Critical study and discrimination of different formulations of electromagnetic force density and consequent stress tensors inside matter

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By examination of the exerted electromagnetic (EM) force on *boundary* of an *object* in a few examples, we look into the compatibility of the stress tensors corresponding to different formulas of the EM force density with special relativity. Ampere-Lorentz's formula of the EM force density is physically justifiable in that the electric field and the magnetic flux density act on the densities of the *total* charges and the *total* currents, unlike Minkowski's formula which completely excludes the densities of the *bounded* charges and the *bounded* currents inside *homogeneous* media. Abraham's formula is fanciful and devoid of physical meaning. Einstein-Laub's formula seems to include the densities of the total charges and the total currents at first sight, but grouping the bounded charges and the bounded currents into pointlike dipoles erroneously results in the hidden momentum being omitted, hence the error in [Phys. Rev. Lett. **108**, 193901 (2012)]. Naturally, the Ampere-Lorentz stress tensor accords with special relativity. The Minkowski sress tensor is also consistent with special relativity. It is worth noting that the mathematical expression of the Minkowski stress tensor can be quite different from the well-known form of this stress tensor in the literature. We show that the Einstein-Laub stress tensor is incompatible with special relativity, and therefore we rebut the Einstein-Laub force density. Since the Abraham momentum density of the EM fields is inherently corresponding to the Einstein-Laub force density [Phys. Rev. Lett. **111**, 043602 (2013)], our rebuttal may also shed light on the controversy over the momentum of light.

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I. INTRODUCTION

The exerted electromagnetic (EM) force on a point charge is given by the Lorentz force law which, together with Maxwell's equations, constitutes one of the five independent principles of EM theory. No matter how Maxwell's equations are mathematically formulated [1], they always express the same notion and yield the same results. In contrast, different mathematical expressions that have been hitherto reported for the EM force density within a material medium could yield different results even when they are applied to one particular problem [2,3].

A survey of the literature shows that the most famous expressions of the EM force density are Ampere-Lorentz's, Einstein-Laub's, Minkowski's, and Abraham's formulas [2–10]. The most physically appealing formula is Ampere-Lorentz's. Still, Einstein has been quoted to disapprove his own formula and endorse Minkowski's [8,11]. More recently, particular attention has been directed toward Ampere-Lorentz's and Einstein-Laub's formulas [4,5,8–10]. It was mistakenly asserted that the former is inconsistent with special relativity [7]. This wrong assertion was soon rectified in scores of publications [12–16].

The Ampere-Lorentz expression of the EM force density is the only one physically justifiable because the electric field ($\vec{E}$) and the magnetic flux density ($\vec{B}$) act on the densities of the *total* charges and the *total* currents. Einstein-Laub's formula seems to have such property, but grouping pairs of charges into pointlike dipoles results in the hidden momentum

being omitted. The lack of the hidden momentum is exactly the same error recently made in [7]. Minkowski's and Abraham's formulas are physically unjustifiable in that they completely exclude the densities of the *bounded* charges and the *bounded* currents inside *homogeneous* media. Moreover, Abraham's formula is based on an esthetic consideration which is devoid of physical meaning.

Once either one of the aforementioned formulas is adopted for the EM force density within a material medium, the overall EM force exerted upon the entire volume of the material can be easily obtained by volume integration of the EM force density. This integral is usually written as a summation of two integrals; first, surface integral of the stress tensor, and second, time rate of decrease of the volume integral representing the momentum of the EM fields inside the volume [2,3]. Obviously different formulations result in different stress tensors and momentum densities. Ampere-Lorentz's force density leads to the Ampere-Lorentz stress tensor and the Livens momentum density. Einstein-Laub's force density leads to the Einstein-Laub stress tensor and the Abraham momentum density. Minkowski's force density leads to the Minkowski stress tensor and the Minkowski momentum density. Abraham's force density leads to the Abraham stress tensor and the Abraham momentum density.

Naturally the debate on the correct form of the EM force density within a material body goes in tandem with the debate on the correct form of the EM stress tensor and the EM momentum density within a material body. Nevertheless, finding the correct form of the EM stress tensor in a material medium has not been a widely studied subject in the literature. Rather, the Minkowski stress tensor, being referred to as the Maxwell stress tensor in Jackson's book [17], other texts [18], and research papers [19–23], has been the most common stress

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tensor in the literature. The correct form of the momentum density in a material medium, on the contrary, has been the subject of controversy in the past [24–28].

In this paper we *theoretically* assess different formulations of the EM stress tensor and thereby evaluate different formulations of the EM force density. To this end, the EM force exerted upon the *boundary* of material *object*s is first calculated by using different stress tensors. In view of the fact that the method of virtual work cannot be accounted a solid criterion for the correct EM force, the compatibility of the obtained results with special relativity is used as the theoretical criterion. Ironically, it is shown that the Einstein-Laub formulation is not necessarily consistent with Einstein's special relativity. This observation is in stark contrast to the assertion that the Einstein-Laub formulation is in perfect harmony with special relativity [7].

The organization of this paper is as follows. First, different formulations of the EM stress tensor are reviewed in Sec. II. Since the mathematical expressions of the stress tensors of Minkowski and Abraham inside a material depend on the constitutive relations that govern the displacement field ($\vec{D}$) and the magnetic flux density ($\vec{B}$), we specifically derive these tensors for three categories of homogeneous materials: linear isotropic dispersive and nondispersive materials, linear reciprocal nondispersive materials with permittivity and/or permeability tensors, and nonlinear materials with permanent electric and/or magnetic dipoles. Peculiarly, the forms of the stress tensors in the latter category of materials are quite different from the well-known forms of the stress tensors used in the literature. This is also in contrast to the assertion of some authors that the applicability of the Minkowski force density is limited to linear materials [8]. In Sec. III, from a viewpoint based on the law of conservation of momentum in EM interactions, the generic problem of the EM force experienced by an object is addressed. Thereby, the total EM force exerted on the object is found together with the distribution of the exerted EM force on the *object boundary*. Despite the opinion of some authors [22], we argue that finding the exact amount of the total EM force exerted on the object in a nonfree space host medium requires more than a macroscopic study of the interface. Besides, all the aforementioned formulations of the EM force density render the same total EM force (or time-averaged EM force) exerted on the object in free space only when the EM fields are static (or time harmonic). It should be noted that even though the contribution of the Helmholtz term in the Minkowski (or Abraham) force density [2,3,29] is no longer zero because there is a discontinuity on the object boundary, finding and integrating the contribution of the Helmholtz term in the exerted Minkowski (or Abraham) force is sidestepped by following the approach of this section. Section IV is devoted to a theoretical assessment of the different formulations of the EM stress tensor. Although the difference among the results obtained by using different formulations for the EM force density is an already known fact [8,30], no conclusive experiment has been carried out to decide on the correct formulation. Regretfully, the result of the method of virtual work cannot be accounted a solid criterion for the correct EM force because the resulting EM force depends on the constitutive relations within the material object in its *virtually expanded or contracted* state. Therefore,

the compatibility of the obtained results with special relativity is exploited as a solid theoretical criterion. Finally, concluding remarks are provided in Sec. V.

II. DIFFERENT FORMULATIONS OF EM FORCE DISTRIBUTION

A. The formulations of Ampere-Lorentz and Einstein-Laub

Ampere and Lorentz naturally, and obviously from the physical point of view, considered that $\vec{E}$ and $\vec{B}$ act on the densities of the total charges and the total currents, i.e., not only the density of the *free* charges (ρ) and the density of the *free* currents ($\vec{J}$), but also the densities of the bounded charges ($-\nabla \cdot \vec{P}$) and the bounded currents ($\frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}$). Therefore, the Ampere-Lorentz force density reads as

$$\vec{f}_{\text{Am}} = (\rho - \nabla \cdot \vec{P})\vec{E} + \left(\vec{J} + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}\right) \times \vec{B}. \quad (1)$$

The Einstein-Laub force density, on the other hand, is as follows:

$$\vec{f}_E = (\rho + \vec{P} \cdot \nabla)\vec{E} + \mu_0\left(\vec{J} + \frac{\partial \vec{P}}{\partial t}\right) \times \vec{H}$$

$$+ \mu_0(\vec{M} \cdot \nabla)\vec{H} - \varepsilon_0\mu_0\frac{\partial \vec{M}}{\partial t} \times \vec{E}. \quad (2)$$

According to [4], the difference between the Ampere-Lorentz and the Einstein-Laub force densities in a nonmagnetic medium ($\vec{M} = 0$) is that the macroscopic electric polarization ($\vec{P}$) is modeled as an aggregate of microscopic charges and microscopic dipoles for the former and latter cases, respectively. It was just the grouping of pairs of separated charges into pointlike dipoles that led Einstein and Laub to an error, because separated charges considered at the same time in their rest frame are subject to forces at different times in a moving frame. The grouping of the bounded charges and the bounded currents into pointlike dipoles erroneously results in the hidden momentum, which is the necessary quantity arising from nonconservation of simultaneity of separate events, being omitted. This is exactly the error made in [7]. Although the error had not been discovered at the Einstein-Laub times, there are papers published decades ago on the subject of relativistic treatment of extended bodies in general [31–33], and magnetic dipoles in particular [34,35].

It can be seen that both of the formulas lead to $(\vec{p}_0 \cdot \nabla)\vec{E} + \mu_0(\partial\vec{p}_0/\partial t) \times \vec{H}$ as the exerted force on a pointlike electric dipole with the electric polarization $\vec{P} = \vec{p}_0\delta(\vec{r})$. The exerted force on a pointlike magnetic dipole with the magnetic polarization $\vec{M} = \vec{m}_0\delta(\vec{r})$ is $\mu_0(\vec{m}_0 \cdot \nabla)\vec{H} + \mu_0\varepsilon_0\vec{m}_0 \times (\partial\vec{E}/\partial t)$, when Ampere-Lorentz's formula is applied, and $\mu_0(\vec{m}_0 \cdot \nabla)\vec{H} - \mu_0\varepsilon_0(\partial\vec{m}_0/\partial t) \times \vec{E}$, when Einstein-Laub's formula is applied. The difference between the obtained forces equals time rate of change of the hidden momentum, i.e., $\partial(\vec{m}_0 \times \vec{E})/(c^2\partial t)$.

Integrations of the Ampere-Lorentz and the Einstein–Laub force densities over an arbitrary volume υ, which is *not* necessarily homogenous, result in a summation of a volume integral of time rate of decrease of the momentum density $\vec{G}$,

and a surface integral of the stress tensor $\overset{\leftrightarrow}{T}$:

$$\int_{\upsilon} \vec{f}d\upsilon = \int_{\upsilon} \frac{-\partial \vec{G}}{\partial t}d\upsilon + \oint_{\partial\upsilon} \hat{n} \cdot \overset{\leftrightarrow}{T}dS, \qquad (3)$$

where $\partial\upsilon$ is the boundary of υ, and $\hat{n}$ is the unit vector normal to the boundary element dS toward the outside of υ. For the Ampere-Lorentz force density, we have the Livens momentum density $\vec{G}_L = \varepsilon_0 \vec{E} \times \vec{B}$, and the Ampere-Lorentz stress tensor $\overset{\leftrightarrow}{T}_{\text{Am}}$ [2,3]:

$$\overset{\leftrightarrow}{T}_{\text{Am}} = \varepsilon_0 \vec{E}\vec{E} + \frac{1}{\mu_0} \vec{B}\vec{B} - \frac{1}{2}\overset{\leftrightarrow}{I}\left(\varepsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \vec{B}\right). \quad (4)$$

For the Einstein-Laub force density, we have the Abraham momentum density $\vec{G}_{\text{Ab}} = \varepsilon_0\mu_0 \vec{E} \times \vec{H}$, and the Einstein-Laub stress tensor $\overset{\leftrightarrow}{T}_E$ [2,3]:

$$\overset{\leftrightarrow}{T}_E = \vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}\overset{\leftrightarrow}{I}(\varepsilon_0 \vec{E} \cdot \vec{E} + \mu_0 \vec{H} \cdot \vec{H}). \qquad (5)$$

It is worth noting that, unlike the stress tensors of Minkowski and Abraham, the forms of the stress tensors of Ampere-Lorentz and Einstein-Laub in Eqs. (4) and (5) are independent of the constitutive relations of $\vec{D}$ and $\vec{B}$.

B. The formulations of Minkowski and Abraham

The Minkowski force density seems to be physically unjustifiable in that it completely excludes the densities of the bounded charges and the bounded currents inside homogeneous media. The Abraham force density is fanciful because it is based on the Minkowski force density together with a merely esthetic consideration which is devoid of physical meaning.

In this subsection the Minkowski stress tensor is derived by integration of the Minkowski force density over an arbitrary volume within homogeneous media. The importance of this derivation lies in the fact that the Minkowski formulation is not limited to linear materials. This is in contrast to what is explicitly stated in some papers [8].

The Minkowski force density, inside a homogeneous medium, is written as

$$\vec{f}_M = \rho\vec{E} + \vec{J} \times \vec{B}, \qquad (6)$$

where ρ and $\vec{J}$ are the densities of the free charges and the free currents, respectively. In *inhomogeneous* media, a Helmholtz term has to be included in the Minkowski force density, which for linear and isotropic media appears as $-(\vec{E} \cdot \vec{E}/2)\nabla\varepsilon - (\vec{H} \cdot \vec{H}/2)\nabla\mu$ [2,3,29]. Despite the opinion of some authors [8], the Minkowski formulation is not limited to linear materials.

As already mentioned, every formula of the EM force density can be integrated over an arbitrary volume υ to obtain the associated EM force exerted on the volume υ. It is shown in Appendix A that the integration of Eq. (6) over the volume υ within a *homogeneous* medium is analogous to the expression in Eq. (3) and therefore includes a momentum density and a stress tensor. $\vec{G}_M = \vec{D} \times \vec{B}$ is the Minkowski momentum density of the EM fields inside the medium, and $\overset{\leftrightarrow}{T}_M$ is the

Minkowski stress tensor which reads as

$$\overset{\leftrightarrow}{T}_M = \vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}\overset{\leftrightarrow}{I}(\vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H}), \qquad (7)$$

for a linear reciprocal medium and

$$\overset{\leftrightarrow}{T}_M = \vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}\overset{\leftrightarrow}{I}(\varepsilon_0 \vec{E} \cdot \vec{E} + 2\vec{P}_0 \cdot \vec{E} + \vec{B} \cdot \vec{H}), \quad (8)$$

for a nonlinear medium with a permanent electric polarization $\vec{P}_0$ and a linear reciprocal magnetic response. In the same manner, the Minkowski stress tensor for other types of homogeneous media, e.g., a nonlinear medium with a permanent magnetization $\vec{M}_0$, can be derived.

The Minkowski stress tensor is not symmetric. The Abraham force density is obtained in such a manner that its corresponding stress tensor becomes symmetric, and its corresponding momentum density becomes the Abraham momentum density $\vec{G}_{\text{Ab}} = \varepsilon_0\mu_0 \vec{E} \times \vec{H}$. It can be shown that the Abraham force density inside a homogeneous medium is as follows:

$$\vec{f}_{\text{Ab}} = \vec{f}_M + \frac{\partial(\vec{G}_M - \vec{G}_{\text{Ab}})}{\partial t} - \frac{1}{2}\sum_{i=x,y,z}\frac{\partial(\vec{E}D_i)}{\partial i}$$
$$+ \frac{1}{2}\sum_{i=x,y,z}\frac{\partial(\vec{D}E_i)}{\partial i}. \qquad (9)$$

The previous remark about the absence of the Helmholtz term for homogeneous media remains true for the Abraham force density as well. The Abraham counterparts of the Minkowski stress tensors in Eq. (7), i.e., for a linear reciprocal medium, and Eq. (8), i.e., for a nonlinear medium with a permanent electric polarization $\vec{P}_0$ and a linear reciprocal magnetic response, can be obtained by following the same steps. It can be shown that these counterparts are

$$\overset{\leftrightarrow}{T}_{\text{Ab}} = \frac{1}{2}(\vec{D}\vec{E} + \vec{E}\vec{D} + \vec{B}\vec{H} + \vec{H}\vec{B}) - \frac{1}{2}\overset{\leftrightarrow}{I}(\vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H})$$
$$(10)$$

and

$$\overset{\leftrightarrow}{T}_{\text{Ab}} = \frac{1}{2}(\vec{D}\vec{E} + \vec{E}\vec{D} + \vec{B}\vec{H} + \vec{H}\vec{B})$$
$$- \frac{1}{2}\overset{\leftrightarrow}{I}(\varepsilon_0 \vec{E} \cdot \vec{E} + 2\vec{P}_0 \cdot \vec{E} + \vec{B} \cdot \vec{H}), \qquad (11)$$

respectively. It is worth noting that, for a linear isotropic material, the stress tensors of Minkowski and Abraham are the same.

III. EM FORCE EXPERIENCED BY AN OBJECT

In this section we investigate how the EM force exerted on an object is distributed over its boundary. Figure 1 schematically shows a typical situation when the object is placed within a host medium. In this figure, υ_I, υ_O, and S_B denote the region inside the object, the region outside the object, and the object boundary, respectively. It is worth noting that the object boundary belongs to both the object and the host medium. Obviously the exact treatment of the interface requires a microscopic study of the interface between inside and outside the object. However, the macroscopic constitutive relations, e.g., electric permittivity or magnetic permeability, become meaningless in the microscopic study of
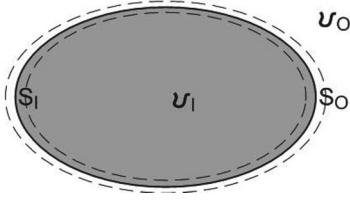
FIG. 1. Schematic of a homogeneous object placed within a homogeneous host medium.

the interface. In the macroscopic point of view adopted here, the necessity of going through a rather meticulous microscopic study of the interface is sidestepped by approaching the interface region once from within the object and once from without the object. Therefore, we consider S_I and S_O, which are imaginary surfaces that approach the object boundary from υ_I and υ_O, respectively.

A. The Ampere-Lorentz force and the Einstein-Laub force

The Ampere-Lorentz (or Einstein-Laub) force experienced by an object inside a host medium is easy to obtain because the Ampere-Lorentz (or Einstein-Laub) stress tensor in Eq. (4) [or Eq. (5)], which is resulting from integration of the Ampere-Lorentz (or Einstein-Laub) force density, is general and independent of whether the integration volume is homogeneous. Therefore, the followings are the exerted Ampere-Lorentz force and the Einstein-Laub force on the object boundary S_B, respectively:

$$\oint_{S_B} \hat{n} \cdot (\ddot{\vec{T}}_{\mathrm{Am}_O} - \ddot{\vec{T}}_{\mathrm{Am}_I}) dS, \qquad (12)$$

$$\oint_{S_B} \hat{n} \cdot (\ddot{\vec{T}}_{E_O} - \ddot{\vec{T}}_{E_I}) dS, \qquad (13)$$

where $\ddot{\vec{T}}_{\mathrm{Am}_I}$ (or $\ddot{\vec{T}}_{E_I}$) and $\ddot{\vec{T}}_{\mathrm{Am}_O}$ (or $\ddot{\vec{T}}_{E_O}$) are the Ampere-Lorentz (or Einstein-Laub) stress tensors inside υ_I and υ_O, respectively, and $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_I. Since the total Ampere-Lorentz (or Einstein-Laub) force exerted on the object boundary S_B is written as a surface integral, its integrand is the distribution of the Ampere-Lorentz (or Einstein-Laub) force exerted on the object boundary S_B.

It is worth noting that the object boundary S_B in our treatment is in fact the macroscopic representation of the microscopic volume in between the object and the host medium. Since the microscopic volume has a nonzero mass, attributing a nonzero force to the two-dimensional object boundary S_B is not paradoxical.

The total Ampere-Lorentz force and the total Einstein-Laub force exerted on υ_I and S_B can be written as

$$\int_{\upsilon_I} \frac{-\partial \vec{G}_L}{\partial t} d\upsilon + \oint_{S_B} \hat{n} \cdot \ddot{\vec{T}}_{\mathrm{Am}_O} dS, \qquad (14)$$

$$\int_{\upsilon_I} \frac{-\partial \vec{G}_{\mathrm{Ab}}}{\partial t} d\upsilon + \oint_{S_B} \hat{n} \cdot \ddot{\vec{T}}_{E_O} dS, \qquad (15)$$

respectively, where $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_I. Regretfully, the macroscopic point of view does not clarify how much of Eq. (14)

[or Eq. (15)] contributes to acceleration of the object center of mass because the object boundary S_B in principle belongs to both the object and the host medium. Nevertheless, there is no ambiguity about the total Ampere-Lorentz (or Einstein-Laub) force that accelerates the object center of mass when the host medium is devoid of matter, i.e., free space. In such a case, the object boundary S_B is not shared between two media (the object and its host medium) but belongs to the object totally. Therefore, Eq. (14) [or Eq. (15)] becomes the overall Ampere-Lorentz (or Einstein-Laub) force exerted on the object.

B. The Minkowski force and the Abraham force

Finding the Minkowski (or Abraham) force experienced by an object inside a host medium is a more delicate process because the form of the Minkowski (or Abraham) stress tensor depends on the constitutive relations. Our derivation of the Minkowski (or Abraham) force experienced by the object is based on the assumption that the object and its host medium are both made of homogeneous materials. The distributions of the free charges and the free currents are arbitrary.

Now, according to the results of Sec. II B the Minkowski force exerted on υ_I, as a homogenous medium, is written as

$$\int_{\upsilon_I} \frac{-\partial \vec{G}_M}{\partial t} d\upsilon + \oint_{S_I} \hat{n} \cdot \ddot{\vec{T}}_M dS, \qquad (16)$$

where $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_I. In a similar fashion, the Minkowski force exerted on υ_O, as the other homogeneous medium, is written as

$$\int_{\upsilon_O} \frac{-\partial \vec{G}_M}{\partial t} d\upsilon + \oint_{S_O} \hat{n} \cdot \ddot{\vec{T}}_M dS, \qquad (17)$$

where $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_O. On the other hand, according to the law of conservation of momentum in EM interactions, the total Minkowski force exerted on the whole system, which comprises υ_I, υ_O, and S_B, should equal time rate of decrease of the total momentum of the EM fields. Comparing the summation of the Minkowski forces exerted on υ_I, and υ_O, i.e., the summation of Eqs. (16) and (17) when the imaginary surfaces S_I and S_O approach the real interface S_B, against time rate of decrease of the total Minkowski momentum of the EM fields reveals that there should be a Minkowski force exerted on the object boundary S_B:

$$\oint_{S_B} \hat{n} \cdot (\ddot{\vec{T}}_{M_O} - \ddot{\vec{T}}_{M_I}) dS, \qquad (18)$$

where $\ddot{\vec{T}}_{M_I}$ and $\ddot{\vec{T}}_{M_O}$ are the Minkowski stress tensors inside υ_I and υ_O, respectively, and $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_I. The integrand in Eq. (18) is the distribution of the Minkowski force exerted on the object boundary S_B. This treatment of the object boundary is clearly distinct from the approach based on the inclusion of the Helmholtz term, which usually appears as $-(\vec{E} \cdot \vec{E}/2)\nabla \varepsilon - (\vec{H} \cdot \vec{H}/2)\nabla \mu$ in the Minkowski force density [2,3,29]. Interestingly, the line integral of the Helmholtz term over an infinitesimal path normal to S_B is

equivalent to $\hat{n} \cdot (\ddot{\vec{T}}_{M_O} - \ddot{\vec{T}}_{M_I})$, where $\hat{n}$ is the local unit vector normal to S_B toward the outside of υ_I.

Now, the total Minkowski force exerted on υ_I and S_B can be written as summation of Eqs. (16) and (18), which is

$$\int_{\upsilon_I} \frac{-\partial \vec{G}_M}{\partial t} d\upsilon + \oint_{S_B} \hat{n} \cdot \ddot{\vec{T}}_{M_O} dS, \qquad (19)$$

where $\hat{n}$ is the unit vector normal to the surface element dS toward the outside of υ_I. Since the object boundary S_B is shared between the object and its host medium, finding the overall Minkowski force exerted on the object—much like finding the overall Ampere-Lorentz and Einstein-Laub forces—requires more than a macroscopic study of the interface unless the host medium is free space. In such a case, Eq. (19) becomes the overall Minkowski force exerted on the object. Therefore, despite the opinion of some authors [22], neither Eq. (19) nor its equivalent forms for static and time-harmonic EM fields [36] can express the overall Minkowski force exerted on the object when the host medium is not free space.

In a similar fashion, the exerted Abraham forces on υ_I, υ_O, and S_B can be obtained. It can be easily shown that the Abraham counterpart of Eq. (18) reads as

$$\oint_{S_B} \hat{n} \cdot (\ddot{\vec{T}}_{\text{Ab}_O} - \ddot{\vec{T}}_{\text{Ab}_I}) dS. \qquad (20)$$

If the object and the host medium are both linear and isotropic, Eqs. (20) and (18) become equivalent to each other and therefore the formulations of Abraham and Minkowski render the same distribution for the exerted EM force on the object boundary.

Along the same line, the overall Abraham force exerted on the object in free space is obtained by replacing $\vec{G}_M$ and $\ddot{\vec{T}}_{M_O}$ with $\vec{G}_{\text{Ab}}$ and $\ddot{\vec{T}}_{\text{Ab}_O}$, respectively, in Eq. (19).

Interestingly, the overall Ampere-Lorentz, Einstein-Laub, Minkowski, and Abraham forces exerted on the object in free space when the EM fields are static (or their time averages when the EM fields are time harmonic) become equal to each other.

IV. ASSESSMENT OF FORMULATIONS OF EM FORCE DISTRIBUTION

Compatibility with special relativity is a solid theoretical criterion for correct formulation of the EM stress tensor and its corresponding force density. Some tried to show that the Ampere-Lorentz force density is inconsistent with special relativity [7]. But this assertion was incorrect because the Ampere-Lorentz force density as a generalization of the Lorentz force law has a covariant four vector [12,13,37], and the Ampere-Lorentz stress tensor has a covariant four tensor [37]. From another point of view adopted by some other authors [14,15], the omission of the hidden momentum [38–40] leads to the wrong result reported in [7].

It is worth noting that one can discard the hidden momentum by employing the asynchronous formulation instead of the synchronous formulation [31–33]. According to the synchronous formulation, the condition of equilibrium in an inertial frame is met when the sum of the forces (and torques) exerted on the constituent parts of the extended body under investigation is

synchronously zero. In the asynchronous treatment; however, the forces (and torques) exerted on the constituent parts are to be considered asynchronously, i.e., at different times. In this latter approach, there is only one inertial frame in which the forces (and torques) are summed at the same time.

If the EM fields are static (or time harmonic) in both the rest frame and a moving inertial frame, we do not have to be worried about the hidden momentum insofar as the EM force (or the time-averaged EM force) is considered. In such a case, there is no need to resort to the asynchronous formulation, and the results of the previous sections for static (or time-harmonic) EM fields are applicable not only in the rest frame but also in the moving frame. In contrast, insofar as torque is considered, even if the forces are static in both the rest and the moving frames, either the hidden entities should be included, or the asynchronous formulation is to be adopted. This is due to the fact that torque is the cross product of the position vector and force, and the positions of the constituent parts vary with time in the moving frame. One famous example is the right-angled lever problem [33].

Here, two examples, named example A and example B, are devised specifically to rebut the assertion that the Einstein-Laub formulation is in harmony with special relativity [7]. The examples are static in not only the rest frame but also in a moving frame. It is worth noting that if the EM fields are time harmonic in the rest frame, they are time harmonic in any other inertial frame. This is not necessarily the case for static EM fields. That is why we emphasize that our examples are static in both frames.

In each of the examples we examine the EM force experienced by a linear isotropic homogeneous object with a permittivity ε and a permeability μ inside free space in the presence of the free charges. For the sake of brevity, the counterparts of the examples in the presence of the free currents are only mentioned in passing. From the remarks made in the previous section it is evident that first, all the formulations of the EM force density agree on the total EM force exerted on the object in each of the examples, and therefore, the discussion focuses on the distribution of the exerted EM force on the object boundary; and second, the results of the formulations of Minkowski and Abraham are identical in each of the examples.

Due to the static nature of our examples, one might be tempted to call upon the method of virtual work rather than special relativity to find the correct force distribution. The method of virtual work, however, is not as straightforward as it seems to be, because the correct form of the constitutive relations after virtual deformation of objects is open to discussion. This point is discussed for each example in Appendix B. The mainstream literature is devoted to *real* deformation of objects, e.g., elastomers, exposed to electrostatic fields [41]. Such study does not necessitate distinction between the shares of mechanical and EM forces. This is fortunate because making such a distinction is a heavy burden [42].

A. Example A

As shown in Fig. 2, the empty space between two parallel metallic planes is partly filled with the object. The lower and the upper metallic planes have predetermined charges $Q_0 > 0$ and $-Q_0$ per unit length of the z axis, respectively. Assuming
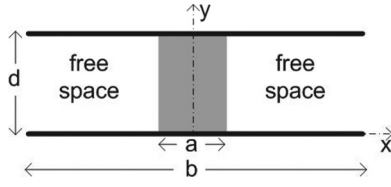
FIG. 2. Two parallel metallic planes partly filled with a linear isotropic homogeneous object with a permittivity ε and a permeability μ. The structure is uniform along the z axis.

FIG. 3. Two parallel metallic planes with a linear isotropic homogeneous slab with a permittivity ε and a permeability μ. The structure is uniform along both the z and the x axes.

that the electric field is uniform within the space enclosed by the metallic planes, it can be analytically calculated and written as $\vec{E} = \hat{y}\frac{Q_0}{a\varepsilon+(b-a)\varepsilon_0}$. The uniformity of the electric field is valid insofar as b is much larger than d.

The total EM force exerted on the object is zero, but the distribution of the exerted EM force on the object boundary hinges upon the choice of the formulation of the EM stress tensor. Both the Ampere-Lorentz force and the Einstein-Laub force exerted on the rightward boundary are zero according to Eqs. (12) and (13), whereas use of Eq. (18) [or Eq. (20)] yields $\hat{x}\frac{Q_0^2(\varepsilon-\varepsilon_0)}{2[a\varepsilon+(b-a)\varepsilon_0]^2}$ for the exerted Minkowski (or Abraham) force per unit area of the rightward boundary of the object. Below, their compatibility with special relativity is examined to find the correct result from the theoretical point of view.

1. Compatibility with special relativity

To provide a relativistic interpretation of example A, which has been hitherto discussed in the rest frame, we consider a new inertial frame moving at a constant velocity $\hat{z}V$ with respect to the rest frame. In this new frame, the electric field and the magnetic flux density are $\vec{E}' = \hat{y}\frac{\gamma Q_0}{a\varepsilon+(b-a)\varepsilon_0}$, and $\vec{B}' = \hat{x}\frac{\gamma V Q_0}{c^2[a\varepsilon+(b-a)\varepsilon_0]}$, respectively, where c denotes the speed of light in free space and $\gamma = \frac{1}{\sqrt{1-V^2/c^2}}$. Similarly, the displacement and the magnetic fields outside the object (and inside the object) are $\vec{D}'_O = \hat{y}\frac{\gamma Q_0\varepsilon_0}{a\varepsilon+(b-a)\varepsilon_0}$ [and $\vec{D}'_I = \hat{y}\frac{\gamma Q_0\varepsilon}{a\varepsilon+(b-a)\varepsilon_0}$] and $\vec{H}'_O = \hat{x}\frac{\gamma V Q_0\varepsilon_0}{a\varepsilon+(b-a)\varepsilon_0}$ [and $\vec{H}'_I = \hat{x}\frac{\gamma V Q_0\varepsilon}{a\varepsilon+(b-a)\varepsilon_0}$], respectively. Therefore, the stress tensors of Ampere-Lorentz, Einstein-Laub, and Minkowski (or Abraham) which appeared in Eqs. (4), (5), and (7) [or (10)], respectively, can now be calculated outside and inside the object in the new inertial frame.

It is worth noting that, generally speaking, in a moving frame, each of the displacement field and the magnetic flux density inside a medium other than free space is intricately dependent on both the electric and the magnetic fields, even if the medium is linear and isotropic in the rest frame [43]. However, in this specific example, it turns out that $\vec{D}'_I = \varepsilon\vec{E}'$ and $\vec{B}' = \frac{1}{c^2\varepsilon}\vec{H}'_I$ inside the object, and therefore the form of the Minkowski (or Abraham) stress tensor in Eq. (7) [or Eq. (10)] is still applicable.

The forces of Ampere-Lorentz, Einstein-Laub, and Minkowski (or Abraham), exerted per unit area of the rightward boundary of the object in the new inertial frame can now be obtained by using Eqs. (12), (13), and (18) [or (20)], respectively. These per unit area forces are expected to be invariant in the two inertial frames because both the force and the unit area decrease by a factor of $1/\gamma$ in the new inertial
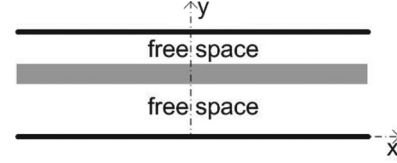
frame. Comparison of the results in the new inertial frame and their counterparts in the rest frame, however, demonstrates that the Einstein-Laub force per unit area, in contrast to the others, is *not* invariant in the two inertial frames. This is an inconsistency between the Einstein-Laub formulation and special relativity. The Einstein-Laub force per unit area in the new inertial frame is $\hat{x}\frac{Q_0^2(\varepsilon-\varepsilon_0)}{2[a\varepsilon+(b-a)\varepsilon_0]^2}\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}(\frac{\gamma V}{c})^2$.

2. Counterpart of example A in the presence of the free currents

In example A, the electric field is parallel to the surface on which the EM force is exerted. The geometry of the counterpart of this example in the presence of the free currents is depicted in Fig. 3. It is assumed that in the rest frame, the metallic planes in Fig. 3 have no charge, but carry predetermined surface current densities $\hat{z}J_{s_0}$ and $-\hat{z}J_{s_0}$. It can be easily shown that each of the forces of Ampere-Lorentz and Minkowski (or Abraham) exerted per unit area of the upper boundary is invariant in the two inertial frames while the Einstein-Laub force per unit area of the upper boundary depends on the velocity of the inertial frame.

B. Example B

As another example, the geometrical arrangement in Fig. 3 is considered when the lower and the upper metallic planes have predetermined surface charge densities $\rho_{s_0} > 0$ and $-\rho_{s_0}$, respectively, but carry no current. Therefore, the displacement field within the space enclosed by the metallic planes is written as $\vec{D} = \hat{y}\rho_{s_0}$.

The total EM force exerted on the object is zero, but different formulations of the EM stress tensor lead to different distributions of the exerted EM force on the object boundary. Use of Eqs. (12), (13), and (18) [or (20)] yield $\hat{y}\frac{\rho_{s_0}^2}{2\varepsilon_0}\frac{\varepsilon^2-\varepsilon_0^2}{\varepsilon^2}$, $\hat{y}\frac{\rho_{s_0}^2}{2\varepsilon_0}(\frac{\varepsilon-\varepsilon_0}{\varepsilon})^2$, and $\hat{y}\frac{\rho_{s_0}^2}{2\varepsilon_0}\frac{\varepsilon-\varepsilon_0}{\varepsilon}$ for the forces of Ampere-Lorentz, Einstein-Laub, and Minkowski (or Abraham), exerted per unit area of the upper boundary of the object, respectively.

1. Compatibility with special relativity

Example B is now reexamined in the new inertial frame moving at the constant velocity $\hat{z}V$ with respect to the rest frame. In the new inertial frame, the displacement and the magnetic fields are $\vec{D}' = \hat{y}\gamma\rho_{s_0}$ and $\vec{H}' = \hat{x}\gamma V\rho_{s_0}$, respectively. Similarly, the electric field and the magnetic flux density outside the object (and inside the object) are $\vec{E}'_O = \hat{y}\frac{\gamma\rho_{s_0}}{\varepsilon_0}$ (and $\vec{E}'_I = \hat{y}\frac{\gamma\rho_{s_0}}{\varepsilon}$) and $\vec{B}'_O = \hat{x}\frac{\gamma V\rho_{s_0}}{c^2\varepsilon_0}$ (and $\vec{B}'_I = \hat{x}\frac{\gamma V\rho_{s_0}}{c^2\varepsilon}$), respectively. The stress tensors in Eqs. (4), (5), and (7) [or (10)] can now be calculated outside and inside the object in the

new inertial frame. It is worth noting that $\vec{D}' = \varepsilon \vec{E}'_I$ and $\vec{B}'_I = \frac{1}{c^2 \varepsilon} \vec{H}'$ inside the object, and therefore, the form of the Minkowski (or Abraham) stress tensor in Eq. (7) [or Eq. (10)] is still applicable.

The forces of Ampere-Lorentz, Einstein-Laub, and Minkowski (or Abraham) exerted per unit area of the upper boundary of the object in the new inertial frame are obtained by using Eqs. (12), (13), and (18) [or (20)], respectively. Once again, only the Einstein-Laub force per unit area is *not* invariant in the two inertial frames. It can be shown that the Einstein-Laub force per unit area in the new inertial frame is $\hat{y} \frac{\rho_{s_0}^2}{2\varepsilon_0} (\frac{\varepsilon - \varepsilon_0}{\varepsilon})^2 \gamma^2$. This is incompatible with special relativity because the force exerted on the upper boundary, and the unit area of the upper boundary both decrease by a factor of $1/\gamma$ in the new inertial frame, and the force per unit area must be invariant in the two inertial frames.

2. Counterpart of example B in the presence of the free currents

In example B, the electric field is perpendicular to the surface on which the EM force is exerted. The geometry of the counterpart of this example in the presence of the free currents is depicted in Fig. 2. It is assumed that in the rest frame, the metallic planes in Fig. 2 have no charge, but carry predetermined current I_0 along the $\hat{z}$ and $-\hat{z}$. Yet again, it can be easily shown that each of the forces of Ampere-Lorentz and Minkowski (or Abraham) exerted per unit area of the rightward boundary is invariant in the two inertial frames while the Einstein-Laub force per unit area of the rightward boundary depends on the velocity of the inertial frame.

V. CONCLUSIONS

We studied the different EM stress tensors affiliated with the different formulations of the EM force density. Since the mathematical expressions of the stress tensors of Minkowski and Abraham inside a material depend on the constitutive relations of the displacement field and the magnetic flux density, we specifically derived these tensors for three categories of homogeneous materials, and show that their forms can be quite different from the well-known forms in the literature.

From a viewpoint based on the law of conservation of momentum in EM interactions, the total EM force exerted on an object in a host medium was addressed together with the distribution of the exerted EM force on the object boundary. The object boundary is in fact the macroscopic representation of the microscopic volume in between the object and the host medium. Since the microscopic volume has a nonzero mass, attributing a nonzero force to the object boundary is not paradoxical. We argued that finding the exact amount of the exerted EM force on the object in a nonfree space host medium would require a detailed study of the object boundary. Besides, the overall Ampere-Lorentz, Einstein-Laub, Minkowski, and Abraham forces exerted on the object in free space when the EM fields are static (or their time averages when the EM fields are time harmonic) become equal to each other. It should be noted that finding and integrating the contribution of the Helmholtz term in the exerted Minkowski (or Abraham) force is sidestepped by following the presented approach.

By a few simple examples, it was shown that the Einstein-Laub formulation is not necessarily consistent with special relativity. In each of the considered examples, we examined the EM force experienced by a linear isotropic homogeneous object inside free space in the presence of the free charges (or currents). It should be noted that EM fields in each of the examples are static in both the rest and the moving frames. Therefore, the issue of the hidden momentum is of no consequence in our examples. The same holds true when EM fields are time harmonic. Otherwise, either hidden entities, e.g., the hidden momentum, come into the play, or the asynchronous formulation [31–33] is to be adopted to circumvent being involved with hidden entities. In the latter approach, the process of synchronously integrating force densities as carried out in Sec. II in the rest frame (the privileged frame in the asynchronous formulation) should be repeated in the moving frame, whereby the stress tensors are obtained. This process is not yet carried out, and thus, can be a good subject for future studies.

APPENDIX A

Here the Minkowski stress tensors of a few categories of homogeneous materials are derived from the Minkowski force density in Eq. (6). According to Maxwell's equations, Eq. (6) can be rewritten as

$$\vec{f}_M = (\nabla \cdot \vec{D})\vec{E} + (\nabla \times \vec{E}) \times \vec{D} + (\nabla \cdot \vec{B})\vec{H}$$
$$+ (\nabla \times \vec{H}) \times \vec{B} - \frac{\partial}{\partial t}(\vec{D} \times \vec{B}). \qquad (A1)$$

To obtain the exerted Minkowski force on a volume υ within a homogeneous medium, Eq. (A1) has to be integrated over the volume. Integration of the fifth term on the right-hand side of Eq. (A1) yields the volume integral in Eq. (3) with $\vec{G} = \vec{G}_M = \vec{D} \times \vec{B}$, but integration of the other terms is to be carried out. It is sufficient to carry out integration of the first two terms since the next two happen to be their magnetic counterparts. The first two terms can be written as

$$(\nabla \cdot \vec{D})\vec{E} + (\nabla \times \vec{E}) \times \vec{D}$$
$$= (\nabla \cdot \vec{D})\vec{E} + \hat{x} \sum_{i=x,y,z} D_i \left(\frac{\partial E_x}{\partial i} - \frac{\partial E_i}{\partial x} \right)$$
$$+ \hat{y} \sum_{i=x,y,z} D_i \left(\frac{\partial E_y}{\partial i} - \frac{\partial E_i}{\partial y} \right)$$
$$+ \hat{z} \sum_{i=x,y,z} D_i \left(\frac{\partial E_z}{\partial i} - \frac{\partial E_i}{\partial z} \right)$$
$$= \nabla \cdot (\vec{D}\vec{E}) - \hat{x} \sum_{i=x,y,z} D_i \frac{\partial E_i}{\partial x}$$
$$- \hat{y} \sum_{i=x,y,z} D_i \frac{\partial E_i}{\partial y} - \hat{z} \sum_{i=x,y,z} D_i \frac{\partial E_i}{\partial z}. \qquad (A2)$$

The last three terms on the right-hand side of Eq. (A2) are behind the fact that the form of the Minkowski stress tensor is influenced by the constitutive relation of $\vec{D}$. Thanks to the similarity among these three terms, we consider the $\hat{x}$ component only. As the first category, we assume that the constitutive relation of $\vec{D}$ is linear and reciprocal, i.e., $\vec{D} = \overset{\leftrightarrow}{\varepsilon}\vec{E}$ with a symmetric tensor $\overset{\leftrightarrow}{\varepsilon}$:

$$\hat{x} \sum_{i=x,y,z} D_i \frac{\partial E_i}{\partial x} = \frac{\hat{x}}{2} \frac{\partial(\vec{D} \cdot \vec{E})}{\partial x}. \tag{A3}$$

From Eqs. (A2), (A3), and its counterparts along the y and the z axes, it becomes evident that

$$(\nabla \cdot \vec{D})\vec{E} + (\nabla \times \vec{E}) \times \vec{D} = \nabla \cdot (\vec{D}\vec{E}) - \tfrac{1}{2}\nabla \cdot [\overset{\leftrightarrow}{I}(\vec{D} \cdot \vec{E})], \tag{A4}$$

where $\overset{\leftrightarrow}{I}$ is the identity tensor. Therefore, if the constitutive relation of $\vec{D}$ is linear and reciprocal, the contribution of the electric and the displacement fields to the form of the Minkowski stress tensor is $\vec{D}\vec{E} - \tfrac{1}{2}\overset{\leftrightarrow}{I}(\vec{D} \cdot \vec{E})$. It can be easily shown that the same is true if the material is linear, isotropic, and dispersive, i.e., $\vec{D}(t) = \varepsilon(t) * \vec{E}(t)$ when $*$ denotes convolution.

As another category, we assume that the constitutive relation of $\vec{D}$ is nonlinear with a permanent electric polarization $\vec{P}_0$, i.e., $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}_0$:

$$\hat{x} \sum_{i=x,y,z} D_i \frac{\partial E_i}{\partial x} = \frac{\hat{x}}{2} \frac{\partial(\varepsilon_0 \vec{E} \cdot \vec{E} + 2\vec{P}_0 \cdot \vec{E})}{\partial x}. \tag{A5}$$

Now from Eqs. (A2), (A5), and its counterparts along the y and the z axes, it becomes evident that the contribution of the electric and the displacement fields to the form of the Minkowski stress tensor when the constitutive relation of $\vec{D}$ is nonlinear with a permanent electric polarization $\vec{P}_0$ is $\vec{D}\vec{E} - \tfrac{1}{2}\overset{\leftrightarrow}{I}(\varepsilon_0 \vec{E} \cdot \vec{E} + 2\vec{P}_0 \cdot \vec{E})$.

APPENDIX B

To obtain the exerted EM force on the rightward boundary of the object in example A by the method of virtual work, the change in the electric energy has to be calculated when the rightward boundary is infinitesimally displaced along the x axis, while the rest of the object boundary remains still. To this end, it is essential to first specify how the constitutive relation of $\vec{D}$ inside the object varies with infinitesimal expansion or contraction of the object along the x axis. This dependence of the result of the method of virtual work on the constitutive relations inside the virtually expanded or contracted object renders the resultant EM force debatable, because the EM force is expected to be influenced only by the constitutive relations when the object is in its original state.

The exerted EM force per unit area of the rightward boundary obtained by the method of virtual work is in agreement with the result of the formulations of Ampere-Lorentz and Einstein-Laub if $\varepsilon = \varepsilon_0(1 + a/L)$, where L is the object's thickness which virtually expands or contracts, namely the object's thickness along the x axis in this example, and a is a constant coefficient whose physical dimension is *length*. The result of the method of virtual work is in agreement with the result of the Minkowski (or Abraham) formulation if ε is assumed to remain unchanged when the object virtually expands or contracts.

To apply the method of virtual work to example B, the change in the electric energy has to be calculated when the upper boundary is infinitesimally displaced along the y axis, while the rest of the object boundary remains unchanged. The exerted EM force per unit area of the upper boundary obtained by the method of virtual work is in agreement with the result of the Ampere-Lorentz formulation if $\varepsilon = \varepsilon_0(1 + bL)$, where L is the object's thickness which virtually expands or contracts, namely the object's thickness along the y axis in this example, and b is a constant coefficient whose physical dimension is 1/length. The result of the method of virtual work is in agreement with the Einstein-Laub formulation if $\varepsilon = \varepsilon_0(1 + a/L)$, and with the result of the Minkowski (or Abraham) formulation if ε does not vary with virtual expansion or contraction of the object.

[1] J. W. Arthur, Antenna Propag. Mag. **55**, 3 (2013).
[2] I. Brevik, Phys. Rep. **52**, 133 (1979).
[3] B. A. Kemp, J. Appl. Phys. **109**, 111101 (2011).
[4] S. M. Barnett and R. Loudon, J. Phys. B **39**, S671 (2006).
[5] M. Mansuripur, A. R. Zakharian, and J. V. Moloney, Proc. SPIE **6326**, 63260G (2006).
[6] M. Mansuripur, Opt. Exp. **16**, 19 (2008).
[7] M. Mansuripur, Phys. Rev. Lett. **108**, 193901 (2012).
[8] M. Mansuripur, A. R. Zakharian, and E. M. Wright, Phys. Rev. A **88**, 023826 (2013).
[9] K. J. Webb, Phys. Rev. Lett. **111**, 043602 (2013).
[10] I. Liberal, I. Ederra, R. Gonzalo, and R. W. Ziolkowski, Phys. Rev. A **88**, 053808 (2013).
[11] A. Einstein, *The Collected Papers of Albert Einstein, Volume 8: The Berlin Years: Correspondence, 1914-1918* (Princeton University Press, Princeton, NJ, 1998).
[12] D. A. T. Vanzella, Phys. Rev. Lett. **110**, 089401 (2013).
[13] S. M. Barnett, Phys. Rev. Lett. **110**, 089402 (2013).
[14] P. L. Saldanha, Phys. Rev. Lett. **110**, 089403 (2013).
[15] M. Khorrami, Phys. Rev. Lett. **110**, 089404 (2013).
[16] M. Mansuripur, Phys. Rev. Lett. **110**, 089405 (2013).
[17] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999), p. 287.
[18] L. Novotny and B. Hecht, *Principles of Nano-Optics* (Cambridge University Press, Cambridge, 2006).
[19] M. I. Antonoyiannakis and J. B. Pendry, Phys. Rev. B **60**, 2363 (1999).
[20] K. Okamoto and S. Kawata, Phys. Rev. Lett. **83**, 4534 (1999).
[21] D. Van Thourhout and J. Roels, Nat. Photon. **4**, 211 (2010).
[22] B. A. Kemp and T. M. Grzegorczyk, Opt. Lett. **36**, 4 (2011).
[23] L. Jia and E. L. Thomas, Phys. Rev. B **84**, 125128 (2011).

[24] R. N. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Rev. Mod. Phys. **79**, 1197 (2007).
[25] S. M. Barnett, Phys. Rev. Lett. **104**, 070401 (2010).
[26] S. M. Barnett and R. Loudon, Philos. Trans. R. Soc. London Sect. A **368**, 927 (2010).
[27] C. Baxter and R. Loudon, J. Mod. Opt. **57**, 830 (2010).
[28] P. W. Milonni and R. W. Boyd, Adv. Opt. Photon. **2**, 519 (2010).
[29] W. Frias and A. I. Smolyakov, Phys. Rev. E **85**, 046606 (2012).
[30] N. G. C. Astrath, G. V. B. Lukasievicz, L. C. Malacarne, and S. E. Bialkowski, Appl. Phys. Lett. **102**, 231903 (2013).
[31] G. Cavalleri and G. Salgarelli, Nuovo Cimento A **62**, 722 (1969).
[32] G. Cavalleri, G. Spavieri, and G. Spinelli, Nuovo Cimento B **53**, 385 (1979).
[33] S. Aranoff, Nuovo Cimento B **10**, 155 (1972).
[34] Y. Aharonov, P. Pearle, and L. Vaidman, Phys. Rev. A **37**, 4052 (1988).
[35] G. Spavieri, Nuovo Cimento B **109**, 1 (1994).
[36] When the EM fields are static (or time harmonic), the first term in Eq. (19) (or its time average) becomes zero.
[37] S. M. Errede, the material is available at http://web.hep.uiuc.edu/home/serrede/P436/Lecture_Notes/P436_Lect_18p75.pdf (2011).
[38] W. Shockley and R. P. James, Phys. Rev. Lett. **18**, 876 (1967).
[39] W. Shockley, Phys. Rev. Lett. **20**, 343 (1968).
[40] S. Coleman and J. H. Van Vleck, Phys. Rev. **171**, 1370 (1968).
[41] X. Zhao and Z. Suo, J. Appl. Phys. **104**, 123530 (2008).
[42] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II.
[43] S. M. Errede, the material is available at http://web.hep.uiuc.edu/home/serrede/P436/Lecture_Notes/P436_Lect_20.pdf (2011).