\psi^\nu$. The essential for properties (see, for e.g., Proposition 4) of the irreducible representations of the algebra $\mathcal{A}_n'(d)$ is the fact that at most one (up to the multiplicity) eigenvalue λ_ν of the matrix $Q(\alpha)$ may be equal to zero [20,21].

The structure of the algebra $\mathcal{A}_n'(d)$ is given in the following theorem.

Theorem 2. The algebra $\mathcal{A}_n'(d)$ is a direct sum of two ideals,

$$\mathcal{A}_n'(d)=\mathcal{M}\oplus\mathcal{N}, \tag{A5}$$

and the ideals $\mathcal{M}$ and $\mathcal{N}$ have different structures.

(a) The ideal $\mathcal{M}$ is of the form

$$\mathcal{M}=\bigoplus_\alpha U(\alpha), \tag{A6}$$

where $U(\alpha)$ are ideals of the algebra $\mathcal{A}_n'(d)$ characterized by the irreducible representations φ^α of the group $S(n-2)$, such that $\varphi^\alpha\in V_d[S(n-2)]$ and

$$U(\alpha)=\operatorname{span}_{\mathbb{C}}\left\{u_{ij}^{ab}(\alpha): a,b=1,\dots,n-1,\ i,j=1,\dots,w^\alpha\right\}, \tag{A7}$$

with

$$u_{ij}^{ab}(\alpha)u_{kl}^{pq}(\beta)=\delta_{\alpha\beta}Q_{ik}^{bp}(\alpha)u_{il}^{aq}(\alpha). \tag{A8}$$

The ideals $U(\alpha)$ are matrix ideals such that

$$U(\alpha)\simeq M(\operatorname{rank}Q(\alpha),\mathbb{C}); \tag{A9}$$

in particular, when $\det Q(\alpha)\neq 0$, we have

$$U(\alpha)\simeq M((n-1)\dim\varphi^\alpha,\mathbb{C}). \tag{A10}$$

(b) The ideal $\mathcal{N}$ has the following structure:

$$\mathcal{N}\simeq\bigoplus_\nu M(\dim\psi^\nu,\mathbb{C}), \tag{A11}$$

where the matrix ideals $M(\dim\psi^\nu,\mathbb{C})$ are generated by irreducible representations ψ^ν of the group $S(n-1)$ that are included in the representation $V_d[S(n-1)]$, i.e., ψ^ν are such that $d\geqslant h(\nu)$.

The matrix ideals contained in the ideals $\mathcal{M}$ and $\mathcal{N}$ contain all minimal left ideals, i.e., all irreducible representations of the algebra $\mathcal{A}_n'(d)$. The next theorems describe all these representations.

The structure of the irreducible representations of the algebra $\mathcal{A}_n'(d)$, included in the ideal $\mathcal{M}$, is completely determined by irreducible representations φ^α of the group $S(n-2)$; therefore we will denote them as Φ_A^α.

Theorem 3. The irreducible representations Φ_A^α of the algebra $\mathcal{A}_n'(d)$ contained in the ideal $U(\alpha)\subset\mathcal{M}$ (see Theorem 2) are indexed by the irreducible representations φ^α of the group $S(n-2)$, such that $\varphi^\alpha\in V_d[S(n-2)]$, and if $\{f_{j_\nu}^\nu:$

$\psi^{\nu} \in \operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha}), \quad j_{\nu} = 1, \ldots, \dim \psi^{\nu}\}$ is the reduced basis of the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$, then the vectors $\{f_{j_\nu}^{\nu} : \lambda_{\nu} \neq 0\}$ from the basis of the irreducible representation of the algebra $\mathcal{A}_n'(d)$ and the natural generators of $\mathcal{A}_n'(d)$ act on it in the following way:

$$V'(an) f_{j_\nu}^{\nu}(\alpha) = \sum_{\rho, j_\rho} \sum_{k} \sqrt{\lambda_\rho} z^{\dagger}(\alpha)_{j_\rho k}^{\rho a} z(\alpha)_{k j_\nu}^{a\nu} \sqrt{\lambda_\nu} f_{j_\rho}^{\rho}(\alpha), \tag{A12}$$

where the summation is over ρ such that $\lambda_\rho \neq 0$. Due to the condition $\varphi^{\alpha} \in V_d[S(n-2)]$ the eigenvalues λ_ν of $Q(\alpha)$ are non-negative. The unitary matrix $Z(\alpha) = (z(\alpha)_{kj_\nu}^{a\nu})$ has the form

$$\left(z(\alpha)_{kj_\nu}^{a\nu}\right) = \frac{\dim \psi^{\nu}}{\sqrt{N_{j_\nu}^{\nu}(n-1)!}} \sum_{\sigma \in S(n-1)} \psi_{j_\nu j_\nu}^{\nu}(\sigma^{-1}) \delta_{a\sigma(q)} \times \varphi_{kr}^{\alpha}[(an-1)\sigma(qn-1)], \tag{A13}$$

with

$$N_{j_\nu}^{\nu} = \frac{\dim \psi^{\nu}}{(n-1)!} \sum_{\sigma \in S(n-1)} \psi_{j_\nu j_\nu}^{\nu}(\sigma^{-1}) \delta_{q\sigma(q)} \times \varphi_{rr}^{\alpha}[(qn-1)\sigma(qn-1)], \tag{A14}$$

where the indices $q = 1, \ldots, n-1, r = 1, \ldots, \dim \varphi^{\alpha}$ are fixed such that $N_{j_\nu}^{\nu} > 0$. For more details, see [21]. Whenever $\sigma_n \in S(n-1)$, we have

$$V(\sigma_n) f_{j_\nu}^{\nu}(\alpha) = \sum_{\rho, j_\rho} \psi_{i_\nu j_\nu}^{\nu}(\sigma_n) f_{i_\nu}^{\nu}(\alpha). \tag{A15}$$

In particular, when $\det Q(\alpha) \neq 0$ (i.e., when all $\lambda_\nu \neq 0$), then the representation Φ_A^{α} is the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$ (in the reduced form) for the subalgebra $V_d[S(n-1)] \subset \mathcal{A}_n'(d)$. In this case the dimension of the irreducible representation is equal to

$$\dim \Phi_A^{\alpha} = (n-1) \dim \varphi^{\alpha} = \dim \left[\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha}) \right]. \tag{A16}$$

When $\det Q(\alpha) = 0$ (i.e., when one eigenvalue λ_θ of $Q(\alpha)$, up to the multiplicity, is equal to zero), then the irreducible representation of $\mathcal{A}_n'(d)$ is defined on a subspace $\{y_{j_\nu}^{\nu} : \lambda_\nu \neq \lambda_\theta\}$ of the representation space $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$, and the representation has a dimension equal to

$$\dim \Phi_A^{\alpha} = \dim_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})] - \dim \psi^{\theta} = \operatorname{rank} Q(\alpha). \tag{A17}$$

This case takes place when

$$d = i - \alpha_i - 1 \tag{A18}$$

for some α_i in the partition $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_k)$ characterizing the irreducible representation φ^{α} under the condition that $\nu = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_k)$ characterizes the representation ψ^{ν} of $S(n-1)$.

The ideal $U(\alpha)$ is a direct sum of $\dim \Phi_A^{\alpha}$ of irreducible representations Φ_A^{α}.

In particular, matrices $z(\alpha)_{kj_\nu}^{a\nu}$ diagonalize matrix $Q(\alpha)_{kl}^{ab}$, i.e., we have the following proposition.

Proposition 2.

$$\sum_{ak} \sum_{bl} z^{\dagger}(\alpha)_{j_\rho k}^{\rho a} Q(\alpha)_{kl}^{ab} z(\alpha)_{l j_\mu}^{b\mu} = \delta^{\rho\mu} \delta_{j_\rho j_\mu} \lambda_\mu, \tag{A19}$$

and the columns of the matrix $Z(\alpha) = (z(\alpha)_{kj_\nu}^{a\nu})$ are eigenvectors of the matrix $Q(\alpha)$.

The formula for the eigenvalues λ_ν of matrices $Q(\alpha)$ is derived in [21].

Remark 3. Note that even if $\dim \varphi^{\alpha} = 1$, we have $\dim \Phi^{\alpha} = n - 1$.

The matrix forms of these representations are the following.

Proposition 3. In the reduced matrix basis $\{f_{j\nu}^{\nu} : \nu \neq \theta\}$ of the ideal $U(\alpha)$ the natural generators $V(\sigma_{ab})^{t_n}$ and $V(\sigma_n)$ of $\mathcal{A}_n'(d)$ are represented by the following matrices:

$$[\mathbb{V}_{\alpha}'(an)]_{j_\rho j_\nu}^{\rho\nu} = \sum_{k=1,\ldots,\dim\varphi^{\alpha}} \sqrt{\lambda_\rho} z^{\dagger}(\alpha)_{j_\rho k}^{\rho a} z(\alpha)_{kj_\nu}^{a\nu} \sqrt{\lambda_\nu} : \rho, \nu \neq \theta, \tag{A20}$$

$$[\mathbb{V}_{\alpha}(\sigma_n)]_{j_{\nu'} j_\nu}^{\nu'\nu} = \delta^{\nu'\nu} \psi_{j_{\nu'} j_\nu}^{\nu}(\sigma_n). \tag{A21}$$

From the properties of the matrix $Q(\alpha)$ ([21]) one gets the following proposition.

Proposition 4. If $d > n - 2$, then $\det Q(\alpha) \neq 0$, and the irreducible representations Φ_A^{α} described in Theorem 3 are the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$ for the subalgebra $V_d[S(n-1)] \subset \mathcal{A}_n'(d)$, so their dimension is equal to $(n-1)\dim \varphi^{\alpha}$. When $d \leqslant n-2$, then for some φ^{α} it may appear that $\det Q(\alpha) = 0$, and consequently, the irreducible representation Φ^{α} of $\mathcal{A}_n'(d)$ is defined on a subspace of the irreducible representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$.

The representations of the algebra $\mathcal{A}_n'(d)$ included in the ideal $\mathcal{N}$ are much simpler.

Theorem 4. Each irreducible representation ψ^{ν} of the group $S(n-1)$, which appears in the decomposition of the ideal $\mathcal{N}$ given in Theorem 2 [i.e., $\psi^{\nu} \in V_d[S(n-1)] \Leftrightarrow d \geqslant h(\nu)$], defines irreducible representations Ψ^{ν} of the algebra $\mathcal{A}_n'(d)$ in the following way:

$$\Psi^{\nu}(a) = \begin{cases} 0 & \text{if} \quad a \in \mathcal{M}, \\ \psi^{\nu}(\sigma_n) & \text{if} \quad a = \sigma_n \in S(n-1). \end{cases} \tag{A22}$$

So in this representation the noninvertible element of the ideal $\mathcal{M}$ is represented trivially by zero, and therefore we call this representation of the algebra $\mathcal{A}_n'(d)$ semitrivial. The matrix forms of these representations are simply matrix forms of the irreducible representations of the group algebra $\mathbb{C}[S(n-1)] \subset A_n'(d)$ and zero matrices for the elements of the ideal $\mathcal{M}$.

Corollary 1. All irreducible representations of the algebra $\mathcal{A}_n'(d)$ of dimension 1 are included in the ideal $\mathcal{N}$. In particular, because the irreducible identity representation ψ^{Id} of $S(n-1)$ is always contained in $V_d[S(n-1)]$, the algebra $\mathcal{A}_n'(d)$ has a trivial representation in which the elements of the ideal M are represented by zero and the elements $V_d(\sigma) : \sigma \in S(n-1)$ are represented by 1.

2. Auxiliary lemmas

After the short summary of [21] given in the previous section we prove here the crucial lemma which says that matrices $z(\alpha)_{kj_\nu}^{a\nu}$ are unitary (real orthogonal), and then we conclude that representation matrices in the reduced matrix basis are Hermitian (symmetric). We start from the following proposition.

Proposition 5. Suppose that all representations ψ^ν of $S(n-1)$ and φ^α of $S(n-2)$ are unitary (real orthogonal); then the matrix

$$z(\alpha)^{a\nu}_{kj_\nu} = \frac{\dim \psi^\nu}{\sqrt{N^\nu_{j_\nu}(n-1)!}} \sum_{\sigma\in S(n-1)} \psi^\nu_{j_\nu j_\nu}(\sigma^{-1})\delta_{a\sigma(q)} \times \varphi^\alpha_{kr}[(an-1)\sigma(qn-1)], \quad (A23)$$

where

$$N^\nu_{j_\nu} = \frac{\dim \psi^\nu}{(n-1)!} \sum_{\sigma\in S(n-1)} \psi^\nu_{j_\nu j_\nu}(\sigma^{-1})\delta_{q\sigma(q)} \times \varphi^\alpha_{rr}[(qn-1)\sigma(qn-1)] \quad (A24)$$

is unitary (real orthogonal).

Proof. We will prove the orthogonal case, proving that

$$\sum_{c,k} z(\alpha)^{c\mu}_{kj_\mu} z(\alpha)^{c\nu}_{kj_\nu} = \delta^{\mu\nu}\delta_{j_\mu j_\nu}. \quad (A25)$$

Using the definition of the matrix $z(\alpha)$, we get that the left-hand side of the above equation is equal to

$$\frac{\dim \psi^\nu \dim \psi^\mu}{\sqrt{N^\nu_{j_\nu}}\sqrt{N^\mu_{j_\mu}}[(n-1)!]^2} \sum_{\sigma,\rho\in S(n-1)} \sum_{c,k} \psi^\mu_{j_\mu j_\mu}(\rho^{-1})\psi^\nu_{j_\nu j_\nu}(\sigma^{-1})\delta_{c\rho(q)} \times \delta_{c\sigma(q)}\varphi^\alpha_{kr}[(cn-1)\rho(qn-1)]\varphi^\alpha_{kr}[(cn-1)\sigma(qn-1)]$$
$$= \frac{\dim \psi^\nu \dim \psi^\mu}{\sqrt{N^\nu_{j_\nu}}\sqrt{N^\mu_{j_\mu}}[(n-1)!]^2} \sum_{\sigma,\rho\in S(n-1)} \psi^\mu_{j_\mu j_\mu}(\rho^{-1})\psi^\nu_{j_\nu j_\nu}(\sigma^{-1}) \times \delta_{\rho^{-1}\sigma(q)q}\varphi^\alpha_{rr}[\rho^{-1}\sigma]. \quad (A26)$$

Substituting $\gamma = \rho^{-1}\sigma \in S(n-2) \subset S(n-1)$ (which follows from $\delta_{\rho^{-1}\sigma(q)q}$), we get

$$\sum_{c,k} z(\alpha)^{c\mu}_{kj_\mu} z(\alpha)^{c\nu}_{kj_\nu} = \frac{\dim \psi^\nu \dim \psi^\mu}{\sqrt{N^\nu_{j_\nu}}\sqrt{N^\mu_{j_\mu}}[(n-1)!]^2} \sum_{\rho\in S(n-1)\gamma\in S(n-2)} \sum_{k_\nu} \psi^\mu_{j_\mu j_\mu}(\rho^{-1}) \times \psi^\nu_{j_\nu k_\nu}(\rho)\psi^\nu_{k_\nu j_\nu}(\gamma^{-1})\delta_{\gamma(q)q}\varphi^\alpha_{rr}[\gamma]. \quad (A27)$$

Now using the orthogonality relations for the irreducible representations ψ^ν of $S(n-1)$, we obtain

$$\sum_{c,k} z(\alpha)^{c\mu}_{kj_\mu} z(\alpha)^{c\nu}_{kj_\nu} = \frac{\dim \psi^\nu}{\sqrt{N^\nu_{j_\nu}}(n-1)!} \sum_{\gamma\in S(n-2)} \delta^{\mu\nu}\delta_{j_\mu j_\nu}\psi^\nu_{j_\nu j_\nu}(\gamma^{-1})\delta_{\gamma(q)q}\varphi^\alpha_{rr}[\gamma] = \delta^{\mu\nu}\delta_{j_\mu j_\nu}. \quad (A28)$$

The proof for the unitary case is similar. ■

Corollary 2. Suppose that all representations ψ^ν of $S(n-1)$ and φ^α of $S(n-2)$ are unitary (real orthogonal); then the representation matrices [in the reduced matrix basis $\{f^\nu_{j_\nu} : \nu \neq \theta\}$ of the ideal $U(\alpha)$]

$$[\mathbb{V}'_\alpha(an)]^{\rho\nu}_{j_\rho j_\nu} = \sum_{k=1,\dots,\dim\varphi^\alpha} \sqrt{\lambda_\rho} z^+(\alpha)^{\rho a}_{j_\rho k} z(\alpha)^{a\nu}_{kj_\nu}\sqrt{\lambda_\nu} : \rho,\nu \neq \theta \quad (A29)$$

are Hermitian (real symmetric). In the orthogonal case we have replace Hermitian conjugation † in Eq. (A29) by normal transposition T.

Indeed, unitarity (orthogonality) of matrices $z(\alpha)^{a\nu}_{kj_\nu}$ from Proposition 5 allows us to write $z^+(\alpha)^{a\nu}_{kj_\nu} = z(\alpha)^{\nu a}_{j_\nu k}$. Now writing explicitly matrix elements for $[\mathbb{V}'_\alpha(an)]^{\rho\nu}_{j_\rho j_\nu}$ and $[\mathbb{V}'_\alpha(an)]^{\nu\rho}_{j_\nu j_\rho}$ together with unitarity (orthogonality) properties from Proposition 5, we obtain the statement of Corollary 2.

3. Proofs of the theorems from the main text

Proof of Lemma 1. From the definition of fidelity we can write

$$F_{1k} = \langle\psi_{1k}|\rho_{1k}|\psi_{1k}\rangle = \mathrm{Tr}(\rho_{1k}|\psi_{1k}\rangle\langle\psi_{1k}|) = \frac{1}{d}\mathrm{Tr}[\rho_{1k}\, V'(1k)], \quad (A30)$$

where $\frac{1}{d}V'(1k) = |\psi_{1k}\rangle\langle\psi_{1k}|$, $\rho_{1k} = \mathrm{Tr}_{\overline{1k}}\,\rho_{1\dots n}$ and $\mathrm{Tr}_{\overline{1k}}$ denote the partial trace over all systems except 1 and k.

Now we can use the decomposition we mentioned in Eq. (14) to represent $V(1k)$ and $\rho_{1\dots n}$:

$$V'(1k) = \bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \mathbb{V}'_\alpha(1k), \quad \rho_{1\dots n} = \bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \widetilde{\rho}^\alpha, \quad (A31)$$

where α runs over all partitions of $n-2$. Inserting (A31) into (A30), we have

$$F_{1k} = \frac{1}{d}\left[\left(\bigoplus_\mu \mathbb{1}_{r(\mu)} \otimes \widetilde{\rho}^\mu\right)\left(\bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \mathbb{V}'_\alpha(1k)\right)\right] = \frac{1}{d}\mathrm{Tr}\left(\bigoplus_\alpha \mathbb{1}_{r(\alpha)} \otimes \widetilde{\rho}^\alpha\,\mathbb{V}'_\alpha(1k)\right) = \frac{1}{d}\sum_\alpha \mathrm{Tr}[\rho^\alpha\,\mathbb{V}'_\alpha(1k)] = \frac{1}{d}\sum_\lambda \mathrm{Tr}[\rho^\alpha\,\mathbb{V}'_\alpha(k-1n)], \quad (A32)$$

where the last equality follows from Eq. (7). Now, one can see that Eq. (A32) can be written as

$$F_{1k} = \sum_\alpha F^\alpha_{1k}, \quad (A33)$$

where $F^\alpha_{1k} = \frac{1}{d}\sum_\alpha \mathrm{Tr}[\rho^\alpha\,\mathbb{V}'_\alpha(k-1n)]$, $\rho^\alpha = d_\alpha\widetilde{\rho}^\alpha$, and d_α stands for the dimension of the irrep labeled by partition α. ■

Proof of Fact 1. The reader can prove this fact by direct calculation. Namely, one has to compute the fidelity between the state which is a product in the $1|2\dots n$ cut and the maximally entangled state $|\psi_{1k}\rangle$:

$$F^{\mathcal{N}}_{1k} = \frac{1}{d}\langle\psi_{1k}|\,\mathrm{Tr}_{\overline{1k}}(\mathbb{1}_1 \otimes \rho_{2\dots n})|\psi_{1k}\rangle = \frac{1}{d}\langle\psi_{1k}|\mathbb{1}_1 \otimes \rho_k|\psi_{1k}\rangle = \frac{1}{d}\mathrm{Tr}\,\rho_k = \frac{1}{d}. \quad (A34)$$

■

Proof of Theorem 1. The proof is similar to that in [19]. The only difference is the fact that now the fidelities look like that in Eq. (16). ■

Proof of Lemma 2. The proof goes as in [19]. The only new thing in the proof is that the matrices of irreps for transpositions (in), where $1 \leqslant i \leqslant n-1$, are symmetric (see Sec. A 2, Corollary 2). ■

4. Fidelity region for each irreducible space and some applications

In this section we provide some technical details regarding the construction of the admissible region of fidelities for $1 \to N$ UQCM. We focus here for clarity on the case when $N = 3$; then we have two nontrivial irreps, $\alpha_1 = (2)$ and $\alpha_2 = (1,1)$. We also restrict ourselves here to dimensions $d \geqslant 3$ to omit a discussion about the dimension of irrep α_2, but, of course, the construction in this situation is the same. For any $d \geqslant 3$ nontrivial irreps have the same dimension, which is equal to 3; thanks to this fact and Lemma 2 we can write an arbitrary pure state as $|\psi^{\alpha_i}\rangle = (a_1,a_2,a_3)^{\mathrm{T}}$ and the corresponding density matrix as

$$\rho^{\alpha_i} = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix},$$

where $a_1^2 + a_2^2 + a_3^2 = 1$ and $i = 1,2$. Now putting, for example, density matrix $\rho^{(2)}$ into Eq. (16) from Lemma 1, together with irreps $\mathbb{V}'_{(2)}(k\ \ n-1)$ from formula (9), we obtain the following set of equations:

$$F_{12}^{(2)} = \frac{1}{18d}\left[a_1^2(d-1) - 2\sqrt{3}a_1a_2(d-1) + 2\sqrt{2}a_1a_3\sqrt{d-1}\sqrt{d+2} + 3a_2^2(d-1) - 2\sqrt{6}a_2a_3\sqrt{d-1}\sqrt{d+2} + 2a_3^2(d+2)\right],$$

$$F_{13}^{(2)} = \frac{1}{18d}\left\{a_1^2(d-1) + 2a_1[\sqrt{3}a_2(d-1) + \sqrt{2}a_3\sqrt{d-1}\sqrt{d+2}] + 3a_2^2(d-1) + 2\sqrt{6}a_2a_3\sqrt{d-1}\sqrt{d+2} + 2a_3^2(d+2)\right\},$$

$$F_{14}^{(2)} = \frac{1}{9d}\left[2a_1^2(d-1) - 2\sqrt{2}a_1a_3\sqrt{d-1}\sqrt{d+2} + a_3^2(d+2)\right]. \quad \text{(A35)}$$

We can also obtain a similar set of equations for partition $(1,1)$. Moreover we know that the fidelity from ideal $\mathcal{N}$ is always equal to $1/d$ (see Fact 1). In the next step we use *Mathematica* to generate parametric plots of regions given by formulas of the form (A35) together with the normalization condition $a_1^2 + a_2^2 + a_3^2 = 1$. Thanks to this we get an admissible range of fidelities in every irreducible space labeled by partition α_i. Due to Theorem 1, to obtain an admissible region of fidelities, we have to generate the convex hull of the allowed regions obtained for every irreducible representation α. To do this we have used the *Mathematica* package CONVEXHULL3D. One can see that to generate admissible regions for a number of clones larger than 3 we need higher-dimensional space to embed the convex hull, so we cannot represent our results in graphical form. There is still some way to omit this problem at least partially. Namely, we can construct some projection which maps convex hulls from d-dimensional space to three-dimensional space, but then of course we lose some information.

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