Closed-form solution of Lindblad master equations without gain

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We present a closed-form solution to the eigenvalue problem of a class of master equations that describe open quantum systems with loss and dephasing but without gain. The method relies on the existence of a conserved number of excitations in the Hamiltonian part and the fact that none of the Lindblad operators describe an excitation of the system. In the absence of dephasing Lindblad operators, the eigensystem of the Liouville operator can be constructed from the eigenvalues and eigenvectors of the effective non-Hermitian Hamiltonian used in the quantum jump approach. Open versions of spin chains, the Tavis-Cummings model, and coupled Harmonic oscillators without gain can be solved using this technique.

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I. INTRODUCTION

Master equations in Lindblad form provide the most general dynamical description of open quantum systems under the Markov assumption [1–3]. Sometimes also called Kossakowski-Lindblad equations, due to pioneering works of Kossakowski [4,5], these type of equations have been extensively used to describe atom cooling [6], decoherence in quantum information theory and quantum engineering of states [7,8].

Many Hamiltonian systems have been extended to include dissipation in Lindblad form; however, even if the Hamiltonian part of the system is solvable, the full solution to the dissipative version is not obvious. The problem results from the fact that the Liouville operator governing the dynamics is a non-Hermitian operator which acts on density matrices. Efforts have been made to tackle this problem and, while there exist analytical steady-state solutions to some problems [9], there are not many solutions to the eigenvalue problem of specific systems. The dissipative version of the Harmonic oscillator, up to two spins, and the Jaynes-Cummings (JC) model are the only open systems for which exact solutions of the eigenvalue problem are known [10–15]. The solutions to the Jaynes-Cummings model that can be found in the literature [10,14] are examples that show how intricate the calculation of the eigensystem of the Liouville operator can be. In particular, the work of Briegel and Englert [10] solves this problem in terms of the eigenbases of the uncoupled subsystems which are formed by a damped harmonic oscillator and a damped two-level atom. The interaction part operates in a nontrivial way on the elements of these combined bases and, therefore, although manageable in this case, this procedure is not suitable to generalize to higher dimensional systems as it would lead to very tedious calculations.

In this work we present a systematic method for solving the eigenvalue problem of a broad class of Lindblad master equations which do not involve any form of gain and that share the characteristic of being solvable in the Hamiltonian part with an additional constant of motion that measures the number of excitations in the system. Under these assumptions we are able to find the eigenvalues and the eigenbasis of the operator that is obtained by subtracting from the Liouville operator, the jump operator of the quantum jumps approach [2,16]. We use an expansion in terms of the elements of this basis to solve for the eigensystem of the complete Liouville operator and obtain a first-order vector recurrence relation that can be solved in an iterative way. In the absence of dephasing Lindblad operators, the eigensystem of the full Liouvillian can be constructed in a systematic way from the eigensystem of the non-Hermitian Hamiltonian of the quantum jumps approach. Specifically, it is shown that each eigenvalue of the complete system is proportional to the sum of two eigenvalues of the corresponding non-Hermitian Hamiltonian. The main difference with respect to the procedure used in [10] is that we use the eigenbasis of the part of the master equation without the jump operator, instead of the eigenbasis without the interaction term. With this approach we are able to reproduce previous specific solutions to the damped harmonic oscillator [11] and the damped Jaynes-Cummings model [10], but most importantly our method is presented in a general way that encompasses systems such as the dissipative version of Heisenberg XXZ spin chains [17], the Affleck-Kennedy-Lieb-Tasaki (AKLT) model [18], the Bose-Hubbard model [7], the Tavis-Cummings model [19], etc. Our construction is performed for systems that do not present any source of gain; nevertheless, similar arguments lead to exact solutions for analog systems without loss.

The paper is organized as follows. In Sec. II we formalize the assumptions that define the class of systems we want to address. Furthermore, we show the procedure to solve the eigensystem of the master equation in terms of the eigensystem of an effective non-Hermitian Hamiltonian. In Sec. III we consider systems that include dephasing Lindblad operators and explain how to solve this class of systems in connection to the method presented in Sec. II. Finally, in Sec. IV we present two examples of systems that can be solved using this technique: The Jaynes-Cummings model and the two-atoms Tavis-Cummings model. As an application we evaluate the atomic spontaneous emission spectrum of the first example.

II. THE MASTER EQUATION

We consider systems whose dynamics are governed by a master equation, which consists of a coherent Hamiltonian

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evolution and a dissipative part in Lindblad form. The dynamical equation is given in terms of the so-called Liouville operator as

$$\mathcal{L}\rho = \frac{1}{i\hbar}[H,\rho] + \sum_s \frac{\gamma_s}{2}(2A_s\rho A_s^\dagger - A_s^\dagger A_s\rho - \rho A_s^\dagger A_s). \quad (1)$$

Concerning the Hamiltonian part we assume that no source of driving is present in the system and that an observable I exists which commutes with H. This additional constant of motion may be interpreted as a measure of the excitations in the system. Furthermore, we consider that the system is defined on a Hilbert space $\mathcal{H}$ and that there exists a complete basis set $\{|n,j\rangle\}$ with $\mathcal{N}$ elements. In this basis I is diagonal and there are d_n states with the same integer eigenvalue of I, that is, $I|n,j\rangle = n|n,j\rangle$, with $n = 0,\dots N$ and $\mathcal{N} = \sum_{n=0}^{N} d_n$. We make all the treatment for finite N, but under the same line of thought of [10,11,15] the results also apply in the limit $N \to \infty$, which also implies the limit of $\mathcal{N} \to \infty$, for countable infinite separable Hilbert spaces. As an immediate consequence of the existence of the conserved quantity I we can identify that the Hamiltonian has a block-diagonal form in the basis where I is diagonal, with each block $H^{(n)}$ of size $d_n \times d_n$. This feature is essential and will be exploited in our construction.

For the dissipative part, we first consider only Lindblad operators A_s which describe losses in the system. We formalize this condition with the commutation relation

$$[A_s, I] = A_s. \quad (2)$$

If we were considering the Jaynes-Cummings model [20], we could take the electromagnetic mode annihilation operator a and the spin lowering operator σ^- as Lindblad operators, corresponding to the situation in which the system interacts with a zero-temperature reservoir. In this example, the additional constant of motion would be $a^\dagger a + \sigma^+\sigma^-$.

From the previous consideration it follows that each Hermitian operator $A_s^\dagger A_s$ also commutes with I, which implies a block-diagonal form in the basis where I is diagonal. This motivates rewriting the master equation in two parts, one that conserves the excitations and the other describing the de-excitation of the system. With the introduction of the non-Hermitian Hamiltonian

$$K = H - i\hbar \sum_s \frac{\gamma_s}{2} A_s^\dagger A_s, \quad (3)$$

which can be recognized as the effective Hamiltonian used in the quantum trajectories technique [2,16], one can rewrite Eq. (1) as the sum of the following two parts $\mathcal{L}\rho = \mathcal{K}\rho + \mathcal{A}\rho$, with

$$\mathcal{K}\rho = \frac{1}{i\hbar}(K\rho - \rho K^\dagger), \quad \mathcal{A}\rho = \sum_s \gamma_s A_s \rho A_s^\dagger. \quad (4)$$

The first term describes the part of the dynamics that conserves excitation, while the second one, the jump operator, describes the de-excitations in the system.

In order to solve the master equation, our strategy will be to find the eigensystem of $\mathcal{K}$. Then, we will deduce how the jump operator acts on each of its eigenvectors. Making a plausible ansatz for the eigenvectors of the complete master equation as a superposition of the eigenvectors of $\mathcal{K}$ and then inserting them into the full master equation will allow us to find a solvable recursion relation for the coefficients of the superposition.

A. Eigensystem of K

Given the fact that $[K,I] = 0$ it follows that $K|n,k\rangle = \sum_{j=1}^{d_n} K_{j,k}^{(n)}|n,j\rangle$, i.e., it does not couple eigenvectors of I with different values of n. In the basis $\{|n,j\rangle\}$, K has a block-diagonal form, with each block given by a matrix of size $d_n \times d_n$. We assume that each block can be diagonalized by the transformation

$$\tilde{K}^{(n)} = Q^{\dagger(n)} K^{(n)} R^{(n)}, \quad \text{with} \quad Q^{\dagger(n)} R^{(n)} = \mathbb{I}_{d_n}, \quad (5)$$

where $\tilde{K}^{(n)}$ is a diagonal matrix with the eigenvalues of the nth block in its diagonal and $\mathbb{I}_{d_n}$ is the identity matrix of dimension d_n. We use a tilde throughout this paper to denote when a matrix is expressed in the eigenbasis of K. The $d_n \times d_n$ matrices $Q^{\dagger(n)}$ and $R^{(n)}$ are the blocks of the operators $Q^\dagger$ and R which diagonalize the operator K. The columns (rows) of R ($Q^\dagger$) are the right (left) eigenvectors of K [21] and in the original basis $\{|n,j\rangle\}$ they can be expanded as

$$|r_j^n\rangle = \sum_{k=1}^{d_n} R_{k,j}^{(n)}|n,k\rangle, \qquad |q_j^n\rangle = \sum_{k=1}^{d_n} Q_{k,j}^{(n)}|n,k\rangle. \quad (6)$$

One can verify that these states are also eigenstates of I, i.e., $I|r_j^n\rangle = n|r_j^n\rangle$, and assuming that the transformation in Eq. (5) exists it follows that they are orthogonal and complete:

$$\langle q_k^n|r_j^m\rangle = \delta_{k,j}\delta_{n,m}, \quad \sum_{n=0}^{N}\sum_{j=1}^{d_n} |r_j^n\rangle\langle q_j^n| = \mathbb{I}. \quad (7)$$

The eigenvalue equation for K is then

$$K|r_j^n\rangle = \varepsilon_j^{(n)}|r_j^n\rangle, \quad K^\dagger|q_j^n\rangle = \varepsilon_j^{*(n)}|q_j^n\rangle, \quad (8)$$

with the complex eigenvalues $\varepsilon_j^{(n)}$.

B. Eigensystem of $\mathcal{K}$

The eigensystem of the operator $\mathcal{K}$ can be constructed from the eigensystem of the non-Hermitian Hamiltonian K. It can be verified by inspection of Eq. (4) that the elements

$$\hat{\varrho}_{j,k}^{(l,n)} = |r_j^{n+l}\rangle\langle r_k^n|, \quad \check{\varrho}_{j,k}^{(l,n)} = |q_j^{n+l}\rangle\langle q_k^n|, \quad (9)$$

with $j = 1,\dots d_{n+l}$ and $k = 1,\dots d_n$, are the right and left eigenvectors of $\mathcal{K}$ and that they solve the eigenvalue equation

$$\mathcal{K}\hat{\varrho}_{j,k}^{(l,n)} = \lambda_{j,k}^{(l,n)}\hat{\varrho}_{j,k}^{(l,n)}, \qquad \mathcal{K}^\dagger\check{\varrho}_{j,k}^{(l,n)} = \lambda_{j,k}^{*(l,n)}\check{\varrho}_{j,k}^{(l,n)}, \quad (10)$$

with eigenvalues

$$\lambda_{j,k}^{(l,n)} = \frac{1}{i\hbar}\left[\varepsilon_j^{(n+l)} - \varepsilon_k^{*(n)}\right]. \quad (11)$$

The dual operator of $\mathcal{K}$ is given by $\mathcal{K}^\dagger\rho = \frac{1}{i\hbar}(\rho K - K^\dagger\rho)$ [10,11]. From Eq. (7), it follows that these eigenvectors are orthogonal with respect to the Hilbert-Schmidt inner product:

$$\mathrm{Tr}\left\{\left(\check{\varrho}_{j,k}^{(l,n)}\right)^\dagger \hat{\varrho}_{j',k'}^{(l',n')}\right\} = \delta_{n,n'}\delta_{l,l'}\delta_{j,j'}\delta_{k,k'}. \quad (12)$$

Now, let us study in more detail the operator $\mathcal{K}$ and how it acts on the elements $|n+l,j\rangle\langle n,k|$ with $n+l,n = 0\dots N$,

$j = 1 \ldots d_{n+l}$ and $k = 1, \ldots d_n$. These elements form a basis for the vector space $\mathcal{B}(\mathcal{H})$ of the operators that act on the Hilbert space $\mathcal{H}$. As K does not couple basis elements of different n it follows that $\mathcal{K}$ does not couple elements with different pairs of excitation numbers $n + l$ and n, that is,

$$\mathcal{K}|n+l,j'\rangle\langle n,k'| = \sum_{j,k=1}^{d_{n+l},d_n} \mathcal{K}^{(l,n)}_{j,k,j',k'}|n+l,j\rangle\langle n,k|. \quad (13)$$

This shows that the operator $\mathcal{K}$ is formed by the uncoupled blocks $\mathcal{K}^{(l,n)}$, where each one of them can be represented by a tensor of rank 4 and dimensions $d_{n+l} \times d_n \times d_{n+l} \times d_n$.

To simplify the evaluation we will adopt the following bijective mapping of indices $j,k \to \nu$, with

$$\nu = d_n(j-1) + k, \quad \nu = 1 \ldots D_{l,n} = d_{n+l}d_n. \quad (14)$$

In this convention that maps two indices to one, the tensor in Eq. (13) can now be expressed as a $D_{l,n} \times D_{l,n}$ matrix that acts on vectors of size $D_{l,n}$ which are obtained by vectorizing row by row a matrix of size $d_{n+l} \times d_n$ using the mapping of indices in Eq. (14). With this convention and using the properties of the tensor product, we can express the blocks of $\mathcal{K}$ as

$$\mathcal{K}^{(l,n)} = \frac{1}{i\hbar}(K^{(n+l)} \otimes \mathbb{I}_{d_n} - \mathbb{I}_{d_{n+l}} \otimes K^{*(n)}). \quad (15)$$

Analogous to Eq. (5), there exists a transformation which diagonalizes each block of $\mathcal{K}$. It has the form

$$\tilde{\mathcal{K}}^{(l,n)} = \mathcal{Q}^{\dagger(l,n)}\mathcal{K}^{(l,n)}\mathcal{R}^{(l,n)}, \quad \mathcal{Q}^{\dagger(l,n)}\mathcal{R}^{(l,n)} = \mathbb{I}_{D_{l,n}}. \quad (16)$$

The eigenvectors of $\mathcal{K}$, given in Eq. (9), provide us with the transformation that diagonalizes each of its blocks $\mathcal{K}^{(l,n)}$, given by the tensor product of the matrices with the eigenvectors of K, i.e.,

$$\mathcal{Q}^{(l,n)} = Q^{(n+l)} \otimes Q^{*(n)},$$
$$\mathcal{R}^{(l,n)} = R^{(n+l)} \otimes R^{*(n)}. \quad (17)$$

C. Jump operator

We proceed to study the action of the jump operator $\mathcal{A}$ on the eigenvectors of $\mathcal{K}$. As it is formed by the Lindblad operators A_s we first focus on how these act on the eigenvectors of the non-Hermitian Hamiltonian K. We assume that the action of each A_s on states in the original basis is known. Considering its commutation relation with I given in Eq. (2) we can deduce that it is of the form $A_s|n,j\rangle = \sum_{k=1}^{d_{n-1}} A^{(n)}_{s;k,j}|n-1,k\rangle$. It is manifested in this way that every A_s connects states of the block n to states in the block $n-1$, meaning that the Lindblad operators are also composed of uncoupled blocks $A^{(n)}_s$ of dimension $d_{n-1} \times d_n$. Using the transformations of Eq. (5) it is possible to transform these blocks to the representation in the eigenbasis of K in the following way:

$$\tilde{A}^{(n)}_s = Q^{\dagger(n-1)}A^{(n)}_s R^{(n)}. \quad (18)$$

Thereby, we find that the action of the Lindblad operators onto the right eigenstates of K can be expressed as

$$A_s|r^n_j\rangle = \sum_{k=1}^{d_{n-1}} \tilde{A}^{(n)}_{s;k,j}|r^{n-1}_k\rangle. \quad (19)$$

With the blocks of the Lindblad operators in the representation of the eigenbasis of K it is now possible to build the blocks of the jump operator using the tensor product. They have the form

$$\tilde{\mathcal{A}}^{(l,n)} = \sum_s \gamma_s \tilde{A}^{(n+l)}_s \otimes \tilde{A}^{*(n)}_s \quad (20)$$

and they are matrices of size $D_{l,n-1} \times D_{l,n}$ that connect vectorized matrices of dimension $D_{l,n}$ to others of dimension $D_{l,n-1}$. It is in this representation that one can identify how the jump operator acts on the eigenbasis of $\mathcal{K}$, that is,

$$\mathcal{A}\hat{\varrho}^{(l,n)}_\nu = \sum_{\nu'}^{D_{l,n-1}} \tilde{\mathcal{A}}^{(l,n)}_{\nu',\nu}\hat{\varrho}^{(l,n-1)}_{\nu'}. \quad (21)$$

Analogously we can find the corresponding equation for the dual jump operator acting on the left eigenvectors as

$$\mathcal{A}^\dagger\check{\varrho}^{(l,n)}_\nu = \sum_{\nu'}^{D_{l,n+1}} \tilde{\mathcal{A}}^{*(l,n+1)}_{\nu,\nu'}\check{\varrho}^{(l,n+1)}_{\nu'}. \quad (22)$$

We have adopted the mapping of indices in Eq. (14) to label the eigenvectors of $\mathcal{K}$ and we introduced the dual of the jump operator, defined as $\mathcal{A}^\dagger\rho = \sum_s \gamma_s A^\dagger_s \rho A_s$.

In an alternative calculation one could start with the evaluation of the blocks $\mathcal{A}^{(l,n)}$ in the original basis based on the blocks $A^{(n)}_s$, in the same manner as in Eq. (20). Then one could change the basis using the transformation of Eq. (17) to find

$$\tilde{\mathcal{A}}^{(l,n)} = \mathcal{Q}^{\dagger(l,n-1)}\mathcal{A}^{(l,n)}\mathcal{R}^{(l,n)}. \quad (23)$$

D. Eigensystem of the full master equation

Noting that the jump operator Eq. (21) couples eigenvectors of $\mathcal{K}$ of definite excitation number n with a superposition of eigenvectors of $n-1$ without changing the value of l, it seems reasonable to take as an ansatz for the eigenvectors of the full Liouvillian $\mathcal{L}$ a superposition of eigenvectors of $\mathcal{K}$ with a fixed value of l. The proposed ansatz, in the vectorized convention is $\hat{\rho}^{l,\Lambda} = \sum_{n,\nu} \tilde{v}^{l,\Lambda;n}_\nu \hat{\varrho}^{(l,n)}_\nu$, where Λ is an eigenvalue of the full master equation and for the moment it labels the eigenvectors and its coefficients. Our next step is to study how the full Liouvillian $\mathcal{L}$ acts on these type of states. From Eqs. (10), (14), and (21) one obtains

$$\mathcal{L}\hat{\rho}^{l,\Lambda} = \Lambda\hat{\rho}^{l,\Lambda} = \sum_{n=0}^{N}\sum_{\nu=1}^{D_{l,n}} \tilde{v}^{l,\Lambda;n}_\nu \lambda^{(l,n)}_\nu \hat{\varrho}^{(l,n)}_\nu$$
$$+ \sum_{n=1}^{N}\sum_{\nu,\nu'=1}^{D_{l,n},D_{l,n-1}} \tilde{v}^{l,\Lambda;n}_\nu \tilde{\mathcal{A}}^{(l,n)}_{\nu',\nu}\hat{\varrho}^{(l,n-1)}_{\nu'}. \quad (24)$$

Reordering of indices and matching the elements $\hat{\varrho}^{(l,n)}_\nu$ leaves us with the following recurrence relation for the coefficients at fixed n:

$$(\Lambda - \lambda^{(l,n)}_\nu)\tilde{v}^{l,\Lambda;n}_\nu = \sum_{\nu'}^{D_{l,n+1}} \tilde{\mathcal{A}}^{(l,n+1)}_{\nu,\nu'}\tilde{v}^{l,\Lambda;n+1}_{\nu'}. \quad (25)$$

The relation holds for any complex value of Λ, but we take the simplest one in which the recurrence ends, using a similar

reasoning as in [11]. We find that the eigenvalues for the complete Liouville operator are $\Lambda = \lambda_\mu^{(l,m)}$, for certain m and μ which label inner blocks in the same way as ν. This result tells us that $\mathcal{L}$ and $\mathcal{K}$ have the same spectrum, a fact that can also be understood as $\mathcal{L}$ has an upper triangular form in the basis where $\mathcal{K}$ is diagonal. Another observation is that for $n = m$ the left-hand side of Eq. (25) vanishes, which means that all coefficients are zero for $n > m$. The first nonvanishing coefficient is $\tilde{v}_\mu^{l,\Lambda;m} = 1$. From here one can proceed to evaluate the rest of the coefficients in a recursive way. Note also that three integers are needed to define each eigenvector: m, μ, and l (or four if instead one uses $\mu \to (j'' - 1)d_m + k''$ in the matrix representation). Hence, we redefine the coefficients as $\tilde{v}_\nu^{l,\Lambda;n} \to \tilde{v}_{\mu;\nu}^{(l,m;n)}$.

The recursion relation can also be cast in terms of matrix multiplication, if one takes a vector of coefficients $\tilde{v}_\mu^{(l,m;n)}$ for each block of n. Let us define the nonzero elements of the $D_{l,n} \times D_{l,n}$ diagonal matrix as

$$\tilde{\mathcal{T}}_{\mu;\nu,\nu}^{(l,m;n)} = \left(\lambda_\mu^{(l,m)} - \lambda_\nu^{(l,n)}\right)^{-1}. \tag{26}$$

With this definition, the recursion in Eq. (25) can be solved to give the nth vector with $D_{l,n}$ entries:

$$\tilde{v}_\mu^{(l,m;n)} = \left(\prod_{i=n}^{m-1} \tilde{\mathcal{T}}_\mu^{(l,m;i)} \tilde{\mathcal{A}}^{(l,i+1)}\right) e_\mu^{(l,m)}. \tag{27}$$

Thereby $\tilde{v}_\mu^{(l,m;m)} = e_\mu^{(l,m)}$ is a column vector of dimensions $D_{l,m}$ with a one in the μth entry and zero elsewhere. All the coefficients for $n > m$ vanish. Now one can write the right eigenvectors of the full Liouvillian as

$$\hat{\rho}_\mu^{(l,m)} = \sum_{n=0}^{m} \sum_{\nu=1}^{D_{l,n}} \tilde{v}_{\mu;\nu}^{(l,m;n)} \hat{\varrho}_\nu^{(l,n)}. \tag{28}$$

The left eigenvectors can be evaluated in a similar way and as we already know the eigenvalues of $\mathcal{L}$ we can use a superposition of the left eigenvectors of $\mathcal{K}$ with fixed l to find

$$\mathcal{L}^\dagger \check{\rho}_\mu^{(l,m)} = \lambda_\mu^{*(l,m)} \check{\rho}_\mu^{(l,m)} = \sum_{n=0}^{N} \sum_{\nu=1}^{D_{l,n}} \tilde{u}_{\mu;\nu}^{(l,m;n)} \lambda_\nu^{*(l,n)} \check{\varrho}_\nu^{(l,n)} + \sum_{n=0}^{N-1} \sum_{\nu,\nu'=1}^{D_{l,n},D_{l,n+1}} \tilde{u}_{\mu;\nu}^{(l,m;n)} \tilde{\mathcal{A}}_{\nu,\nu'}^{*(l,n+1)} \check{\varrho}_{\nu'}^{(l,n+1)}. \tag{29}$$

Again, reordering indices and matching the coefficients for each $\check{\varrho}_\nu^{(l,m)}$ we find the recursion relation

$$\left(\lambda_\mu^{*(l,m)} - \lambda_\nu^{*(l,n)}\right) \tilde{u}_{\mu;\nu}^{(l,m;n)} = \sum_{\nu'=1}^{D_{l,n-1}} \tilde{\mathcal{A}}_{\nu',\nu}^{*(l,n)} \tilde{u}_{\mu;\nu'}^{(l,m;n-1)}, \tag{30}$$

which can be iterated to give the solution for the coefficients as

$$\tilde{u}_\mu^{(l,m;n)} = \left(\prod_{i=n-1}^{m} \tilde{\mathcal{T}}_\mu^{\dagger(l,m;i+1)} \tilde{\mathcal{A}}^{\dagger(l,i+1)}\right) e_\mu^{(l,m)}. \tag{31}$$

In this way, we find the following expression for the left eigenvectors:

$$\check{\rho}_\mu^{(l,m)} = \sum_{n=m}^{N} \sum_{\nu=1}^{D_{l,n}} \tilde{u}_{\mu;\nu}^{(l,m;n)} \check{\varrho}_\nu^{(l,n)}. \tag{32}$$

To express the eigenvectors of the full Liouvillian in the original basis, one can apply the transformation in Eq. (17) one by one to each of the vectors in Eqs. (27) and (31) as

$$v_\mu^{(l,m;n)} = \mathcal{R}^{(l,n)} \tilde{v}_\mu^{(l,m;n)}, \quad u_\mu^{(l,m;n)} = \mathcal{Q}^{(l,n)} \tilde{u}_\mu^{(l,m;n)}. \tag{33}$$

Using the mapping of indices in Eq. (14), one finally finds the left and right set of eigenvectors in the original basis:

$$\hat{\rho}_{j,k}^{(l,m)} = \sum_{n=0}^{m} \sum_{j',k'=1}^{d_{n+l},d_n} v_{j,k;j',k'}^{(l,m;n)} |n+l,j'\rangle\langle n,k'|,$$

$$\check{\rho}_{j,k}^{(l,m)} = \sum_{n=m}^{N} \sum_{j',k'=1}^{d_{n+l},d_n} u_{j,k;j',k'}^{(l,m;n)} |n+l,j'\rangle\langle n,k'|. \tag{34}$$

In a matrix representation the right eigenvectors take the following form:

$$\hat{\rho}_{j,k}^{(l,m)} = \begin{array}{c|ccccc|c} & 0 & \dots & m & m+1 & \dots & \\ \hline 0 & & & & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ \hline v_{j,k}^{(l,m;0)} & & & & & & l \\ & \ddots & & & & & \vdots \\ & & v_{j,k}^{(l,m;m)} & & & & m+l \\ & & & 0 & & & m+l+1 \\ & & & & \ddots & & \vdots \end{array}$$

where it is manifested that they are formed by uncoupled blocks that lie in the lth diagonal and there are nonzero entries only until the excitation value m. The left eigenvectors can be represented as

$$\check{\rho}_{j,k}^{(l,m)} = \begin{array}{c|ccccc|c} & 0 & \dots & m & m+1 & \dots & \\ \hline 0 & & & & & & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & & & & & & l+1 \\ & \ddots & & & & & \vdots \\ & & u_{j,k}^{(l,m;m)} & & & & m+l \\ & & & u_{j,k}^{(l,m;m+1)} & & & m+l+1 \\ & & & & \ddots & & \vdots \end{array}.$$

In this case they also lie in the lth diagonal, but in contrast to the right eigenvectors the nonzero entries start at excitation number m.

In this construction we have only focused on elements that lie below the main diagonal $l = 0$. The elements above the main diagonal can be evaluated from those which have $l \neq 0$

by taking the Hermitian adjoint as $\hat{\rho}^{\dagger(l,m)}_{j,k}$ and $\check{\rho}^{\dagger(l,m)}_{j,k}$, with corresponding eigenvalues $\lambda^{*(l,m)}_{j,k}$. Including them completes a basis set that spans the vector space $\mathcal{B}(\mathcal{H})$.

III. INCLUSION OF DEPHASING OPERATORS

In this section we briefly comment on the inclusion of dephasing Lindblad operators C_s in the master equation. These have the commutation relation $[C_s, I] = 0$ with the constant of motion I. Its inclusion results in a Liouville operator that we write in the form $\mathcal{L} = \mathcal{M} + \mathcal{A}$, with $\mathcal{M} = \mathcal{K} + \mathcal{C}$, which is written in terms of the operators in Eq. (4) and the new term

$$\mathcal{C}\rho = \sum_s \frac{\kappa_s}{2}(2C_s\rho C_s^\dagger - C_s^\dagger C_s\rho - \rho C_s^\dagger C_S). \tag{35}$$

The Liouvillian $\mathcal{C}$ also preserves the excitation numbers n and l, but cannot be constructed solely out of a non-Hermitian Hamiltonian because of the dephasing term $\sum_s \kappa_s C_s\rho C_s^\dagger$. Nevertheless, as we have separated the parts of the Liouvillian $\mathcal{L}$ that conserves excitations from the jump operator, the diagonalization in this case can be carried out in a similar manner as it was shown in the previous section. In this case the diagonalization of the blocks of $\mathcal{M}$ is required. These can be evaluated using Eq. (15) with $\mathcal{M}^{(l,n)} = \mathcal{K}^{(l,n)} + \mathcal{C}^{(l,n)}$ and the blocks of $\mathcal{C}$ which can be constructed as

$$\mathcal{C}^{(l,n)} = \sum_s \frac{\kappa_s}{2}\big(2C_s^{(n+l)} \otimes C_s^{*(n)} - [C_s^\dagger C_s]^{(n+l)} \otimes \mathbb{I}_{d_n} - \mathbb{I}_{d_{n+l}} \otimes [C_s^\dagger C_s]^{\top(n)}\big). \tag{36}$$

Here we have assumed that the action of each C_s onto the basis $\{|n,j\rangle\}$ is known and so the $d_n \times d_n$ matrices $C_s^{(n)}$ are known. Thereby, the transformations to be found are those that diagonalize each block as $\mathcal{Q}^{\dagger(l,n)}\mathcal{M}^{(l,n)}\mathcal{R}^{(l,n)}$ and form the eigenvectors of $\mathcal{M}$. From this point, the diagonalization procedure of the full Liouvillian follows the same steps as in Sec. II D, with the eigensystem of $\mathcal{K}$ being replaced by the eigensystem of $\mathcal{M}$ and the blocks of the jump operator evaluated as in Eq. (23). The eigenvalues of $\mathcal{L}$ are the eigenvalues of the excitation-preserving part $\mathcal{M}$.

IV. EXAMPLES

In this section we present examples of systems with physical relevance that can be solved using the technique introduced in Secs. II and III.

A. Jaynes-Cummings model

The first example we consider is the damped Jaynes-Cummings model, which describes the interaction of a two-level system (TLS) with one mode of the electromagnetic field inside an optical cavity [20]. We use this model to test and show our method and compare with the solution introduced by Briegel and Englert in [10].

The Hamiltonian of the JC model in the interaction picture with respect to the cavity mode energy is given by

$$H = \hbar\delta\sigma^+\sigma^- + \hbar g(a\sigma^+ + a^\dagger\sigma^-), \tag{37}$$

where a and $a^\dagger$ are the cavity mode creation and annihilation operators, and $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ are the raising and lowering operators of the two-level system and are defined in terms of the Pauli matrices σ^x, σ^y, and σ^z. The detuning between the TLS frequency gap and the cavity mode is given by δ.

In this case one can recognize that the constant of motion is given by

$$I = a^\dagger a + \sigma^+\sigma^-, \tag{38}$$

and its eigenbasis $\{|n,j\rangle\}$ is given by the states

$$\begin{aligned} |n,1\rangle &= |n\rangle \otimes |g\rangle, \quad n \geqslant 0, \\ |n,2\rangle &= |n-1\rangle \otimes |e\rangle, \quad n > 0, \end{aligned} \tag{39}$$

with the number state $|n\rangle$ describing n photons in the cavity and the atomic excited $|e\rangle$ and ground state $|g\rangle$. The state $|n,2\rangle$ is only permissible for $n > 0$ and so we note that the basis is formed by a singlet state, which is the eigenvector of I with eigenvalue zero and a family of pairs with eigenvalue n. Therefore, $d_0 = 1$ and $d_{n>0} = 2$ in this example.

The Lindblad operators we consider are σ^- and a, which fulfill the commutation relation of Eq. (2). Using them we can construct the effective Hamiltonian of Eq. (3), which reads

$$K = H - i\tfrac{1}{2}\hbar\gamma\sigma^+\sigma^- - i\tfrac{1}{2}\hbar\kappa a^\dagger a. \tag{40}$$

It is a non-Hermitian operator which commutes with I, so that one can write it as a block-diagonal matrix in the basis of Eq. (39), with its blocks given by

$$\begin{aligned} K^{(0)} &= 0, \\ K^{(n>0)} &= \hbar\begin{pmatrix} -i\frac{n\kappa}{2} & g\sqrt{n} \\ g\sqrt{n} & \frac{2\delta - i(n-1)\kappa - i\gamma}{2} \end{pmatrix}. \end{aligned} \tag{41}$$

The eigenvalues can be computed and are given by

$$\varepsilon_j^{(n)} = \hbar\frac{2\delta - i(2n-1)\kappa - i\gamma}{4} + (-1)^j\hbar\sqrt{g^2 n + \frac{(2\delta + i\kappa - i\gamma)^2}{16}}. \tag{42}$$

It can be checked that they are degenerate only in the special case $\delta = 0$ and $16g^2 n = (\kappa - \gamma)^2$. Apart from this case, the eigenvalues of each block are different and the diagonalization of the matrices $K^{(n)}$ can be accomplished with the transformation

$$\begin{aligned} R^{(n>0)} &= \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix}, \\ \theta_n &= \arctan\left(\frac{2\varepsilon_1^{(n)} + i\hbar n\kappa}{2\hbar g\sqrt{n}}\right). \end{aligned} \tag{43}$$

In this example $Q^\dagger = R^\top$, because K is a complex symmetric operator. Together with Eqs. (6) and (39) the right and left eigenvectors of K can be obtained. For $n = 0$ no transformation is needed as the singlet $|0,1\rangle$ is already an eigenstate of K.

Using Eq. (15) together with Eq. (41) one can build the blocks $\mathcal{K}^{(l,n)}$ of the generator $\mathcal{K}$ defined in Eq. (4), which represents the part of the Liouvillian $\mathcal{L}$ which conserves the number of excitations. These blocks can be diagonalized by the transformation $\mathcal{R}^{(l,n)}$ which can be obtained from Eqs. (17)

and (43). It is again an orthonormal transformation in this example. The eigenvalues of $\mathcal{K}$ are also the eigenvalues of the full Liouvillian $\mathcal{L}$ and can be evaluated from Eq. (11), by inserting the result of Eq. (42).

The Lindblad operators can also be expressed in terms of blocks in the basis of Eq. (39). They have the following form:

$$\sigma^{-(1)} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \sigma^{-(n>1)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$a^{(1)} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad a^{(n>1)} = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n-1} \end{pmatrix}. \tag{44}$$

Using them as explained in Sec. II C one can also construct the blocks of the jump operator $\mathcal{A}$. All the information is now assembled and the full solution can be obtained as explained in Sec. II D.

In particular, the part of the eigensystem with lowest excitation number is given by

$$\hat{\rho}^{(0,0)}_{1,1} = |0,1\rangle\langle 0,1|, \qquad \breve{\rho}^{(0,0)}_{1,1} = \mathbb{I},$$
$$\hat{\rho}^{(1,0)}_{j,1} = |r^1_j\rangle\langle 0,1|, \qquad \breve{\rho}^{(1,0)}_{j,1} = |q^1_j\rangle\langle 0,1| + \ldots,$$
$$\hat{\rho}^{(0,1)}_{j,k} = |r^1_j\rangle\langle r^1_k| - \langle r^1_k|r^1_j\rangle|0,1\rangle\langle 0,1|, \qquad \breve{\rho}^{(0,1)}_{j,k} = |q^1_j\rangle\langle q^1_k| + \ldots,$$

with the corresponding eigenvalues $\lambda^{(0,0)}_{1,1} = 0$, $\lambda^{(1,0)}_{j,1} = -i\varepsilon^{(1)}_j/\hbar$ and $\lambda^{(0,1)}_{j,k} = -i(\varepsilon^{(1)}_j - \varepsilon^{*(1)}_k)/\hbar$.

We close this subsection by remarking that our method can solve the Jaynes-Cummings model by solving 2×2 complex matrices which are the blocks of the effective Hamiltonian widely used in the quantum trajectories approach. The solutions can be manageable at least for few excitation numbers, as will be shown in the following application.

1. Spontaneous emission spectrum

As an application of the solution to the damped Jaynes-Cummings model, we present the evaluation of the spontaneous emission spectrum [22,23] of the atom, which can be written as

$$S(\omega) = \frac{s(\omega)}{2\pi\varsigma} = \frac{\int_0^\infty dt \int_0^\infty dt' e^{i\omega(t'-t)}\langle\sigma^+(t)\sigma^-(t')\rangle}{2\pi \int_0^\infty dt\, \langle\sigma^+(t)\sigma^-(t)\rangle}, \tag{45}$$

where we have introduced ς, the constant integral which appears in the denominator and serves as the normalization factor. To evaluate the correlations functions which appear in Eq. (45), we can use the relation (see [2])

$$\langle\sigma^+(t)\sigma^-(t')\rangle = \mathrm{Tr}\{\sigma^- e^{\mathcal{L}(t'-t)}[\rho(t)\sigma^+]\}, \tag{46}$$

which can be evaluated with the aid of the eigenbasis of $\mathcal{L}$ as one can express the time evolution of any given initial state ρ_0 in the following form:

$$\rho(t) = e^{\mathcal{L}t}\rho_0 = \sum_\lambda \mathrm{Tr}\{\breve{\rho}^\dagger_\lambda \rho_0\} e^{\lambda t}\hat{\rho}_\lambda. \tag{47}$$

The sum runs over all the eigenvalues λ which at this stage have not been specified. Using the results of Eqs. (46) and (47) together with the definition of the emission spectrum $S(\omega)$ in Eq. (45) and performing the integration, one obtains

$$s(\omega) = \sum_{\lambda,\lambda'} \frac{T_{\lambda,\lambda'}}{(\lambda - \lambda' - i\omega)(\lambda' + i\omega)}, \quad \varsigma = \sum_{\lambda,\lambda'} \frac{T_{\lambda,\lambda'}}{-\lambda}, \tag{48}$$

where we have introduced the weight factors $T_{\lambda,\lambda'} = \mathrm{Tr}\{\breve{\rho}^\dagger_\lambda\rho_0\}\mathrm{Tr}\{\breve{\rho}^\dagger_{\lambda'}\hat{\rho}_\lambda\sigma^+\}\mathrm{Tr}\{\sigma^-\hat{\rho}_{\lambda'}\}$. If we assume that initially the atom is in the excited state and the cavity is in the vacuum state, that is, a state given by $\rho_0 = |1,2\rangle\langle 1,2|$, one can realize that the only contributions to those traces are given by the eigenvectors with seven different eigenvalues, namely, $\lambda^{(0,1)}_{j,k}$, $\lambda^{(1,0)}_{k,1}$, and $\lambda^{(0,0)}_{0,0}$ $(j,k = 1,2)$. With this, one gets the solution to the emission spectrum as

$$s(\omega) = \left| \frac{\hbar\sin^2\theta_1}{\varepsilon^{(1)}_1 - \hbar\omega} + \frac{\hbar\cos^2\theta_1}{\varepsilon^{(1)}_2 - \hbar\omega} \right|^2 = \left| \frac{2(2\omega + i\kappa)}{4g^2 + (2\delta - 2\omega - i\gamma)(2\omega + i\kappa)} \right|^2, \tag{49}$$

with the normalization factor given by the expression

$$\varsigma = \frac{\hbar|\sin\theta_1|^4}{i\left(\varepsilon^{(1)}_1 - \varepsilon^{*(1)}_1\right)} + \frac{\hbar|\cos\theta_1|^4}{i\left(\varepsilon^{(1)}_2 - \varepsilon^{*(1)}_2\right)} + 2\mathrm{Re}\left[\frac{\hbar\sin^2\theta_1\cos^2\theta^*_1}{i\left(\varepsilon^{(1)}_1 - \varepsilon^{*(1)}_2\right)}\right] = \frac{4g^2(\gamma+\kappa) + \kappa(4\delta^2 + (\gamma+\kappa)^2)}{4g^2(\gamma+\kappa)^2 + \gamma\kappa(4\delta^2 + (\gamma+\kappa)^2)}. \tag{50}$$

This result can be checked to be in agreement with the ones obtained previously using different methods in [22,23].

2. Including dephasing Lindblad operators

The situation is slightly different when dephasing Lindblad operators are included, those which have the property of commuting with the constant of motion I. In this example we additionally consider σ^z as a Lindblad operator, which clearly has the desired property. To the full Liouvillian one has to include the following dissipator:

$$\mathcal{C}\rho = \gamma_z\left(\sigma^z\rho\sigma^z - \rho\right). \tag{51}$$

As all the Lindblad operators in $\mathcal{C}$ commute with I, one can no longer use the eigensystem of an effective non-Hermitian Hamiltonian K to construct the eigenvalues of the full Liouvillian. There is a dephasing operator $\sigma^z\rho\sigma^z$ which conserves excitations and whose effect cannot be included in K. However, one can exploit the fact that $\mathcal{C}$ conserves the excitations and express it in blocks $\mathcal{C}^{(l,n)}$. To this end one needs the representations of σ^z in the basis of Eq. (39). One can verify that these are $\sigma^{z(0)} = -1$ and $\sigma^{z(n>0)} = -\sigma^z = \mathrm{diag}(-1,1)$, a diagonal matrix with entries -1 and 1. With them one is able to construct the blocks of $\mathcal{C}$ like in

Eq. (36) as

$$\mathcal{C}^{(0,0)} = 0, \quad \mathcal{C}^{(l>0,0)} = \gamma_z(\sigma^z - \mathbb{I}_2),$$
$$\mathcal{C}^{(l\geqslant 0,n>0)} = \gamma_z \sigma^z \otimes \sigma^z - \gamma_z \mathbb{I}_4. \tag{52}$$

The operator $\mathcal{M} = \mathcal{K} + \mathcal{C}$ represents the part of the Liouville operator that conserves excitations and it can be expressed by the uncoupled blocks $\mathcal{M}^{(l,n)} = \mathcal{K}^{(l,n)} + \mathcal{C}^{(l,n)}$, with each block given by

$$\mathcal{M}^{(0,0)} = 0,$$
$$\mathcal{M}^{(l>0,0)} = \frac{1}{i\hbar} K^{(l)} + \gamma_z(\sigma^z - \mathbb{I}_2),$$
$$\mathcal{M}^{(l\geqslant 0,n>0)} = -\tfrac{1}{2}\left((2n + l - 1)\kappa + \gamma\right)\mathbb{I}_4 - i \begin{pmatrix} i\frac{\gamma-\kappa}{2} & -g\sqrt{n} & g\sqrt{l+n} & 0 \\ -g\sqrt{n} & -i2\gamma_z - \delta & 0 & g\sqrt{l+n} \\ g\sqrt{l+n} & 0 & -i2\gamma_z + \delta & -g\sqrt{n} \\ 0 & g\sqrt{l+n} & -g\sqrt{n} & i\frac{\kappa-\gamma}{2} \end{pmatrix}. \tag{53}$$

It can be verified that the characteristic polynomial of the 4×4 blocks shown above coincide with the one presented in the work by Briegel and Englert for the same situation [24]. The diagonalization of these blocks $\mathcal{M}^{(l,n)}$ gives the corresponding transformation $\mathcal{R}^{(l,n)}$, which in this case is again orthogonal as the Liouvillian remains complex symmetric and so $\mathcal{Q} = \mathcal{R}^\top$. These transformations have to be used together with Eq. (23) and the blocks of the jump operator defined $\mathcal{A}^{(l,n)}$ formed with the blocks of Eq. (44). The eigenvalues $\lambda_\nu^{(l,n)}$ of $\mathcal{M}^{(l,n)}$ are eigenvalues of the full Liouvillian as well. With these one can compute the left and right eigenvectors of the full Liouvillian as indicated in Sec. II D.

B. Two-atoms Tavis-Cummings model

As a second example we consider the two-atoms Tavis-Cummings model [19] with damping. The model describes two two-level atoms interacting with one mode of the electromagnetic field of an optical cavity. The Hamiltonian that describes this situation can be written as

$$H = \sum_{\ell=1}^{2} \hbar(\delta_\ell \sigma_\ell^+ \sigma_\ell^- + g_\ell(a\sigma_\ell^+ + a^\dagger \sigma_\ell^-)). \tag{54}$$

where ℓ labels the operators for each of the two atoms, which can be of different species as we consider possible distinct couplings strengths g_ℓ and detunings δ_ℓ. The number of excitations in the system can be described by the constant of motion:

$$I = a^\dagger a + \sigma_1^+ \sigma_1^- + \sigma_2^+ \sigma_2^-. \tag{55}$$

The states of the eigenbasis $\{|n,j\rangle\}$ of I can be organized as follows:

$$|n,1\rangle = |n\rangle|g\rangle_1|g\rangle_2,$$
$$|n,2\rangle = |n-1\rangle|g\rangle_1|e\rangle_2, \quad |n,3\rangle = |n-1\rangle|e\rangle_1|g\rangle_2, \quad n > 0,$$
$$|n,4\rangle = |n-2\rangle|e\rangle_1|e\rangle_2, \quad n > 1, \tag{56}$$

where one can note that the degeneracy of I can be divided in blocks of $d_0 = 1$, $d_1 = 3$, and $d_{n>1} = 4$ states.

As Lindblad operators we consider σ_1^-, σ_2^-, and a, and doing so allows us to write the non-Hermitian Hamiltonian as

$$K = H - i\hbar\tfrac{1}{2}\kappa a^\dagger a - i\hbar\tfrac{1}{2}\sum_{\ell=1}^{2} \gamma_\ell \sigma_\ell^+ \sigma_\ell^-. \tag{57}$$

In the representation of the basis states of Eq. (56) the non-Hermitian Hamiltonian has a block-diagonal form with blocks given by

$$K^{(0)} = 0, \quad K^{(1)} = \hbar \begin{pmatrix} -i\frac{1}{2}\kappa & g_2 & g_1 \\ g_2 & \delta_2 - i\frac{1}{2}\gamma_2 & 0 \\ g_1 & 0 & \delta_1 - i\frac{1}{2}\gamma_1 \end{pmatrix},$$
$$K^{(n>1)} = -i\hbar\tfrac{1}{2}\left(n\kappa - \kappa + \gamma_1 + \gamma_2\right)\mathbb{I}_4 + \hbar \begin{pmatrix} i\frac{\gamma_1+\gamma_2-\kappa}{2} & g_2\sqrt{n} & g_1\sqrt{n} & 0 \\ g_2\sqrt{n} & \delta_2 + i\frac{1}{2}\gamma_1 & 0 & g_1\sqrt{n-1} \\ g_1\sqrt{n} & 0 & \delta_1 + i\frac{1}{2}\gamma_2 & g_2\sqrt{n-1} \\ 0 & g_1\sqrt{n-1} & g_2\sqrt{n-1} & \delta_1 + \delta_2 + i\frac{1}{2}\kappa \end{pmatrix}. \tag{58}$$

Because these are at most 4×4 matrices, their eigensystem can be evaluated in analytical form. The solutions are lengthy and we do not present them here but just comment that, as this system can be solved in an analytical way, also the full eigensystem of $\mathcal{L}$ can be evaluated analytically.

The blocks of the Lindblad operators can be expressed as

$$\sigma_1^{-(1)} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^{-(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_1^{-(n>2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2^{-(1)} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_2^{-(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^{-(n>2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{59}$$

and for the Lindbald operator a we obtain the following blocks:

$$a^{(1)} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad a^{(2)} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$a^{(n>2)} = \begin{pmatrix} \sqrt{n} & 0 & 0 & 0 \\ 0 & \sqrt{n-1} & 0 & 0 \\ 0 & 0 & \sqrt{n-1} & 0 \\ 0 & 0 & 0 & \sqrt{n-2} \end{pmatrix}. \tag{60}$$

This is all the information one needs to evaluate the solution of the eigensystem of the Liouville operator of this problem following the steps of Sec. II.

We would like to mention that one could also consider Heisenberg XXZ interaction between the atoms, as the terms $(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+)$ and $\sigma_1^z\sigma_2^z$ also commute with the constant of motion I and the block structure is preserved [25].

C. Other extensions

In principle one could consider higher dimensional systems that belong to the class of systems proposed to be solved using the construction explained in this paper. The solution would involve the diagonalization of blocks of larger dimension and thus is not suitable for analytical calculations. Nevertheless this gives a systematic method of solving the problem efficiently in a numerical way and gives insight into the behavior of open systems. Therefore, we briefly comment on two important classes of systems that can be treated in this way.

1. M interacting spins

For a system composed of M interacting spins we could consider as a constant of motion $I = \sum_\ell^M \sigma_\ell^+\sigma_\ell^-$. In this case the eigenstates of I with eigenvalue n can be chosen as all the states with n spins up, which amounts to $d_n = \binom{M}{n}$ states. This follows from the fact that one has to arrange n spin-up particles out of a total of M. A possible Hamiltonian in this case could be $H = \sum_{\ell>J} J_{\ell,J}\sigma_\ell^z\sigma_J^z + \eta_{\ell,J}(\sigma_\ell^+\sigma_J^- + \sigma_\ell^-\sigma_J^+)$. One could also include the term $(\vec{\sigma}_\ell \cdot \vec{\sigma}'_\ell)^2$ like in the AKLT model [18], because such a term also commutes with I. As Lindblad operators one could consider $\{\sigma_\ell^-\}_{\ell=1}^M$ and the dephasing Lindblad operators $\{\sigma_\ell^z\}_{\ell=1}^M$.

2. M spins interacting with one oscillator

The situation is similar for M interacting spins with an oscillator, such as the general Tavis-Cummings model [19]. The constant of motion is in this case $I = a^\dagger a + \sum_{\ell=1}^M \sigma_\ell^+\sigma_\ell^-$. The degeneracy of the excitation n can be divided in two classes, one with $d_{n<M} = \sum_{n'=0}^n \binom{M}{n'}$ and the other with $d_{n\geqslant M} = 2^M$. This sum can be understood as follows: for each $n - n'$ excitation in the oscillator, there are $\binom{M}{n'}$ spin states with excitation n'. A typical Hamiltonian in this scenario would be the M atoms Tavis-Cummings Hamiltonian $\sum_{\ell=1}^M \frac{\delta_\ell}{2}\sigma_\ell^z + g_\ell(a^\dagger\sigma_\ell^- + a\sigma_\ell^+)$, and one could also think of interaction between the spins like in the previous example, as its contribution to the Hamiltonian also commutes with the I for this case. Plausible Lindblad operators are $\{a\} \cup \{\sigma_\ell^-\}_{\ell=1}^M$, whereas for dephasing operators one could take $\{\sigma_\ell^z\}_{\ell=1}^M$.

V. CONCLUSIONS

We have presented a systematic method to solve a broad class of master equations in Lindblad form. Our approach is an alternative but also an extension to previous work [10,11]. If the Liouville operator presents exclusively loss Lindblad operators, we obtained the remarkable result that the full set of eigenvalues of the Liouville operator is given by a sum of eigenvalues of the effective non-Hermitian Hamiltonian used in the quantum trajectories approach. When dephasing Lindblad operators are also present, the eigensystem can be obtained by diagonalizing blocks of finite size of the full Liouvillian. We presented two examples of systems that can be solved in an analytical way, as the solution involves the diagonalization of matrices of dimension of at most 4. For blocks larger than 4×4, this approach can be used to improve the numerical diagonalization, as one has to diagonalize blocks of smaller size than the dimension of the whole Hilbert space.

We also show as an application the analytical evaluation of the spontaneous emission spectrum of an excited atom in an empty cavity. This shows that the introduced method can be manageable to deal with analytical calculations. In this sense, it also offers the possibility to deal with other more general classes of Liouvillians in a perturbative manner. In particular the extension to nonzero temperature baths comes to mind, as the Lindbladian term that accounts for the incoherent pumping or gain acts typically with a smaller strength. In this way the eigensystem obtained with this construction might be used as a starting point to solve more general problems using perturbation theory such as the evaluation resonance fluorescence spectra [26,27] or the derivation of reduced master equations [6].

To conclude we would like to mention that these ideas could be also helpful in solving systems like the coupled oscillators that describe optomechanical systems [28]. In this case the free Hamiltonian of the optical mode serves also as a constant of motion and the Lindblad operators of the mechanical mode are of the dephasing type.

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