Recurrent construction of optimal entanglement witnesses for 2N-qubit systems

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We provide a recurrent construction of entanglement witnesses for a bipartite systems living in a Hilbert space corresponding to $2N$ qubits (N qubits in each subsystem). Our construction provides a method of generalization of the Robertson map that naturally meshes with $2N$-qubit systems, i.e., its structure respects the 2^{2N} growth of the state space. We prove that for $N > 1$ these witnesses are indecomposable and optimal. As a byproduct we provide a family of PPT (Positive Partial Transpose) entangled states.

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I. INTRODUCTION

Entanglement witnesses (EWs) provide universal tools for analyzing and detecting quantum entanglement [1,2]. Let us recall that a Hermitian operator $\mathcal{W}$ defined on a tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called an EW if and only if (iff) $\langle \psi_A \otimes \phi_B|\mathcal{W}|\psi_A \otimes \phi_B\rangle \geqslant 0$ for all product vectors $\psi_A \otimes \phi_B$ in $\mathcal{H}$ and $\mathcal{W}$ possesses at least one negative eigenvalue. It turns out that a state ρ in $\mathcal{H}$ is entangled iff it is detected by some EW [3]; that is, iff there exists an EW $\mathcal{W}$ such that $\mathrm{Tr}\,(\mathcal{W}\rho) < 0$. In recent years there has been a considerable effort in constructing and analyzing the structure of EWs (see, e.g., Refs. [4–20] and a recent review [21]). However, the general construction of an EW is not known. Let us recall that an entanglement witness $\mathcal{W}$ is decomposable if

$$\mathcal{W} = A + B^{\Gamma}, \tag{1}$$

where $A,B \geqslant 0$ and B^{Γ} denotes the partial transposition of B. EWs that cannot be represented as Eq. (1) are called indecomposable. Indecomposable EWs are necessary to detect positive partial transpose (PPT) entangled states (a state ρ is PPT if $\rho^{\Gamma} \geqslant 0$). If ρ is PPT, $\mathcal{W}$ is an EW and $\mathrm{Tr}\,(\mathcal{W}\rho) < 0$, then ρ is entangled and $\mathcal{W}$ is necessarily indecomposable. The optimal EW is defined as follows: if $\mathcal{W}_1$ and $\mathcal{W}_2$ are two entanglement witnesses then, following Ref. [5], we call $\mathcal{W}_1$ finer than $\mathcal{W}_2$ if $D_{\mathcal{W}_1} \supseteq D_{\mathcal{W}_2}$, where

$$D_{\mathcal{W}} = \{\rho|\mathrm{Tr}\,(\rho\mathcal{W}) < 0\,\}$$

denotes the set of all entangled states detected by $\mathcal{W}$. Now, an EW $\mathcal{W}$ is optimal if there is no other witness that is finer than $\mathcal{W}$. One proves that $\mathcal{W}$ is optimal iff for any $\alpha > 0$ and a positive operator P the operator $\mathcal{W} - \alpha P$ is no longer an EW [5]. The authors of Ref. [5] provided the following sufficient condition of optimality: for a given EW $\mathcal{W}$ one defines

$$P_{\mathcal{W}} = \{|\psi \otimes \phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B|\langle \psi \otimes \phi|\mathcal{W}|\psi \otimes \phi\rangle = 0\}. \tag{2}$$

If $P_{\mathcal{W}}$ spans $\mathcal{H}_A \otimes \mathcal{H}_B$, then $\mathcal{W}$ is optimal.

By using the well-known duality between bipartite operators in $\mathcal{H}_A \otimes \mathcal{H}_B$ and linear maps $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, one associates with a given EW $\mathcal{W}$ a linear positive map by $\Lambda_{\mathcal{W}}$ such that $\mathcal{W} = (\mathcal{I} \otimes \Lambda_{\mathcal{W}})P_A^+$, where P_A^+ denotes the maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_A$, and $\mathcal{I}$ denotes the identity map. Due to the fact that $\mathcal{W} \ngeqslant 0$ the corresponding map $\Lambda_{\mathcal{W}}$ is not completely positive (CP).

In the present paper we provide a recurrent construction of a family of positive maps $\Psi_N : \mathbb{M}_2^{\otimes N} \to \mathbb{M}_2^{\otimes N}$ for $N \geqslant 1$. Equivalently, we define a family of EWs $\mathcal{W}_N$ in $(\mathbb{C}^2)^{\otimes N} \otimes (\mathbb{C}^2)^{\otimes N}$. Interestingly, Ψ_1 reproduces the well-known reduction map and for $N = 2$ our construction reproduces the Robertson map [22]. However, for $N \geqslant 3$ it provides a different class of positive maps (equivalently EWs). Moreover, we show that for $N > 1$ these EWs are indecomposable and optimal and hence may be used to detect PPT entangled states. Finally, we show that the so-called structural physical approximation to $\mathcal{W}_N$ is a separable state [23]. As a byproduct we provide PPT entangled states detected by our witnesses.

II. RECURRENT CONSTRUCTION

In what follows we provide a recurrent construction of linear positive maps

$$\Psi_N : \mathbb{M}_2^{\otimes N} \longrightarrow \mathbb{M}_2^{\otimes N},$$

where $\mathbb{M}_2^{\otimes N}$ denotes a tensor product of N copies of $\mathbb{M}_2$ (a space of 2×2 complex matrices). Let us start with a "vacuum" map $\Psi_0 : \mathbb{C} \to \mathbb{C}$ defined by $\Psi_0(z) = 0$ which is evidently positive but not very interesting. Out of Ψ_0 we construct a family of nontrivial positive maps via the following formula:

$$\Psi_{N+1}\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2^N}\left(\begin{array}{c|c} D_{11} & -A_N \\ \hline -B_N & D_{22} \end{array}\right), \tag{3}$$

with the diagonal blocks defined as

$$D_{ii} = \mathbb{1}_2^{\otimes N}(\mathrm{Tr}\,X - \mathrm{Tr}\,X_{ii}),$$

and the off-diagonal blocks given recursively by

$$A_N = X_{12} + \Psi_N(X_{21}), \quad B_N = X_{21} + \Psi_N(X_{12}).$$

In Eq. (3) one uses $\mathbb{M}_2^{\otimes(N+1)} = \mathbb{M}_2 \otimes \mathbb{M}_2^{\otimes N}$ and hence we can rewrite $X = \sum_{i,j=1}^2 e_{ij} \otimes X_{ij}$, with $X_{ij} \in \mathbb{M}_2^{\otimes N}$ and $e_{ij} = |i\rangle\langle j|$. It is clear from the construction that each Ψ_N is trace preserving and unital, i.e., $\Psi_N(\mathbb{1}_2^{\otimes N}) = \mathbb{1}_2^{\otimes N}$.

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Interestingly, one finds $\Psi_1 : \mathbb{M}_2 \to \mathbb{M}_2$ to be

$$\Psi_1 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix},$$

which reconstructs the reduction map in $\mathbb{M}_2$, i.e.,

$$\Psi_1(X) \equiv \mathcal{R}(X) = \mathbb{1}_2 \mathrm{Tr}\, X - X.$$

This map is known to be positive, decomposable, and optimal (even extremal) [15]. Similarly one can reproduce the Robertson map:

$$\Psi_2 \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array} \right) = \frac{1}{2} \left(\begin{array}{c|c} \mathbb{1}_2 \mathrm{Tr}\, X_{22} & -A_1 \\ \hline -B_1 & \mathbb{1}_2 \mathrm{Tr}\, X_{11} \end{array} \right),$$

with

$$A_1 = X_{12} + \mathcal{R}(X_{21}), \quad B_1 = X_{21} + \mathcal{R}(X_{12}),$$

which is known to be positive, indecomposable, and extremal [13]. Recently, this map has been generalized to higher-dimensional bipartite systems in several ways [13–16]. In all cases these generalizations lead to families of indecomposable and optimal maps. The main difference with Refs. [13–16] is that the present construction is recurrent; that is, each step uses the map constructed a step before. In Refs. [13–16] each family of positive maps is constructed via different generalizations of the same basic (reduction or Robertson) map.

III. PROPERTIES OF Ψ_N

In this section we analyze the basic properties of the family of maps Ψ_N. We already noted that Ψ_N is positive for $N = 0$, 1, and 2 (actually, the vacuum map Ψ_0 is even CP). The crucial result of this paper consists in the following:

Theorem 1. The map Ψ_N is positive for any N.

Proof. See the appendix.

Note that for $N \geqslant 1$ the map Ψ_N is not CP. Indeed, the corresponding EW $\mathcal{W}_N = (\mathbb{1}_N \otimes \Psi_N)P^+$ possesses exactly one negative eigenvalue,

$$\mathcal{W}_N \phi^+ = -\frac{1}{2^N}\phi^+,$$

where $\phi^+ = \sum_{i=1}^{2^N} e_i \otimes e_i$ denotes the (unnormalized) maximally entangled state. The existence of a negative eigenvalue of $\mathcal{W}_N$ proves that Ψ_N is not CP and hence $\mathcal{W}_N$ is a legitimate entanglement witness.

We already noticed that Ψ_1, corresponding to the reduction map, is decomposable while Ψ_2, corresponding to the Robertson map, is indecomposable. One has the following theorem:

Theorem 2. The map Ψ_N is indecomposable for $N > 1$.

Proof. To prove indecomposability of Ψ_N it is enough to find a PPT state ρ such that $\mathrm{Tr}\,(\mathcal{W}_N \rho) < 0$. Let us consider the following construction of a family of (unnormalized) matrices parametrized by $t \in \mathbb{R}$:

$$\rho_t = \sum_{i,j=1}^{2^N} e_{ij} \otimes \rho_{ij}, \tag{4}$$

with the $2^N \times 2^N$ blocks ρ_{ij} defined as follows:

(i) $\rho_{ii} = \frac{1}{2^N}\mathbb{1}_{2^N} - (2^{N-1} - 1)W_{ii}$ for $i = 1, \ldots, 2^N$,

(ii) $\rho_{ij} = \mathbb{O}_{2^N}$ if $i \neq j$ and $i,j \leqslant 2^{N-1}$ or $i,j > 2^{N-1}$,

(iii) $\rho_{i,i+2^{N-1}} = -t W_{i,i+2^{N-1}}$ for $i = 1, \ldots, 2^{N-1}$,

(iv) $\rho_{ij} = \frac{1}{2^N 2^{N-1}} e_{ij}$ in the remaining cases,

and $W_{ij} = \frac{1}{2^N}\Psi_N(e_{ij})$. Figure 1 shows how the minimal eigenvalue of the state ρ_t and the minimal eigenvalue of the partially transposed state ρ_t^Γ depends on the parameter t. The smallest eigenvalue of ρ_t^Γ becomes strictly negative for $t < -1$ and $t > 1$. Thus ρ_t is PPT iff $|t| \leqslant 1$. This statement is true for all $N > 1$.

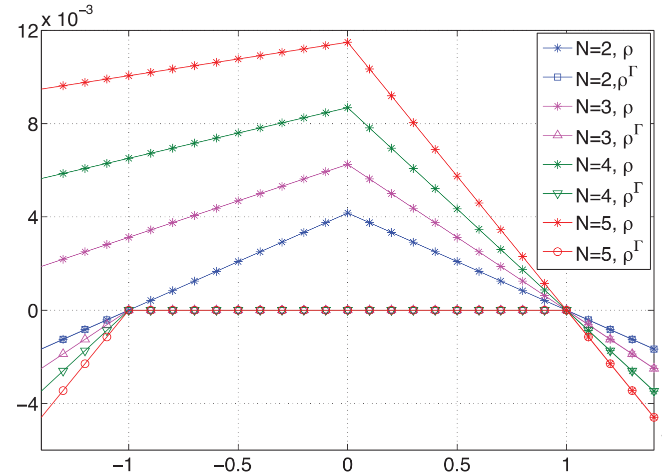

FIG. 1. (Color online) Smallest eigenvalues of the matrix ρ_t defined by Eq. (4) and ρ_t^Γ as a function of the parameter $t \in [-1.5; 1.5]$ for four different N. In the case of $N = 2,4,5$ eigenvalues are scaled so that everything can be shown on one plot. It does not affect the positivity of eigenvalues.

One shows that for any N the expectation value of $\mathcal{W}_N$ in the state ρ_t is given by

$$\mathrm{Tr}\,(\mathcal{W}_N \rho_t) = \frac{-4t(2^N + 4) + 2^{N+2}}{2^{4N}},$$

and hence ρ_t is entangled for $t \in (\frac{2^N}{2^N+4}, 1]$. The analysis of the few first cases is shown in Fig. 2.

Proposition 1. ρ_t is PPT iff $|t| \leqslant 1$.

Proof. Let us start with $N = 2$. One finds the following matrix representation of an entanglement

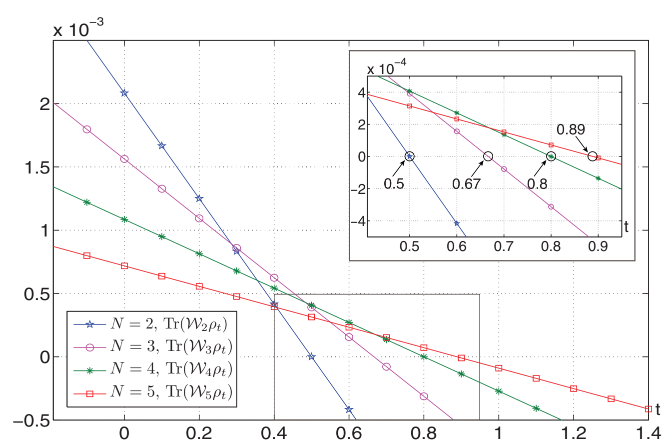

FIG. 2. (Color online) Expectation value of $\mathcal{W}_N$ in state ρ_t for four different values of N.

witness $\mathcal{W}_2$:

$$
\frac{1}{8}\left(\begin{array}{cccc|cccc|cccc|cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right),
$$

and the (unnormalized) matrix ρ_t

$$
\left(\begin{array}{cccc|cccc|cccc|cccc}
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & t \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2
\end{array}\right),
$$

where $2^N \times 2^N$ blocks are separated by horizontal and vertical lines. Moreover, to make the picture more transparent we denote zeros by dots. Note that positivity of ρ is controlled by 2×2 and 4×4 matrices:

$$
A_1 = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \quad A_2 = \left(\begin{array}{cc|cc} 2 & 0 & t & 1 \\ 0 & 2 & 1 & t \\ \hline t & 1 & 2 & 0 \\ 1 & t & 0 & 2 \end{array}\right),
$$

with the corresponding eigenvalues: $1 \pm t$ for A_1, and $1 \pm t$, $3 \pm t$ for A_2. Clearly, $\rho \geqslant 0$ iff $|t| \leqslant 1$. Similarly, the positivity of ρ^Γ is controlled by A_1 and hence $\rho^\Gamma \geqslant 0$. In this case one finds

$$
\mathrm{Tr}\,(\mathcal{W}_2 \rho_t) = (1 - 2t)/16,
$$

which shows that ρ_t is entangled for $t > 1/2$.

For $N = 3$ the positivity of ρ_t is controlled again by 2×2 matrix A_1 and the corresponding 8×8 matrix A_2

$$
A_2 = \left(\begin{array}{cccc|cccc}
4 & 0 & 0 & 0 & t & 1 & 1 & 1 \\
0 & 4 & 0 & 0 & 1 & t & 1 & 1 \\
0 & 0 & 4 & 0 & 1 & 1 & t & 1 \\
0 & 0 & 0 & 4 & 1 & 1 & 1 & t \\
\hline
t & 1 & 1 & 1 & 4 & 0 & 0 & 0 \\
1 & t & 1 & 1 & 0 & 4 & 0 & 0 \\
1 & 1 & t & 1 & 0 & 0 & 4 & 0 \\
1 & 1 & 1 & t & 0 & 0 & 0 & 4
\end{array}\right).
$$

Its eigenvalues are given by

$$
1 - t, \quad 5 - t, \quad 3 - t, \quad 7 + t,
$$

where $5 - t$ and $7 + t$ are three-fold degenerate. Hence $\rho_t \geqslant 0$ iff $|t| \leqslant 1$. Now, positivity of ρ^Γ is again controlled by the 2×2 matrix A_1 and hence ρ_t is PPT iff $|t| \leqslant 1$.

In the general case, the corresponding $2^N \times 2^N$ matrix A_2 has the following structure:

$$
A_2 = \left(\begin{array}{c|c} X & Y \\ \hline Y & X \end{array}\right),
$$

and the $2^{N-1} \times 2^{N-1}$ matrices X and Y read as follows:

$$
X = \begin{pmatrix} 2^N & 0 & \dots & 0 \\ 0 & 2^N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2^N \end{pmatrix}, \quad Y = \begin{pmatrix} t & 1 & \dots & 1 \\ 1 & t & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & t \end{pmatrix}.
$$

One finds the corresponding eigenvalues

$$
1 - t, \quad 2^{N-1} + 1 - t, \quad 2^{N-1} - 1 - t, \quad 2^N - 1 + t,
$$

where $2^{N-1} + 1 - t$ and $2^{N-1} - 1 - t$ are $(2^{N-1} - 1)$-fold degenerate. Hence if $t \leqslant 1$, then ρ_t is positive. Again, positivity of ρ^Γ is controlled by the 2×2 matrix A_1 and hence ρ_t is PPT iff $|t| \leqslant 1$, which ends the proof.

One can observe that the larger the number of "qubits", the smaller the range of t for which the witness detects entanglement of ρ_t. The decreasing range of t can intuitively be ascribed to the fact that having more qubits in our system leads to spreading out the same "amount" of entanglement between more particles. As a consequence, our witness $\mathcal{W}_N$ might not become strong enough to detect it. In order to fully understand how the entanglement is being distributed in states ρ_t and ρ_t^Γ further and more detailed analysis is necessary.

Following Ref. [5] to prove that Ψ_N are optimal for $N > 1$ it is enough to find for each N a set of linearly independent product vectors $\psi_i \otimes \phi_i \in \mathbb{C}^{2\otimes N} \otimes \mathbb{C}^{2\otimes N}$ satisfying Eq. (2). Let us consider a set of vectors introduced in Ref. [13]:

$$
\mathcal{G}_\mathcal{W} := \{\psi_\alpha \otimes \psi_\alpha^*, \quad \alpha = 1, \dots, 2^{2N}\},
$$

with $\psi_\alpha \in \{e_l, f_{mn}, g_{mn}\}$, where $\{e_i\}$ stands for an orthonormal basis and

$$
f_{mn} = e_m + e_n, \quad g_{mn} = e_m + i e_n,
$$

for $1 \leqslant m < n \leqslant N$. Direct calculations show that elements of $\mathcal{G}_\mathcal{W}$ are linearly independent and that

$$
\forall\, \alpha = 1, \dots, N, \quad \langle \psi_\alpha \otimes \psi_\alpha^* | \mathcal{W}_N | \psi_\alpha \otimes \psi_\alpha^* \rangle = 0,
$$

which is sufficient to prove the following theorem:

Theorem 3. For all $N \geqslant 1$, Ψ_N defines a class of optimal maps.

Positive, but not completely positive maps, unlike entanglement witnesses, cannot be directly implemented in the laboratory. One way to tackle this problem is to approximate the positive map by a completely positive one which may serve as a quantum operation. Given a positive map $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ one defines a family of maps

$$\widetilde{\Lambda}(p) = p\mathcal{I} + (1-p)\Lambda.$$

Let p_* be the smallest p such that $\widetilde{\Lambda}(p_*)$ is completely positive. One calls $\widetilde{\Lambda}(p_*)$ the structural physical approximation (SPA) of Λ. It was conjectured [23,24] that the structural physical approximation to an optimal positive map defines an entanglement-breaking (EB) map [a completely positive map $\mathcal{E}$ is entanglement breaking if $(\mathcal{I}\otimes\mathcal{E})\rho$ is separable for an arbitrary state ρ, see Ref. [25]]. In the language of EWs SPA conjecture states that if $\mathcal{W}$ is an optimal EW, then the corresponding SPA

$$\mathcal{W}(p_*) = \frac{p_*}{d_A d_B}\mathbb{1}_A \otimes \mathbb{1}_B + (1-p_*)\mathcal{W},$$

defines a separable state. Recently, SPA conjecture has been disproved for indecomposable EWs in Ref. [26] and for decomposable ones in Ref. [27] (see also recent papers [28,29]). Interestingly, the SPA for Ψ_N provides an EB map. To show this let us recall the following result from Ref. [15]:

Corollary 1. If $\Lambda : \mathbb{M}_n \to \mathbb{M}_n$ is a unital map, and the smallest eigenvalue of the corresponding entanglement witness W satisfies $\xi_{\min} \leqslant -\frac{1}{n}$, then the SPA to W defines a separable state.

Since for any $N \geqslant 1$ an entanglement witness $\mathcal{W}_N$ corresponding to Ψ_N possesses only one negative eigenvalue $\xi = -\frac{1}{2^N}$, thus the SPA to Ψ_N indeed defines an entanglement-breaking channel.

IV. CONCLUSIONS

We provided a class of linear positive, but not completely positive, maps in $\mathbb{M}_2^{\otimes N}$. These maps are indecomposable and optimal, and their structural physical approximation gives rise to an entanglement-breaking channel. Equivalently, our construction provides entanglement witnesses for bipartite systems where each subsystem lives in the N-qubit Hilbert space. It would be interesting to generalize the current recursive construction from $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C})$ to $\mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C})$ with arbitrary $d > 2$.

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APPENDIX: PROOF OF THEOREM 1

Proof. We prove the theorem by induction. We already know that it holds for $N = 1$ and $N = 2$. Now, assuming that it is true for Ψ_N we prove it for Ψ_{N+1}. We shall use the fact that Ψ_N is contractive, i.e.,

$$\|\Psi_N(X)\| \leqslant \|X\|, \tag{A1}$$

where $\|X\|$ denotes an operator norm of X, i.e., the maximal eigenvalue of $|X| = \sqrt{XX^\dagger}$. Recall that any unital map is positive iff it is contractive in the operator norm [30]. To show that Ψ_{N+1} defines a positive map it is enough to show that it maps any rank-1 projector into a positive element. Let us consider $P = |\psi\rangle\langle\psi|$ with ψ being an arbitrary vector in $\mathbb{C}^{2^{N+1}}$. Since $\mathbb{C}^{2^{N+1}} = \mathbb{C}^{2^N} \oplus \mathbb{C}^{2^N}$ one can rewrite $\psi = \bigoplus_{i=1}^2 \sqrt{\alpha_i}\psi_i$, with $\psi_1,\psi_2 \in \mathbb{C}^{2^N}$ and $\alpha_1 + \alpha_2 = 1$. Without loosing generality one can assume $\langle\psi_i|\psi_i\rangle = 1$ and hence

$$\Psi_{N+1}(P) = \frac{1}{2^N}\left(\begin{array}{c|c} \mathbb{1}_{2^N}\alpha_2 & -\sqrt{\alpha_1\alpha_2}A_N \\ \hline -\sqrt{\alpha_1\alpha_2}A_N^\dagger & \mathbb{1}_{2^N}\alpha_1 \end{array}\right),$$

with $A_N = |\psi_1\rangle\langle\psi_2| + \Psi_N(|\psi_2\rangle\langle\psi_1|)$. It is clear that $\Psi_{N+1}(P) \geqslant 0$ iff

$$A_N A_N^\dagger \leqslant \mathbb{1}_{2^N}. \tag{A2}$$

Lemma 1. The map Ψ_N satisfies

$$\Psi_N(|x\rangle\langle y|)|x\rangle = 0\ , \quad \langle y|\Psi_N(|x\rangle\langle y|) = 0, \tag{A3}$$

for any vectors $|x\rangle,|y\rangle \in \mathbb{C}^{2^N}$.

We prove this by induction. For $N = 1$ one immediately verifies Eq. (A3). Now, assuming that Eq. (A3) holds for Ψ_N we prove it for Ψ_{N+1}. By using

$$|x\rangle = |x_1 \oplus x_2\rangle, \quad |y\rangle = |y_1 \oplus y_2\rangle,$$

one finds for $2^N\Psi_{N+1}(|x\rangle\langle y|)$

$$\left(\begin{array}{c|c} \langle y_2|x_2\rangle\mathbb{1}_{2^N} & -|x_1\rangle\langle y_2| - \Psi_N(|x_2\rangle\langle y_1|) \\ \hline -|x_2\rangle\langle y_1| - \Psi_N(|x_1\rangle\langle y_2|) & \langle y_1|x_1\rangle\mathbb{1}_{2^N} \end{array}\right),$$

and hence

$$\Psi_{N+1}(|x\rangle\langle y|)|x\rangle \equiv \Psi_{N+1}(|x\rangle\langle y|)\begin{pmatrix}|x_1\rangle\\|x_2\rangle\end{pmatrix} = 0,$$

where we have used $\Psi_N(|x_2\rangle\langle y_1|)|x_2\rangle = 0$. Similarly, $\langle y|\Psi_N(|x\rangle\langle y|) = 0$.

Now, using Lemma A one arrives at

$$A_N A_N^\dagger = |\psi_1\rangle\langle\psi_1| + Q_N,$$

where $Q_N = \Psi_N(|\psi_2\rangle\langle\psi_1|)\Psi_N(|\psi_1\rangle\langle\psi_2|)$. Note that Q_N is supported on the subspace orthogonal to $|\psi_1\rangle$ and hence the set of eigenvalues of $A_N A_N^\dagger$ consists of eigenvalues of Q_N and 1. Now, using contractivity (A1), one obtains

$$\|\Psi_N(|\psi_1\rangle\langle\psi_2|)\| \leqslant \||\psi_1\rangle\langle\psi_2|\| \leqslant 1,$$

which shows that the maximal eigenvalue of Q_N is not greater than 1. This finally proves Eq. (A2).

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