Weakly nonadditive Polychronakos statistics

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A two-parameter fractional statistics is proposed, which can be used to model a weakly interacting Bose system. It is shown that the parameters of the introduced weakly nonadditive Polychronakos statistics can be linked to the effects of interactions as well as to finite-size corrections. Calculations are made of the specific heat and condensate fraction of the model system corresponding to harmonically trapped Rb-87 atoms. The behavior of the specific heat of three-dimensional isotropic harmonic oscillators with respect tothe statistics parameters is studied in the temperature domain including the BEC-like phase-transition point.

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I. INTRODUCTION

In recent decades, a number of modifications of the conventional Bose–Einstein and Fermi–Dirac statistics have been proposed. Quantum-mechanical generalizations include anyons [1–3] as well as so-called q-deformed algebras [4,5]. Approaches rooted more in the statistical mechanics are represented by Refs. [6–8]. A special subbranch is the nonextensive statistical mechanics based on the Tsallis entropy [9] and its generalizations [10–13].

Methods involving fractional statistics concepts proved to be successful in the studies of the fractional quantum Hall effect, high-temperature superconductivity [14], interacting systems in low dimensions [15,16], cold atomic gases [17], in the analysis of nuclear matter [18], and even in models of dark matter [19]. Nonextensive generalizations for Bose–Einstein and Fermi–Dirac statistics were also developed [20–23]. Note the long-existing terminological confusion between nonextensivity and nonadditivity, which is discussed in detail in Ref. [24]. When the former is sporadically referred to in this work, the nonadditive nature of entropy and arising in this context Tsallis q exponentials are generally meant. The terms *nonextensive statistics* and *nonadditive statistics*, however, continue to be used interchangeably in the scientific literature (cf. Ref. [25]).

The idea of this paper is to suggest a fractional-statistical model with parameters being linked to interactions and finite-size effects. To be more specific, a bosonic system is considered and thus the modifications of statistics proceed from a reference Bose distribution. This primary attention to weakly interacting Bose systems is reasoned by an ongoing interest in this issue. It can be demonstrated by recent studies of trapped two-dimensional systems [26], finite systems [27–29], bosonic mixtures [30], and some other approaches to analyze the influence of interactions [31–33].

The proposed model is based on the Polychronakos statistics [34,35] with the Tsallis q exponential standing instead of the conventional one in the expressions for occupation numbers. The functional form of the distribution function is thus introduced phenomenologically with the following physical motivation: While the nonextensivity can be explained by long-range interactions [36] and the Polychronakos statistics parameter is related to the number-of-states counting, one should not expect that influences of interactions and finite-size effects can be easily attributed to separate modifications of statistics. For instance, the very Polychronakos statistics parameter with a small imaginary part allows us to model a weak dissipative branch of the excitation spectrum of a Bose system [37,38]. The phenomenological introduction of the Tsallis statistics is known in various aspects [39–41] as well.

The paper is organized as follows: The statistics is defined and calculations are outlined in Sec. II. Series expansions of occupation numbers as well as energy with respect to small parameters are presented in Sec. III. Equations for the statistics parameters of the three-dimensional (3D) system of isotropic harmonic oscillators are obtained in Sec. IV. Analysis of the critical temperature of a weakly interacting finite Bose system and respective calculations of the specific heat in Sec. V are followed by calculations for model systems obeying the weakly nonadditive Polychronakos statistics in Sec. VI. Conclusions are given in Sec. VII.

II. STATISTICS DEFINITION

Define the occupation number of the jth level of a system with the elementary excitation spectrum ε_j as

$$n_j = \frac{1}{z^{-1}e_q^{\varepsilon_j/T} - \alpha}, \tag{1}$$

where T is the temperature and z is the fugacity. The Tsallis q exponential is given by

$$e_q^x = [1 + (1-q)x]^{1/(1-q)} \quad \text{for } 1 + (1-q)x > 0. \tag{2}$$

Properties of this and other related functions are well described in Refs. [22,42,43].

Further in this work small deviations from the Bose distribution are considered, so the parameters q and α are represented in the form

$$q = 1 - b, \quad \alpha = 1 + a, \tag{3}$$

with a and b being small corrections.

Since the q exponential originating from the nonextensive or nonadditive statistical mechanics are used, the statistics with the occupation numbers defined by Eqs. (1)–(3) will hereafter be referred to as the *weakly nonadditive Polychronakos statistics* (WNAPS).

While it might seem more natural to introduce the exponential deformation as $e_q^{(\varepsilon_j-\mu)/T}$ instead of $z^{-1}e_q^{\varepsilon_j/T}$, which uses the chemical potential μ (cf. Refs. [20,21]), expressions with fugacity z appear to be mathematically simpler for further analysis. In the weakly nonadditive limit $q \to 1$,

$$e_q^{(\varepsilon_j-\mu)/T} = e_q^{\varepsilon_j/T} e_q^{-\mu/T}\left(1+(1-q)\frac{\varepsilon_j\mu}{T^2}\right), \tag{4}$$

because no simple factorization of the Tsallis q exponentials exists [42]. The fugacity introduced in Eq. (1) can thus be approximately related to the chemical potential as follows:

$$z^{-1} \simeq e_q^{-\mu/T}\left(1+(1-q)\frac{\langle\varepsilon\rangle\mu}{T^2}\right), \tag{5}$$

where the j dependence on the right-hand side is suppressed by substituting ε_j with the energy of the reference system (cf. below) per particle $\langle\varepsilon\rangle = E/N$.

Let the reference Bose system be an ideal gas with spectrum ε_j and degeneracy g_j; the fugacity $z_B = e^{\mu_B/T}$, where μ_B is the chemical potential, is defined by

$$N = \sum_j \frac{g_j}{z_B^{-1}e^{\varepsilon_j/T}-1}, \tag{6}$$

which is understood in the thermodynamic limit. The condition defining the thermodynamic limit itself depends on the system under consideration and will be specified later.

Calculation of thermodynamic functions is made by a simple procedure. First, the fugacity is defined as a function of T and N from

$$N = \sum_j g_j n_j, \tag{7}$$

and then it is inserted into the definition of energy

$$E = \sum_j \varepsilon_j g_j n_j, \tag{8}$$

from which the heat capacity is calculated as the temperature derivative:

$$C = \frac{dE}{dT}. \tag{9}$$

III. SERIES EXPANSIONS

Since small deviations from the reference Bose system are considered, the fugacities also must be expanded around z_B. Let

$$z = z_B + \Delta z_1 \tag{10}$$

for the weakly nonadditive statistics and

$$z = z_B + \Delta z \tag{11}$$

for a weakly interacting finite Bose system.

In the approximation linear with respect to small corrections, the occupation numbers in the WNAPS read

$$\begin{aligned} n_j &= \frac{1}{(z_B+\Delta z_1)^{-1}e_{1-b}^{\varepsilon_j/T}-(1+a)} \\ &= \frac{1}{z_B^{-1}e^{\varepsilon_j/T}-1} + a\frac{1}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2} \\ &\quad + \frac{b\varepsilon_j^2}{2T^2}\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2} + \frac{\Delta z_1}{z_B}\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2}. \end{aligned} \tag{12}$$

This can be compared to the occupation number of the interacting Bose system:

$$\begin{aligned} n_j &= \frac{1}{(z_B+\Delta z)^{-1}e^{(\varepsilon_j+\Delta\varepsilon_j)/T}-1} \\ &= \frac{1}{z_B^{-1}e^{\varepsilon_j/T}-1} - \frac{\Delta\varepsilon_j}{T}\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2} \\ &\quad + \frac{\Delta z}{z_B}\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2}. \end{aligned} \tag{13}$$

Comparing the summands with b in Eq. (12) and with $\Delta\varepsilon_j$ in Eq. (13) one can suggest that the nonadditivity parameter b is (chiefly) responsible for effective accounting of the interaction.

The above expansions can be written in a "macroscopic" form by using

$$N = \sum_j g_j n_j = \sum_j \frac{g_j}{z_B^{-1}e^{\varepsilon_j/T}-1}, \tag{14}$$

with an auxiliary notation

$$Q = \sum_j \frac{g_j}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2} \tag{15}$$

as follows:

$$\begin{aligned} N &= N + aQ + \frac{\Delta z_1}{z_B}(N+Q) \\ &\quad + \frac{b}{2T^2}\sum_j g_j\varepsilon_j^2\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2} \end{aligned} \tag{16}$$

and

$$N = N + \frac{\Delta z}{z_B}(N+Q) - \frac{1}{T}\sum_j g_j\Delta\varepsilon_j\frac{z_B^{-1}e^{\varepsilon_j/T}}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2}. \tag{17}$$

On the other hand, for the energy of a weakly interacting Bose system one has

$$\begin{aligned} E &= \sum_j(\varepsilon_j+\Delta\varepsilon_j)g_j n_j \\ &= E_B + \frac{\Delta z}{z_B}(E_B+D_B) \\ &\quad + \sum_j g_j\Delta\varepsilon_j\frac{z_B^{-1}e^{\varepsilon_j/T}(1-\varepsilon_j/T)-1}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2}, \end{aligned} \tag{18}$$

where

$$E_B = \sum_j \varepsilon_j g_j n_j = \sum_j \frac{g_j\varepsilon_j}{z_B^{-1}e^{\varepsilon_j/T}-1}, \tag{19}$$

$$D_B = \sum_j \frac{g_j\varepsilon_j}{\left[z_B^{-1}e^{\varepsilon_j/T}-1\right]^2}. \tag{20}$$

The energy of the WNAPS system is given by

$$\begin{aligned} E &= \sum_j \varepsilon_j g_j n_j \\ &= E_{\rm B} + aD_{\rm B} + \frac{\Delta z_1}{z_{\rm B}}(E_{\rm B}+D_{\rm B}) \\ &\quad + \frac{b}{2T^2}\sum_j g_j \varepsilon_j^3 \frac{z_{\rm B}^{-1}e^{\varepsilon_j/T}}{\left[z_{\rm B}^{-1}e^{\varepsilon_j/T}-1\right]^2}. \end{aligned} \tag{21}$$

In order to link the parameters a and b to the quantities which characterize the weakly interacting finite system (namely, the spectrum correction $\Delta\varepsilon_j$ and the fugacity correction Δz), one can use the following set of equations:

(i)

$$0 = aQ + \frac{\Delta z_1}{z_{\rm B}}(N+Q) + \frac{b}{2}\sum_j g_j \left(\frac{\varepsilon_j}{T}\right)^2 \frac{z_{\rm B}^{-1}e^{\varepsilon_j/T}}{\left[z_{\rm B}^{-1}e^{\varepsilon_j/T}-1\right]^2}, \tag{22}$$

(ii)

$$0 = \frac{\Delta z}{z_{\rm B}}(N+Q) - \frac{1}{T}\sum_j g_j \Delta\varepsilon_j \frac{z_{\rm B}^{-1}e^{\varepsilon_j/T}}{\left[z_{\rm B}^{-1}e^{\varepsilon_j/T}-1\right]^2},$$

(iii)

$$\begin{aligned} &aD_{\rm B} + \frac{\Delta z_1}{z_{\rm B}}(E_{\rm B}+D_{\rm B}) + \frac{b}{2}\sum_j g_j\varepsilon_j \left(\frac{\varepsilon_j}{T}\right)^2 \frac{z_{\rm B}^{-1}e^{\varepsilon_j/T}}{\left[z_{\rm B}^{-1}e^{\varepsilon_j/T}-1\right]^2} \\ &= \frac{\Delta z}{z_{\rm B}}(E_{\rm B}+D_{\rm B}) + \sum_j g_j \Delta\varepsilon_j \frac{z_{\rm B}^{-1}e^{\varepsilon_j/T}(1-\varepsilon_j/T)-1}{\left[z_{\rm B}^{-1}e^{\varepsilon_j/T}-1\right]^2}. \end{aligned}$$

Equation (22)(ii) just allows expressing the Δz correction directly via $\Delta\varepsilon_j$ in the linear approximation. A third equation is thus required because the WNAPS correction to fugacity Δz_1 is in fact the third parameter.

Before proceeding to the calculations of thermodynamic functions of the WNAPS system, it is worth estimating the values of a and b for some model or real physical systems.

IV. THREE-DIMENSIONAL HARMONIC OSCILLATORS

Further in this work, the calculations are performed for isotropic three-dimensional (3D) harmonic oscillators with frequency ω. Such a model describes a system of particles trapped in an isotropic harmonic potential. For convenience, the summation is substituted by an integration according to the following rule:

$$\sum_j g_j \cdots = \int_0^\infty d\varepsilon g(\varepsilon)\cdots, \tag{23}$$

where the density of states is

$$g(\varepsilon) = \frac{1}{(\hbar\omega)^3}\frac{\varepsilon^2}{2}. \tag{24}$$

Since $g(0)=0$, the contribution of the ground state $j=0$ must be written explicitly for temperatures corresponding to the Bose–Einstein condensation (BEC) phase.

Equation (14) becomes

$$\begin{aligned} N &= n_0 + \frac{1}{(\hbar\omega)^3}\int_0^\infty d\varepsilon \frac{\varepsilon^2/2}{z_{\rm B}^{-1}e^{\varepsilon/T}-1} \\ &= n_0 + \left(\frac{T}{\hbar\omega}\right)^3 \mathrm{Li}_3\, z_{\rm B}, \end{aligned} \tag{25}$$

where the polylogarithm function

$$\mathrm{Li}_s\, z = \sum_{k=1}^\infty \frac{z^k}{k^s}. \tag{26}$$

Energy (19) equals

$$E_{\rm B} = \hbar\omega\left(\frac{T}{\hbar\omega}\right)^4 \mathrm{Li}_4\, z_{\rm B}. \tag{27}$$

The condition defining the thermodynamic limit of a 3D harmonic oscillator system reads [44]

$$\omega N^{1/3} = \text{const}. \tag{28}$$

The correction to the spectrum from the interaction in the case of a δ-like interatomic potential $\Phi(\mathbf{r}) = \lambda\delta(\mathbf{r})$, where $\lambda = 4\pi\hbar^2 a_s/m$ is a coupling constant and m is the mass of an atom, can be majorized by the following expression (cf. Ref. [45]):

$$\Delta\varepsilon_j = \hbar\omega N\frac{\gamma}{j+1}, \tag{29}$$

with

$$\gamma = \frac{4}{\sqrt{2\pi}}\frac{a_s}{a_{\rm ho}}. \tag{30}$$

In the above equations, a_s is the s-wave scattering length and $a_{\rm ho} = \sqrt{\hbar/(m\omega)}$ is the harmonic oscillator length.

Performing integrations in Eqs. (22) one can reduce the equations to

$$0 = a\,(\mathrm{Li}_2\, z_{\rm B} - \mathrm{Li}_3\, z_{\rm B}) + \frac{\Delta z_1}{z_{\rm B}}\mathrm{Li}_2\, z_{\rm B} + 6b\,\mathrm{Li}_4\, z_{\rm B},$$

$$\begin{aligned} &a\,(\mathrm{Li}_3\, z_{\rm B} - \mathrm{Li}_4\, z_{\rm B}) + \frac{\Delta z_1}{z_{\rm B}}\mathrm{Li}_3\, z_{\rm B} + 10b\,\mathrm{Li}_5\, z_{\rm B} \\ &= \frac{\Delta z}{z_{\rm B}}\mathrm{Li}_3\, z_{\rm B} + \int_0^\infty d\xi\,\xi^2 \frac{\Delta\varepsilon(\xi)}{T}\frac{z_{\rm B}^{-1}e^{\xi}(1-\xi)-1}{\left[z_{\rm B}^{-1}e^{\xi}-1\right]^2}, \end{aligned}$$

with

$$\frac{\Delta z}{z_{\rm B}} = \frac{1}{\mathrm{Li}_2\, z_{\rm B}}\int_0^\infty d\xi\,\xi^2\frac{\Delta\varepsilon(\xi)}{T}\frac{z_{\rm B}^{-1}e^{\xi}}{\left[z_{\rm B}^{-1}e^{\xi}-1\right]^2}. \tag{31}$$

This yields

$$\begin{aligned} &aA(z_{\rm B},T) + bB(z_{\rm B},T) \\ &\quad = \left(\frac{\hbar\omega}{T}\right)^2 N\frac{\gamma}{2}[X(z_{\rm B},T)+Y(z_{\rm B},T)], \end{aligned} \tag{32}$$

where

$$A(z_{\rm B},T) = \frac{\mathrm{Li}_3\, z_{\rm B}}{\mathrm{Li}_2\, z_{\rm B}} - \frac{\mathrm{Li}_4\, z_{\rm B}}{\mathrm{Li}_3\, z_{\rm B}}, \tag{33}$$

$$B(z_{\rm B},T) = 10\frac{\mathrm{Li}_5\, z_{\rm B}}{\mathrm{Li}_3\, z_{\rm B}} - 6\frac{\mathrm{Li}_4\, z_{\rm B}}{\mathrm{Li}_2\, z_{\rm B}}, \tag{34}$$

$$X(z_{\rm B},T) = \frac{1}{\mathrm{Li}_2\, z_{\rm B}} \int_0^\infty d\xi\, \frac{\xi^2}{\xi + \hbar\omega/T} \frac{z_{\rm B}^{-1} e^{\xi}}{\left[z_{\rm B}^{-1} e^{\xi} - 1\right]^2}, \tag{35}$$

$$Y(z_{\rm B},T) = \frac{1}{\mathrm{Li}_3\, z_{\rm B}} \int_0^\infty d\xi\, \frac{\xi^2}{\xi + \hbar\omega/T} \frac{z_{\rm B}^{-1} e^{\xi}(1-\xi) - 1}{\left[z_{\rm B}^{-1} e^{\xi} - 1\right]^2}. \tag{36}$$

The parameters a and b appear thus to be temperature dependent. However, the coefficient functions in Eq. (32) are smooth enough, so for calculations in a specific temperature domain the value of T can be fixed, as shown in the next section.

On the other hand, it is straightforward to show that, in the limit of $T \to \infty$, the fugacity tends to zero as

$$z_{\rm B}|_{T\to\infty} = N \left(\frac{\hbar\omega}{T}\right)^3. \tag{37}$$

Coefficient functions $A(z_{\rm B},T)$ and $B(z_{\rm B},T)$ in this limit are

$$A(z_{\rm B},T) = -\tfrac{1}{16} z_{\rm B}, \quad B(z_{\rm B},T) = 4, \tag{38}$$

and for $X(z_{\rm B},T)$ and $Y(z_{\rm B},T)$ one has

$$X(z_{\rm B},T),\ Y(z_{\rm B},T) \to \text{const}. \tag{39}$$

From Eq. (32) it is thus clear that

$$-\frac{1}{16} a N \left(\frac{\hbar\omega}{T}\right)^3 + 4b \propto \frac{1}{T^2}, \tag{40}$$

which means the high-temperature limiting behavior of the parameters a and b is as follows:

$$a \propto T^{\nu} \quad \text{with} \quad \nu \leqslant 0, \quad b \propto \frac{1}{T^2}, \tag{41}$$

and thus classical results are expected without any influence of the statistics deformation as $T \to \infty$.

V. CRITICAL TEMPERATURE IN THREE-DIMENSIONAL CASE

An equation to complement Eq. (32) can be found, for instance, from the definition of the critical temperature of a finite weakly interacting Bose system.

In the thermodynamic limit, the critical temperature T_c of the WNAPS system corresponding to a BEC-like transition is defined by the condition which, in the 3D case, reads

$$N = \left(\frac{T_c}{\hbar\omega}\right)^3 \int_0^\infty \frac{\xi^2/2}{(1+a)e_{1-b}^{\xi} - (1+a)} d\xi, \tag{42}$$

where the critical value of the fugacity is given by $z_c^{-1} = 1 + a$. With linear corrections only, Eq. (42) becomes

$$N = \left(\frac{T_c}{\hbar\omega}\right)^3 \zeta(3) \left[1 - a + b\frac{6\zeta(4)}{\zeta(3)}\right], \tag{43}$$

where $\zeta(s)$ is the Riemann ζ function, $\zeta(s) = \mathrm{Li}_s\, 1$. The critical temperature of the reference Bose system is

$$T_c^{\rm B} = \hbar\omega \left(\frac{N}{\zeta(3)}\right)^{1/3}, \tag{44}$$

and for T_c one easily obtains

$$T_c = T_c^{\rm B}\left[1 + \frac{a}{3} - b\frac{2\zeta(4)}{\zeta(3)}\right]. \tag{45}$$

The shift of the critical temperature in a finite Bose system of N particles is given by [46,47]

$$\frac{\Delta T_c^{\rm fin}}{T_c^{\rm B}} = -\frac{1}{2}\frac{\zeta(2)}{[\zeta(3)]^{2/3}} N^{-1/3}, \tag{46}$$

and the shift due to interaction effects is [46]

$$\frac{\Delta T_c^{\rm int}}{T_c^{\rm B}} = -1.33 \frac{a_s}{a_{\rm ho}} N^{1/6}, \tag{47}$$

where, as above, the harmonic oscillator length $a_{\rm ho} = \sqrt{\hbar/(m\omega)}$ and a_s is the s-wave scattering length. Note that, in the thermodynamic limit,

$$\frac{a_s}{a_{\rm ho}} N^{1/6} \propto (\omega N^{1/3})^{1/2} = \text{const}. \tag{48}$$

does not depend on the number N of particles.

Comparing Eqs. (45)–(47), the relation linking a and b with the system parameters is obtained:

$$\frac{a}{3} - b\frac{2\zeta(4)}{\zeta(3)} = -\frac{1}{2}\frac{\zeta(2)}{[\zeta(3)]^{2/3}} N^{-1/3} - 1.33\frac{a_s}{a_{\rm ho}} N^{1/6}. \tag{49}$$

For a system of 5000 Rb-87 atoms [46], the ratio $a_s/a_{\rm ho} \simeq 2.6 \times 10^{-3}$. Assuming that the a parameter is entirely due to the finite-size correction and that the b parameter is entirely due to interactions, the following numbers are obtained from Eq. (49):

$$a = -0.13, \quad b = 0.022. \tag{50}$$

On the other hand, if Eqs. (32) and (49) are solved simultaneously, the above numbers change slightly:

$$a = -0.16, \quad b = 0.0027. \tag{51}$$

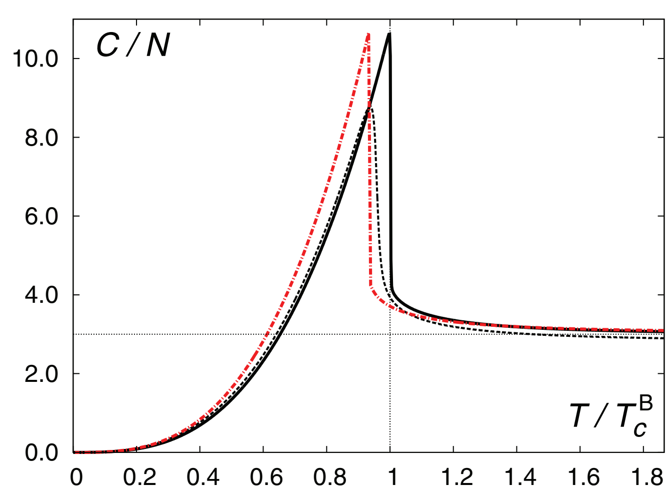

FIG. 1. (Color online) Specific heat of the ideal Bose system of 3D harmonic oscillators (black solid line is thermodynamic limit, black dotted line is for $N = 5000$) compared to the WNAPS system (red dashed-dotted line) in the thermodynamic limit with parameters given by Eq. (51). The discontinuity of the C/N curves is observed in the thermodynamic limit at the critical temperatures but continuous lines are drawn for easier visualization.

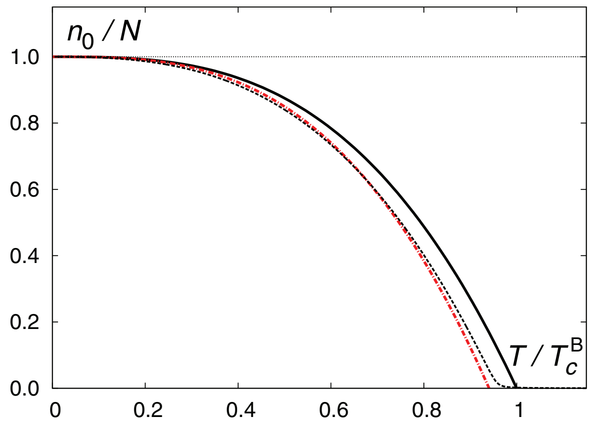

FIG. 2. (Color online) Condensate fraction n_0/N of an ideal Bose system of 3D harmonic oscillators (black solid line is thermodynamic limit, black dotted line is for $N = 5000$) compared to the WNAPS system (red dashed-dotted line) in the thermodynamic limit with parameters given by Eq. (51).

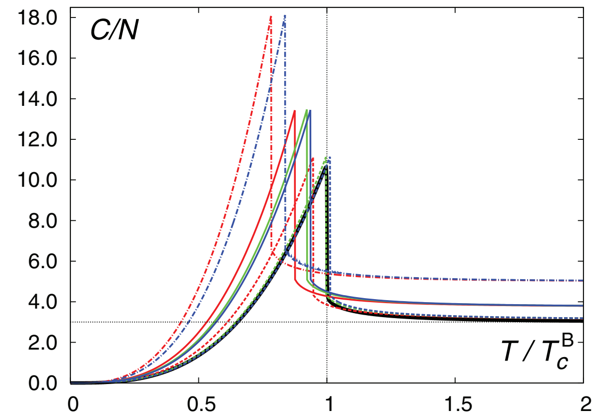

FIG. 3. (Color online) Specific heat of a model WNAPS system with $b =$ const. in the thermodynamic limit for different values of the statistics parameters. Dashed-dotted lines are for $b = 0.1$, solid lines are for $b = 0.05$, and dashed lines are for $b = 0.01$. Red lines are for $a = -0.1$, green lines are for $a = +0.05$, and blue lines are for $a = +0.1$. Black solid line corresponds to the reference Bose system.

The results of calculations of the specific heat C/N for a system with parameters corresponding to the above values are shown in Fig. 1.

Note that a smooth behavior of the specific heat in the vicinity of the critical temperature for a finite Bose system cannot be modeled correctly by the proposed model. A possible solution is to consider a finite WNAPS system as well, which would ensure such a dependence.

The fraction of particles in the ground state n_0, which for the WNAPS system is an analog of a Bose condensate, can be calculated quite easily as

$$\frac{n_0}{N} = 1 - \frac{1}{N}\frac{1}{1+a}\left(\frac{T}{\hbar\omega}\right)^3 \int_0^\infty \frac{\xi^2/2}{e_{1-b}^{\xi} - 1} d\xi. \tag{52}$$

By using the parameters from Eq. (51), the following temperature dependence is obtained for $N = 5000$:

$$\frac{n_0}{N} = 1 - \frac{1.45}{N}\left(\frac{T}{\hbar\omega}\right)^3. \tag{53}$$

The comparison of the above result with reference Bose systems is shown in Fig. 2. The shapes of these dependencies are very similar to those reported in Refs. [46,48].

VI. SPECIFIC HEAT OF MODEL WNAPS SYSTEMS

Having estimated the values of the statistics parameters a and b, it is possible to present some results illustrating the behavior of thermodynamic functions (namely, the specific heat) of model systems obeying the weakly nonadditive Polychronakos statistics. Below, the calculations are made for two modifications of the statistics. In the first one, the parameters a and b are kept constant with respect to temperature. In the second one, the parameter a remains constant but the parameter $b = 2\eta T/(\hbar\omega)$, so that the summands with b in Eq. (12) and with $\Delta\varepsilon_j$ in Eq. (13) become similar.

Indeed, as we assume the second of the above-mentioned statistics modifications, the spectrum shift is

$$\Delta\varepsilon_j = -\frac{b\varepsilon_j^2}{2T} = -\eta\frac{\varepsilon_j^2}{\hbar\omega} = -\eta\hbar\omega j^2. \tag{54}$$

Curiously, such a dependence of the excitation spectrum appears in the problems within deformed Heisenberg algebras. Namely, for the harmonic oscillator with the commutation relation for the coordinate and momentum operators given by

$$[\hat{x},\hat{p}] = i\hbar(1 + \beta\hat{p}^2), \tag{55}$$

the spectrum is [49,50]

$$\varepsilon_j = \hbar\bar{\omega} j + \frac{\beta}{2} j^2, \tag{56}$$

where $\bar{\omega}$ denotes some constant. Due to an extremely small estimated value of β, however, its effect on the thermodynamic properties is unobservable.

The results of calculations of the specific heat are shown in Figs. 3 and 4 in comparison to a reference Bose system. The choice of the values of the statistics parameters is made according to the estimates from the previous section. Note that the $b \propto T$ model is valid only in a limited temperature domain since the system ceases to be weakly nonadditive as $T \to \infty$.

In the model with $b =$ const., the asymptotic value of the specific heat at $T \to \infty$ can be estimated as follows: The fugacity tends to zero as $T \to \infty$, so Eq. (25) in the case of the harmonic oscillator problem under consideration simplifies to

$$N = \left(\frac{T}{\hbar\omega}\right)^3 \frac{z}{2}\int_0^\infty d\xi \frac{\xi^2}{e_{1-b}^{\xi}}. \tag{57}$$

Applying the relation [42]

$$\left[e_q^{f(x)}\right]^p = e_{1-(1-q)/p}^{pf(x)}, \tag{58}$$

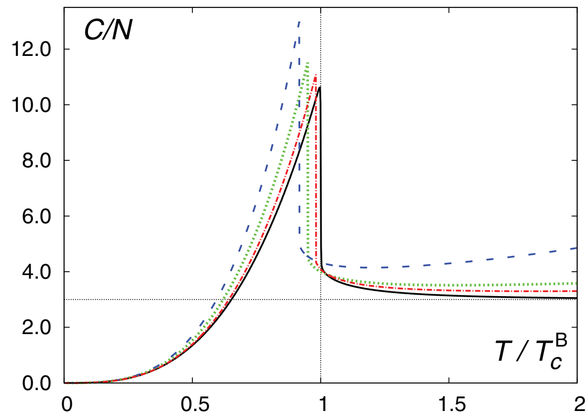

FIG. 4. (Color online) Specific heat of a model WNAPS system with $b = 2\eta T/(\hbar\omega)$ in the thermodynamic limit for different values of the statistics parameters. Red dashed-dotted line is for $a = 0, \eta = 0.0005$; green dotted line is for $a = -0.05, \eta = 0.001$; and blue dashed line is for $a = -0.01, \eta = 0.0025$. Black solid line corresponds to the reference Bose system.

this integral can be calculated by using

$$\int_0^\infty d\xi\, \xi^{k-1} e_{1+b}^{-\xi} = \frac{b^{-k}\Gamma(1/b - k)\Gamma(k)}{\Gamma(1/b)}. \tag{59}$$

In the same limit, the energy equals

$$E = \hbar\omega \left(\frac{T}{\hbar\omega}\right)^4 \frac{z}{2} \int_0^\infty d\xi \frac{\xi^3}{e_{1-b}^{\xi}}. \tag{60}$$

After simple transformations, one obtains for z

$$z = N \left(\frac{\hbar\omega}{T}\right)^3 \frac{(1 - 6b + 11b^2 - 6b^3)}{2}, \tag{61}$$

and the heat capacity is given by

$$\left.\frac{C}{N}\right|_{T\to\infty} = \frac{3}{1 - 4b}, \tag{62}$$

showing no dependence on the value of the a parameter.

VII. CONCLUSIONS

In summary, a two-parameter modification of statistics was proposed, which can be used in particular to model a weakly interacting Bose system. It was shown that the parameters of the introduced weakly nonadditive Polychronakos statistics can be linked to effects of interactions as well as to finite-size corrections.

A simplified WNAPS model was used to describe a system of 5000 harmonically trapped Rb-87 atoms. The calculations of the specific heat C/N of the 3D isotropic harmonic oscillators were also made for several sets of values of the statistics parameters a and b to demonstrate the temperature behavior of C/N in the domain including the BEC-like phase-transition point.

It is expected that WNAPS can provide an alternative mathematical model for Bose systems with weak interatomic interactions and/or a finite number of particles. Its application to correctly reproduce the critical behavior in the vicinity of the transition point requires additional tests based on experimental observations. With minor modifications, the model can be employed for other related systems, in particular lower-dimensional oscillators corresponding to trapped bosons.

Some other physical objects, for which the proposed two-parameter statistics can be used, include anyons and particles in spaces with minimal length. While the latter correspond to the deformed Heisenberg algebra discussed in Sec. VI, the application to the anyonic statistics is elucidated by the possibility of establishing an approximate correspondence with the nonadditive Polychronakos statistics from expressions for virial coefficients. An effective description of long-range interactions and other complex behavior can be expected from this statistics and these issues are the subjects of further studies.

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