Entanglement classification of three fermions with up to nine single-particle states

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Based on results well known in the mathematics literature but not yet common knowledge in the physics literature, we conduct a study on three-fermionic systems with six, seven, eight, and nine single-particle states. Via introducing special polynomial invariants playing the role of entanglement measures the structure of the stochastic local operations and classical communication (SLOCC) entanglement classes is investigated. The SLOCC classes of the six- and seven-dimensional cases can elegantly be described by special subconfigurations of the Fano plane. Some special embedded systems containing distinguishable constituents are arising naturally in our formalism, namely, three-qubits and three-qutrits. In particular, the three fundamental invariants I_6, I_9, and I_{12} of the three-qutrits system are shown to arise as special cases of the four fundamental invariants of three-fermions with nine single-particle states.

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I. INTRODUCTION

Quantum entanglement is a key resource for implementing tasks for processing quantum information [1]. It is well known by now that this resource can be based on manipulating composite quantum systems with both distinguishable and indistinguishable constituents. Although historically the study of entanglement based on systems belonging to the former class has received much greater scrutiny, investigations focusing on the latter have gained considerable attention too [2–7]. Quite recently, fermionic systems started to play a key role in studies revisiting the so-called N representability [8] and quantum marginal problem [9] centered around studies employing the important notion of entanglement polytopes [10–12], an idea having roots in the work of Klyachko [13]. The introduction of this notion was partly motivated by the study of special tripartite fermionic systems having six, seven, and eight single-particle states [9]. These systems provide simple special examples for multifermionic wave functions with physical properties easy to investigate. On the other hand, they also give rise to mathematical structures, namely, three-forms in six-, seven-, eight-, and nine-dimensional vector spaces over a field, well known to mathematicians [14–21]. Although the results in these papers on the classification of trivectors bear a relevance on the so-called stochastic local operations and classical communication (SLOCC) classification of entanglement classes [22,23] in quantum information, apart from scattered remarks [24,25] and our recent paper on Hitchin functionals [26], to our best knowledge these systems have not made their full debut to the literature on quantum entanglement.

The aim of this paper is to present a study on these special entangled fermionic systems based on these findings. In quantum information, one wishes to quantify and classify different types of entanglement regarded as a resource. There are different classification schemes. In the SLOCC classification scheme of multipartite systems, the representative *pure* states are equivalent if they can be mutually converted to each other with a finite probability of success using only local operations and classical communication. It can be shown [23] that for a system consisting of n distinguishable subsystems, SLOCC equivalence mathematically means that the equivalent pure states representing the system are on the same orbit under the action of the local group $\mathrm{GL}(N_1)\times\mathrm{GL}(N_2)\times\ldots\times\mathrm{GL}(N_n)$, where $(N_1,N_2,\ldots,N_n)$ are the local dimensions of the Hilbert spaces associated to the subsystems. For systems with indistinguishable constituents, the corresponding orbit should be formed under the n-fold *diagonal* action of $\mathrm{GL}(N)$ where N is the number of single-particle states. Although due to proliferation of entanglement classes solving the SLOCC classification problem in its full generality is a hopeless task, we still have a number of important special cases for which the structure of the SLOCC classes is known. These special entangled systems can provide a convenient theoretical framework to see multipartite entanglement in action.

Now, although these special systems have already been studied by mathematicians, however, physicists are either not aware of these results or they are reluctant to apply them, or they are rediscovering them from time to time in different contexts. For example, the classification problem equivalent to the classification of SLOCC entanglement types for three-qubits has already been solved in 1881 by mathematicians [27] (see also the paper of Schwartz [28] and the book of Gelfand [29]), the result has later been independently rediscovered in the influential paper by physicists [23]. As another example one can consider, the case of three-fermions with six single-particle states is a system used by Borland and Dennis in their seminal paper [9]. Using results known from cubic Jordan algebras, the SLOCC classes for this case have been rediscovered by one of us [30]. We have learned later that the solution to this problem dates back as early as 1907 via the work of Reichel. Moreover, it also turns out that this case is also well known from the theory of prehomogeneous vector spaces [31] where in the full classification of these spaces as given by Sato and Kimura, these types of fermionic systems show up as an important special case [32]. As we already mentioned, this case also constituted the archetypical example for further studies on entanglement polytopes and the N-representability problem [10–12]. Moreover, elevating a *real* three-fermionic

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state with six single-particle states to a three-form living on a six-dimensional manifold renders the square root of the magnitude of the quartic entanglement measure [30] to a functional on the manifold [26]. As shown by Hitchin [33,34] in an important special case the critical points of this functional correspond to Calabi-Yau manifolds. On the other hand, the evaluation of this functional at the critical point gives the semiclassical Bekenstein-Hawking entropy of certain black hole solutions in string theory [26,35].

This wide variety of physical applications justifies an attempt to present a self-contained entanglement-based reformulation of the results on the classification of three-forms in six, seven, eight, and nine dimensions. Apart from shedding new light on special fermionic systems and presenting some of their invariants serving as measures of entanglement in a unified manner, this approach also facilitates an embedding of special entangled systems of distinguishable constituents such as three-qubits and three-qutrits. In this philosophy, systems with distinguishable constituents are just special cases of systems with indistinguishable ones.

For clarity, we would like to note that the methods presented here are not directly applicable when one considers entanglement between modes [36–38] of indistinguishable systems. Mode entanglement is particulary useful when one wants to classify entanglement between different momenta or different regions of space. However, entanglement in this notion involves the splitting of fermionic mode operators $f_i^\dagger$ into subsets which are not invariant under local unitary transformations of the form $f_i^\dagger \mapsto U_i^j f_j^\dagger$ (e.g., the Fourier transformation on a lattice) which is a key ingredient in conventional entanglement classification between particles.

This paper is organized as follows. In Sec. II, we give a brief introduction to the language of multilinear algebra for the reader unfamiliar with it. This language turns out to be a particulary useful tool for generating SLOCC covariants and invariants. In Sec. III, we introduce a family of linear maps or *covariants* derived from the amplitudes of a fermionic state. All the invariants considered in this paper are derived from this construction. In Sec. IV, we present the SLOCC classification for three-fermion systems in dimensions six, seven, eight, and nine. For the six- and seven-dimensional cases, we present the structure of the SLOCC classes in a new manner based on the structure of the Fano plane. In addition to a discussion of the SLOCC classes, we present all the algebraically independent continuous invariants of these systems. Most of these invariants are known and used in different fields of physics and mathematics although except for the case of six dimensions, they have not made their debut in quantum information theory yet. We also discuss the embedding of three-qubits into the system of three fermions with six single-particle states and show how the measures of entanglement are related. There is a similar possibility of embedding three-qutrits into the system of three-fermions with nine single-particle states. We consider this case in Sec. IV D 1, and relate the invariants of three-qutrits to those of the corresponding fermionic system. In Sec. V, we outline some of the connections of these results with the entanglement polytopes of Klyachko in particular with the pinning of fermionic occupation numbers, which is a concept of huge interest recently [11]. Our conclusions are left to Sec. VI. For the convenience of the reader, we included two Appendices with some proofs and calculational details.

II. MULTILINEAR ALGEBRA

In this section, we give a brief summary of the language of multilinear algebra, which is a useful tool for attacking the entanglement classification problem of multifermion systems. The reader familiar with these concepts may skip to the next section.

Let $V \cong \mathbb{C}^N$ be an N-dimensional complex vector space. Denote the Cartesian product of V with itself by $V \times V$. There are two canonical ways of defining a vector space from $V \times V$. The first is the direct product, the second is the direct sum. The direct product of vectors is defined by the relations $(v+u)\otimes w = v\otimes w + u\otimes w$, $v\otimes(u+w) = v\otimes u + v\otimes w$, $(cv)\otimes w = v\otimes(cw) = c(v\otimes w)$ where $u,v,w \in V$, $c \in \mathbb{C}$. The vector space spanned by elements of the form $v\otimes w$ is denoted by $V\otimes V$ or $V^{\otimes 2}$. If $\{e_i\}_{i=1}^N$ is a basis in V, then $\{e_i \otimes e_j\}_{i,j=1}^N$ is a basis of $V\otimes V$. Obviously, $V\otimes V$ has dimension N^2. Similarly one can define the kth tensor power of V denoted by $V^{\otimes k}$ spanned by elements of the form $v_1\otimes v_2\otimes\ldots\otimes v_k$. This has dimension N^k. The tensor product is now a map $\otimes:\ V^{\otimes k}\times V^{\otimes m} \to V^{\otimes(k+m)}$.

The wedge product of $k \leqslant N$ vectors is defined as

$$v_1\wedge\ldots\wedge v_k = \frac{1}{k!}\sum_{\pi\in S_k}\sigma(\pi)v_{\pi(1)}\otimes\ldots\otimes v_{\pi(k)}, \qquad (1)$$

where S_k is the symmetric (permutation) group and σ is its alternating representation, namely, $\sigma(\pi)=1$ for even, $\sigma(\pi)=-1$ for odd permutations. The vector space spanned by elements of the form $v_1\wedge\ldots\wedge v_n$ is denoted by $\wedge^k V$ and has dimension $\binom{N}{k}$. Its elements are denoted with $\alpha,\beta,\gamma,\ldots$ and we will call them k vectors.

The direct sum is defined from $V\times V$ with the relations $(v+u)\oplus(w+z) = v\oplus w + u\oplus z$, $(cv)\oplus(cw) = c(v\oplus w)$. The vector space obtained in this way is denoted by $V\oplus V$. If $\{e_i\}_{i=1}^N$ is a basis of V, then $\{e_i\oplus 0, 0\oplus e_i\}_{i=1}^N$ is a basis in $V\oplus V$ (here, 0 denotes the zero vector in V). Thus, the dimension of $V\oplus V$ is simply $2N$.

Define now the vector space

$$\wedge(V) = \mathbb{C}\oplus V\oplus\wedge^2 V\oplus\ldots\oplus\wedge^N V. \qquad (2)$$

Now, $\wedge(V)$ can be elevated into an algebra via extending linearly the exterior product

$$\begin{aligned}\wedge : \wedge(V)\times\wedge(V) &\to \wedge(V),\\ \alpha,\beta &\mapsto \alpha\wedge\beta.\end{aligned} \qquad (3)$$

Endowed with this product $\wedge(V)$ is called an exterior algebra or Grassman algebra. The exterior product is a graded anticommutative product, meaning that for $\alpha\in\wedge^k V$ and $\beta\in\wedge^m V$ we have

$$\alpha\wedge\beta = (-1)^{km}\beta\wedge\alpha. \qquad (4)$$

Fixing a basis $\{e_i\}_{i=1}^N$ in V allows one to write $\alpha \in \wedge^k V$ in the form

$$\alpha = \frac{1}{k!}\alpha^{i_1\ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}, \tag{5}$$

where $\alpha^{i_1\ldots i_k}$ is totally antisymmetric in all of its indices and summation for the indices is understood.

Let V^* be the dual space of V comprising the linear functionals acting on V. If $\{e_j\}_{j=1}^N$ refers to a basis of V and $\{e^i\}_{i=1}^N$ a basis of V^*, then we have $\langle e^i, e_j\rangle = \delta^i_j$. One can also define the exterior algebra of V^* denoted by $\wedge(V^*)$. Its elements $P,Q,R,\ldots$ will be called k forms. An element P of $\wedge^k V^* \cong (\wedge^k V)^*$ is a multilinear functional $P : V \times \ldots \times V \to \mathbb{C}$ on V satisfying $P(v_1,\ldots,v_k) = \sigma(\pi)P(v_{\pi(1)},\ldots,v_{\pi(k)})$ for all $\pi \in S_k$. A general element $P \in \wedge^k V^*$ can be written as

$$P = \frac{1}{k!}P_{i_1\ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k}. \tag{6}$$

The pairing $\langle \ldots, \ldots\rangle$ between one-forms and vectors gives rise to a natural pairing between k forms and k vectors. In terms of basis vectors, it reads as

$$\langle e^{i_1} \wedge \cdots e^{i_k}, e_{j_1} \wedge \cdots e_{j_N}\rangle = \mathrm{Det}\begin{pmatrix} \delta^{i_1}{}_{j_1} & \ldots & \delta^{i_1}{}_{j_N} \\ \vdots & \ddots & \vdots \\ \delta^{i_N}{}_{j_1} & \ldots & \delta^{i_N}{}_{j_N}\end{pmatrix}. \tag{7}$$

There is a useful structure connecting the exterior algebra and its dual, called the interior product or contraction. For a vector $v \in V$, the interior product ι_v is a $\wedge^k V^* \to \wedge^{k-1}V^*$ linear mapping given by the defining formula

$$\iota_v e^{i_1} \wedge \ldots \wedge e^{i_k} = \sum_{n=1}^{k}(-1)^{k-1}\langle e^{i_n}, v\rangle e^{i_1} \wedge \ldots \wedge \check{e}^{i_n} \wedge \ldots \wedge e^{i_k}, \tag{8}$$

where the notation $\check{e}^{i_n}$ means that e^{i_n} has to be omitted from the product. For a k form P having the form (6), we have the explicit expression for the contraction:

$$\iota_v P = \frac{1}{(k-1)!} v^{i_1} P_{i_1 i_2 \ldots i_k} e^{i_2} \wedge \ldots \wedge e^{i_k}. \tag{9}$$

The definition of the contraction is a natural notion justified by the important identity

$$\langle \iota_{e_a} e^{i_1} \wedge e^{i_2} \ldots \wedge e^{i_k}, e_{j_2} \wedge \ldots \wedge e_{j_k}\rangle = \langle e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}, e_a \wedge e_{j_2} \wedge \ldots \wedge e_{j_k}\rangle. \tag{10}$$

This definition can be extended by linearity to one featuring a contraction by an arbitrary m vector

$$\begin{aligned}\iota : \wedge^m V \times \wedge^k V^* &\to \wedge^{k-m}V^*,\\ \alpha &= \frac{1}{m!}\beta^{i_1\ldots i_m} e_{i_1} \wedge \ldots \wedge e_{i_m},\\ P &= \frac{1}{k!}P_{i_1\ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k},\\ \alpha, P \mapsto \iota_\alpha P &= \frac{1}{(k-m)!}\alpha^{i_1\ldots i_m} P_{i_1\ldots i_m i_{m+1}\ldots i_k}\\ &\quad \times e^{i_{m+1}} \wedge \ldots \wedge e^{i_k} \in \wedge^{k-m}V^*,\end{aligned} \tag{11}$$

where $k \geqslant m$. We have the useful properties

$$\begin{aligned}&\iota_\alpha \circ \iota_\beta = (-1)^{km}\iota_\beta \circ \iota_\alpha, \quad \alpha \in \wedge^k V,\ \beta \in \wedge^m V,\\ &\iota_\alpha(P \wedge Q) = \iota_\alpha(P) \wedge Q + (-1)^{kp} P \wedge \iota_\alpha(Q),\\ &\qquad \alpha \in \wedge^k V,\ P \in \wedge^p V^*,\ Q \in \wedge^q V^*,\\ &\qquad k \leqslant p,q.\end{aligned} \tag{12}$$

There is an important isomorphism relating m forms and $N-m$ vectors. It reads as

$$\wedge^m V^* \cong \wedge^{N-m} V \otimes \wedge^N V^*. \tag{13}$$

This isomorphism is based on the definition of the $\star$ operation defined as follows:

$$\begin{aligned}&Q \wedge R = \langle Q, \star R\rangle \quad Q \in \wedge^{N-m}V^*,\\ &R \in \wedge^m V^*, \quad \star R \in \wedge^{N-m} \otimes \wedge^N V^*.\end{aligned} \tag{14}$$

Using the (7) identity, one can show that

$$\star R = \frac{1}{(N-m)!}(\star R)^{i_1\ldots i_{N-m}} e_{i_1} \wedge \ldots \wedge e_{i_{N-m}} \otimes \mathbb{E}, \tag{15}$$

where

$$(\star R)^{i_1\ldots i_{N-m}} = \frac{1}{m!}\varepsilon^{i_1\ldots i_{N-m} j_1 \ldots j_m} R_{j_1\ldots j_m}, \quad \mathbb{E} = e^1 \wedge \ldots \wedge e^N. \tag{16}$$

It should be emphasized that $\star$ is not the Hodge star; until this point we did not equip V with any metric.

Let $g = g_i^j e^i \otimes e_j \in \mathrm{GL}(V)$ be an invertible linear map from V to itself acting on a $v \in V$ as $gv = g_i^j v^k \langle e^i, e_k\rangle \otimes e_j = g_k^j v^k e_j$. For this action on the basis vectors we write

$$g e_i = e_j g_i^j, \quad g \in \mathrm{GL}(V). \tag{17}$$

Given this action on V, an action g^* on V^* is induced via the formula

$$\langle g^* e^i, g e_j\rangle = \langle e^i, e_j\rangle = \delta^i_j. \tag{18}$$

Explicitly, we have

$$g^* e^i = e^j g'^i_j, \quad g'^i_k g^k_j = \delta^i_j, \tag{19}$$

i.e., the matrix of g' is just the inverse transpose of the matrix of g:

$$g' = (g^t)^{-1}. \tag{20}$$

Now, this dual action induces an action $\varrho(g)$ on $\wedge^k V^*$. However, by an abuse of notation we use again g^* for this action

$$g^* : \wedge^k V^* \to \wedge^k V^*, \quad P \mapsto g^* P. \tag{21}$$

For the components, this reads as

$$P_{i_1\ldots i_k} \mapsto (g^* P)_{i_1\ldots i_k} = g'^{j_1}_{i_1} g'^{j_2}_{i_2} \ldots g'^{j_k}_{i_k} P_{j_1\ldots j_k}. \tag{22}$$

Similarly, the action on the components of a k vector α reads as

$$\alpha^{i_1\ldots i_k} \mapsto (g\alpha)^{i_1\ldots i_k} = g^{i_1}_{j_1} g^{i_2}_{j_2} \ldots g^{i_k}_{j_k} \alpha^{j_1\ldots j_k}. \tag{23}$$

By virtue of Eq. (19) in the special case of the top form $\mathbb{E}$ we have the transformation formula

$$g^* \mathbb{E} = (\mathrm{Det} g)^{-1}\mathbb{E}. \tag{24}$$

III. SLOCC INVARIANTS FOR FERMIONIC SYSTEMS

Now, let us identify $V = \mathbb{C}^N$ with the finite-dimensional single-particle Hilbert space. The full Hilbert space of a system with an indefinite number of fermions is called the Fock space. Let us denote the vacuum state of the Fock space as $|0\rangle$. Let us moreover define the fermionic operators f_i, $f_j^\dagger$ as the ones satisfying the canonical anticommutation relations

$$\{f_i, f_j^\dagger\} = \delta_{ij}, \quad \{f_i, f_j\} = \{f_i^\dagger, f_j^\dagger\} = 0, \quad i,j = 1,\dots,N. \tag{25}$$

Then, the Fock space is spanned by vectors of the form $f_{i_1}^\dagger f_{i_2}^\dagger \dots f_{i_k}^\dagger|0\rangle$ with $k = 0,1,2,\dots,N$.

Now, this space can alternatively be represented [39] as the exterior algebra $\wedge(V)$ or $\wedge(V^*)$. For later convenience, we chose $\wedge(V^*)$. In this picture, the operators f_i and $f_i^\dagger$ acting on the Fock space are mapped to the ones $e^i\wedge$ and ι_{e_i} acting on $\wedge(V^*)$. If we use $P \in \wedge^k V^*$ of (6) as the representative of the unnormalized k-fermion state

$$|P\rangle = P_{i_1 i_2 \dots i_k} f_{i_1}^\dagger f_{i_2}^\dagger \dots f_{i_k}^\dagger |0\rangle, \tag{26}$$

then the action of the fermionic operators on the usual Fock space can be represented as the

$$f_i^\dagger |P\rangle \mapsto e^i \wedge P, \quad f_i|P\rangle \mapsto \iota_{e_i} P \tag{27}$$

action on k forms. This map clearly gives a representation of the (25) anticommutation relations. Indeed, from (12) one sees that

$$\iota_{e_i}(e^j \wedge P) = \delta_i^j P - e^j \wedge (\iota_{e_i} P), \tag{28}$$

hence, $\{\iota_{e_i}, e^j\wedge\} = \delta_i^j$.

Let then V be an N-dimensional complex vector space representing the one-particle states of a fermionic system and the unnormalized k-fermion states be represented as in Eq. (6). The $P_{i_1\dots i_k}$ in this formula are the $\binom{N}{k}$ complex amplitudes characterizing the k-fermion state. Here, we are dealing with a system of indistinguishable constituents, hence, SLOCC transformations are acting via the same $\mathrm{GL}(V) = \mathrm{GL}(N,\mathbb{C})$ map on each slot as defined in (22).

Two fermionic states P and P' are called SLOCC equivalent if there exists an element of $g \in \mathrm{GL}(V)$ such that $P' = g^* P$. The abbreviation SLOCC refers to stochastic local operations and classical communication [22,23], the type of physical manipulations represented mathematically by invertible linear transformations $g \in \mathrm{GL}(V)$. Sometimes, the unimodular subgroup $\mathrm{SL}(V) = \mathrm{SL}(N,\mathbb{C})$ is also used to define new equivalence classes. The subgroup

$$\mathrm{Stab}(P) = \{g \in \mathrm{GL}(V) | g^* P = P\} \tag{29}$$

is called the stabilizer subgroup of the multifermion state. Under the SLOCC equivalence relation, one can form the corresponding equivalence classes. We will refer to these classes as the SLOCC *entanglement classes*.

In order to distinguish between different types (classes) of entanglement, one can introduce entanglement measures. An entanglement measure is a real-valued function f of the amplitudes $P_{i_1\dots i_k}$ satisfying a number of physically useful properties [40]. Here, we will be content merely with one of such properties, namely, that our measures should be coming from relative invariants under the SLOCC group (invariants under the unimodular SLOCC group). A rational function $I : \wedge^k V^* \to \mathbb{C}$ is a SLOCC relative invariant if there exists a rational character $\chi : \mathrm{GL}(V) \to \mathrm{GL}(1,\mathbb{C})$, i.e., a one-dimensional rational representation such that

$$I(g^* P) = \chi(g) I(P). \tag{30}$$

If $\chi \equiv 1$, then I is called an invariant. The entanglement measures $f : \wedge^* V \to \mathbb{R}$ studied here are arising as magnitudes of relative invariants with respect to the SLOCC group (invariants under the unimodular SLOCC group).

There are a number of covariants that can be defined to form such invariants. For a study of covariants and invariants useful in the fermionic context, see Gurevich [41]. Here, we will be content with some of his constructions suitably modified and adapted to our purposes. For a multifermionic state P, one can review a collection of SLOCC invariants as follows.

Degree 1 invariants. These are the ranks of the linear maps that can be constructed from P and are linear in the amplitudes. Let ι denote the interior product of Eq. (11). Define the set of linear maps

$$P^{(l)} : \wedge^l V \to \wedge^{k-l} V^*, \quad \alpha \mapsto \iota_\alpha P. \tag{31}$$

Now, $P^{(l)}$ is a linear map from a vector space of dimension $\binom{N}{l}$ to a vector space of $\binom{N}{k-l}$, thus it has a SLOCC invariant rank at most $\min(\binom{N}{l},\binom{N}{k-l})$. However, not all of these are independent. Obviously, $P^{(k-l)}$ is the transpose of $P^{(l)}$, thus their rank is equal.

Degree 2 invariants. These are ranks of linear maps which are quadratic in the amplitudes of P. Let

$$\tilde{\kappa}_P^{(l)} : \wedge^l V \to \wedge^{2k-l} V^*, \quad \alpha \mapsto \iota_\alpha P \wedge P. \tag{32}$$

Now, by virtue of the (13) isomorphism one can define a new quantity

$$\kappa_P^{(l)} \equiv \star \circ \tilde{\kappa}_P^{(l)} \tag{33}$$

which is a *linear* map from $\wedge^l V$ to $\wedge^{N-2k+l} V \otimes \wedge^N V^*$. The appearance of the one-dimensional space $\wedge^N V^*$ means that according to Eq. (24), this object picks up a determinant factor under a SLOCC transformation. Obviously, this construction only makes sense if $0 \leqslant l \leqslant k$ satisfies

$$0 \leqslant 2k - l \leqslant N. \tag{34}$$

Let us give the explicit form of $\kappa_P^{(l)}(\alpha)$:

$$\kappa_P^{(l)}(\alpha) = \frac{1}{(N-2k+l)!} \big(K_P^{(l)}\big)^{a_1\dots a_{N-2k+l}}{}_{b_1\dots b_l} \alpha^{b_1\dots b_l} \times e_{a_1} \wedge \dots \wedge e_{a_{N-2k+l}} \otimes \mathbb{E}, \tag{35}$$

where

$$\big(K_P^{(l)}\big)^{a_1\dots a_{N-2k+l}}{}_{b_1\dots b_l} = \frac{1}{(k-l)!k!} \varepsilon^{a_1\dots a_{N-2k+l} i_1 \dots i_{k-l} i_{k-l+1} \dots i_{2k-l}} \times P_{b_1\dots b_l i_1 \dots i_{k-l}} P_{i_{k-l+1}\dots i_{2k-l}}. \tag{36}$$

Clearly, the $\binom{N}{2k-l} \times \binom{N}{l}$ matrices $K_P^{(l)}$ have a SLOCC invariant rank. The index structure of $K_P^{(l)}$ shows that under SLOCC transformations, the upper indices are transformed via the use of $N - 2k + l$ matrices g_j^i and the lower indices via the use of l matrices $g_i^{\prime j}$; moreover, due to the presence of the Levi-Civitá symbol [compare also with the transformation rule of Eq. (24)], an extra factor of $\mathrm{Det}g' = (\mathrm{Det}g)^{-1}$ appears.

Proposition 1. $\tilde{\kappa}_P^{(k-1)} = 0$ if and only if P is separable.

Proof. Let $\alpha \in \wedge^{k-1}V$. By definition

$$\tilde{\kappa}_P^{(k-1)}(\alpha) = \alpha^{i_1\ldots i_{k-1}} P_{i_1\ldots i_{k-1}i_k} P_{j_1\ldots j_k} e^{i_k} \wedge e^{j_1} \wedge \ldots \wedge e^{j_k}. \quad (37)$$

Since α is arbitrary, then our condition reads as

$$P_{i_1\ldots i_{k-1}[i_k} P_{j_1\ldots j_k]} = 0, \quad (38)$$

where the brackets denote antisymmetrization. It can be shown (see, e.g., Proposition 3.5.30 of Penrose and Rindler [42]) that Eq. (38) is a sufficient and necessary condition for $P_{i_1\ldots i_k}$ to be separable, i.e., of the form $P_{i_1\ldots i_k} = a_{[i_1} b_{i_2} \ldots z_{i_k]}$. ■

These amplitudes can be expressed in terms of a single Slater determinant, hence they represent separable multifermion states. Note that for these sufficient and necessary conditions of separability, an equivalent form is provided by the set of Plücker relations usually expressed [43] in the

$$\Pi_{\mathcal{A},\mathcal{B}} = \sum_{n=1}^{k+1} (-1)^{n-1} P_{i_1 i_2 \ldots i_{k-1} j_n} P_{j_1 j_2 \ldots j_{k+1} \hat{j}_n} = 0 \quad (39)$$

form. Here, $\mathcal{A} = \{i_1, i_2, \ldots, i_{k-1}\}$ and $\mathcal{B} = \{j_1, j_2, \ldots, j_{k+1}\}$ are $k-1$ and $k+1$ element subsets of the set $\{1,2,\ldots,N\}$, and where the number $\hat{j}_n$ has to be omitted.

Degree $n+1$ invariants. Define

$$\begin{aligned} \kappa_P^{(l_1 l_2 \ldots l_n)} : \otimes_{j=1}^n (\wedge^{l_j} V) &\to \wedge^{k+nk-\sum_{j=1}^n l_j} V^* \\ &\cong \wedge^{N-k(n+1)+\sum_{j=1}^n l_j} V \otimes \wedge^N V^*, \\ \alpha_1, \ldots, \alpha_n &\mapsto \star \left(\iota_{\alpha_1} P \wedge \ldots \wedge \iota_{\alpha_n} P \wedge P \right), \\ \alpha_i &\in \wedge^{l_i} V. \end{aligned} \quad (40)$$

Just like the ones of Eq. (35), these quantities are based on $\prod_{j=1}^n \binom{N}{l_j}$ times $\binom{N}{(n+1)k - \sum_{j=1}^n l_j}$ matrices $K_P^{(l_1\ldots l_n)}$ with a SLOCC invariant rank. For the definition to make sense, we have the constraint for $0 \leqslant l_j \leqslant k$:

$$0 \leqslant (n+1)k - \sum_{j=1}^n l_j \leqslant N. \quad (41)$$

These covariants with degree over 2 can have extra symmetry properties if there exists $l_i = l_j$ for some $i \neq j$. Consider, for example, $\kappa_P^{(ll)}$. Then, we have

$$\kappa_P^{(ll)}(\alpha_1, \alpha_2) = (-1)^{k-l} \kappa_P^{(ll)}(\alpha_2, \alpha_1), \quad \alpha_1, \alpha_2 \in \wedge^l V. \quad (42)$$

As an example needed later on, let us consider the special case of $\kappa_P^{(l_1\ldots l_n)}$ with $l_1 = \ldots = l_n = 1$ for three-fermion systems with N single-particle states. In this case, $k = 3$, $\alpha = \alpha^b e_b \in V$, and we define m via $3 + 2n + m = N$. For simplicity in this case we will refer to $\kappa_P^{(l_1\ldots l_n)}$ as $\kappa_P^{[m,n]}$. Then, $\kappa_P^{[m,n]} \wedge^n V \to \wedge^m V \otimes \wedge^N V^*$ has the form

$$\begin{aligned} &\kappa_P^{[m,n]}(\alpha_1, \ldots, \alpha_n) \\ &\quad = \frac{1}{m!} \left(K_P^{[m,n]}\right)^{a_1\ldots a_m}{}_{b_1\ldots b_n} \alpha_1{}^{b_1} \ldots \alpha_n{}^{b_n} e_{a_1} \wedge \ldots \wedge e_{a_m} \otimes \mathbb{E}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} &\left(K_P^{[m,n]}\right)^{a_1\ldots a_m}{}_{b_1\ldots b_n} \\ &\quad = \frac{1}{2^n 3!} \varepsilon^{a_1\ldots a_m i_1 \ldots i_{2n+3}} P_{b_1 i_1 i_2} \ldots P_{b_n i_{2n-1} i_{2n}} P_{i_{2n+1} i_{2n+2} i_{2n+3}}. \end{aligned} \quad (44)$$

Notice that $(K_P^{[m,n]})^{a_1\ldots a_m}{}_{b_1\ldots b_n}$ is totally antisymmetric in its upper, and symmetric in its lower, indices.

The ranks of the linear maps defined above are SLOCC invariants because a SLOCC transformation on them simply means an invertible change of basis in the domain and the range and a multiplication by some power of the SLOCC determinant. However, these ranks are not continuous invariants in the amplitudes $P_{i_1 i_2 i_3}$. We can also use the above-defined linear maps to define continuous relative SLOCC invariants. The idea is to utilize the trace and determinant defined on linear automorphisms of vector spaces. In order to do this, we need to construct square matrices. This can be done by composing maps with each other to have the same dimensional range and domain. As we will see, the simplest case arises when the above-defined maps are square matrices themselves.

It is also worth noting that a system of k qudits with Hilbert space $\mathcal{H} = \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ can be embedded in this special fermionic system [44] in the following way:

$$\begin{aligned} |\psi\rangle &= \sum_{\mu_1,\ldots,\mu_k=1}^{d} \psi_{\mu_1\ldots\mu_k} |\mu_1\rangle \otimes \ldots \otimes |\mu_k\rangle \in \mathcal{H}, \\ P_\psi &= \sum_{\mu_1,\ldots,\mu_k=1}^{d} \psi_{\mu_1\ldots\mu_k} e^{\mu_1} \wedge e^{d+\mu_2} \wedge \ldots \wedge e^{(k-1)d+\mu_k}. \end{aligned} \quad (45)$$

Obviously, a SLOCC transformation on $\mathcal{H}$ of the form $g_1 \otimes \ldots \otimes g_k \in \mathrm{GL}(d,\mathbb{C})^{\otimes k}$ acting on ψ like

$$\psi_{\mu_1\ldots\mu_k} \mapsto (g_1)_{\mu_1}^{\nu_1} \ldots (g_k)_{\mu_k}^{\nu_k} \psi_{\mu_1\ldots\nu_k} \quad (46)$$

can be embedded in the SLOCC group $\mathrm{GL}(V)$ of our fermionic system via

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \in \mathrm{GL}(dk,\mathbb{C}) = \mathrm{GL}(V). \quad (47)$$

As a consequence, embedded states on different $\mathrm{GL}(V)$ orbits must be in different $\mathrm{GL}(d,\mathbb{C})^{\otimes k}$ orbits as well. However, the converse is not generally true: entanglement classes of the fermionic system may split into different classes when just the embedded system is considered. However, when we consider the generalized SLOCC group, i.e., the SLOCC group combined with permutations, some important exceptions arise. In the case of three-qubits, the embedding into three-fermions with six single-particle states is bijective between the SLOCC classes of the two systems. As was shown in the case of four-qubits embedded into the system of four-fermions with eight single-particle states, two inequivalent four-qubit states remain inequivalent under the fermionic SLOCC group [45]. As pointed out in Sec. IV D 1, splitting does not occur between *families* of entanglement classes for the embedding of three-qutrits into the system of three-fermions with nine single-particle states. Most likely this is true for the entanglement classes too. In the cases when splitting of fermionic entanglement classes does occur, one can still use the ranks of the maps $\kappa_{P_\psi}^{(l_1\dots l_n)}$ in order to obtain a coarse graining of the entanglement classes of $\mathcal{H}$.

Finally, note that one can see from the isomorphism (27) that the anticommutation relations (25) are invariant under invertible SLOCC transformations. Based on this property, one can extend the group $\mathrm{GL}(V)$ acting on fermionic states to a bigger one which also enables the implementation of Bogoliubov transformations. This way, one can obtain a classification of states on the whole fermionic Fock space, not just on the fixed-particle-number subspaces. For details on this subject, see our recent work [46].

IV. ENTANGLEMENT OF THREE-FERMIONS

A. Six single-particle states

The entanglement classification of three-fermions with six single-particle states is already well known and has a broad connection with several mathematical and physical structures [9–11,25] in the literature. It was first recognized as a quantum information theory problem in Ref. [30], where also the connection to Freudenthal triple systems has been revealed. Later, it has been realized that the corresponding mathematical problem has already been solved long ago [14] and that the generic SLOCC orbit is precisely the one which shows up in the theory of prehomogeneous vector spaces [31,32]. Moreover, within such three-fermionic systems, three-qubit systems can be embedded [25,30,44,45] in this case this generic SLOCC class corresponds to the famous Greenberger-Horne-Zeilinger (GHZ) class [23] of three-qubit entanglement. Furthermore, recently it has been shown that the problem is even connected to string theory via the so-called Hitchin functionals [26,33,34].

Let V be a the six-dimensional complex vector space $\mathbb{C}^6$. Then, an unnormalized three-fermion state can be represented as

$$P=\frac{1}{3!}P_{i_1i_2i_3}e^{i_1}\wedge e^{i_2}\wedge e^{i_3}\in\wedge^3V^*. \tag{48}$$

The $P_{i_1i_2i_3}$ are the 20 complex amplitudes describing the three-fermion state. The SLOCC transformations act with the same $\mathrm{GL}(V^*)=\mathrm{GL}(6,\mathbb{C})$ map on each slot as

$$P_{i_1i_2i_3}\mapsto g'^{j_1}_{i_1}g'^{j_2}_{i_2}g'^{j_3}_{i_3}P_{j_1j_2j_3} \tag{49}$$

just as defined in Eqs. (22) and (20).

In the following, we show that the SLOCC orbits of this system are completely characterized by the ranks of the degree one $P^{(2)}$ [Eq. (31)] and the degree two $\kappa_P^{(l_1=1)}=\kappa_P^{[1,1]}$ [Eq. (35)] covariants. In order to see this, let us consider the latter one. According to Eq. (36), its underlying 6×6 matrix has the explicit form

$$\left(K_P^{[1,1]}\right)^a_{\ b}=\frac{1}{2!3!}\varepsilon^{ai_1i_2i_3i_4i_5}P_{bi_1i_2}P_{i_3i_4i_5}, \tag{50}$$

where we also used the notation introduced in Eq. (44). By construction, $K_P^{[1,1]}$ transforms under SLOCC transformations as

$$\left(K_P^{[1,1]}\right)^a_{\ b}\mapsto \mathrm{Det}(g')g^a_{\ c}g'^{\ d}_{b}\left(K_P^{[1,1]}\right)^c_{\ d},\quad g\in\mathrm{GL}(V). \tag{51}$$

According to Eq. (20), the matrix g' is just the inverse transpose of the one g, hence, this transformation rule is of the form $K_P^{[1,1]}\mapsto[\mathrm{Det}(g)]^{-1}gK_P^{[1,1]}g^{-1}$. It follows that any power of the trace of $K_P^{[1,1]}$ is a relative invariant. One can immediately check that $\mathrm{Tr}K_P^{[1,1]}=0$, hence, the next item in line to experiment with is $\mathrm{Tr}(K_P^{[1,1]})^2$.

It is well known that this quantity suitably normalized

$$\mathcal{D}(P)=\tfrac{1}{6}\mathrm{Tr}\left(K_P^{[1,1]}\right)^2 \tag{52}$$

is indeed a relative invariant and its magnitude defines a good measure of entanglement. That $\mathcal{D}$ is a relative invariant transforming as

$$\mathcal{D}(P)\mapsto[\mathrm{Det}(g')]^2\mathcal{D}(P) \tag{53}$$

can immediately be seen from the transformation property of Eq. (51) and the definition of Eq. (52). In order to see the last property, namely, that its magnitude provides a measure of entanglement, let us give this relative invariant another look [30]. First, we reorganize the 20 independent complex amplitudes $P_{i_1i_2i_3}$ into two complex numbers η,ξ and two complex 3×3 matrices X and Y as follows. As a first step, we change our labeling convention by using the symbols $\dot{1},\dot{2},\dot{3}$ instead of 4,5,6, respectively; hence, we have

$$(1,2,3,4,5,6)\leftrightarrow(1,2,3,\dot{1},\dot{2},\dot{3}). \tag{54}$$

Hence, for example, we can alternatively refer to P_{456} as $P_{\dot{1}\dot{2}\dot{3}}$ or to P_{125} as $P_{12\dot{2}}$. Now, we define

$$\eta\equiv P_{123},\quad \xi\equiv P_{\dot{1}\dot{2}\dot{3}}, \tag{55}$$

$$X=\begin{pmatrix}X_{11}&X_{12}&X_{13}\\X_{21}&X_{22}&X_{23}\\X_{31}&X_{32}&X_{33}\end{pmatrix}\equiv\begin{pmatrix}P_{1\dot{2}\dot{3}}&P_{1\dot{3}\dot{1}}&P_{1\dot{1}\dot{2}}\\P_{2\dot{2}\dot{3}}&P_{2\dot{3}\dot{1}}&P_{2\dot{1}\dot{2}}\\P_{3\dot{2}\dot{3}}&P_{3\dot{3}\dot{1}}&P_{3\dot{1}\dot{2}}\end{pmatrix}, \tag{56}$$

$$Y=\begin{pmatrix}Y_{11}&Y_{12}&Y_{13}\\Y_{21}&Y_{22}&Y_{23}\\Y_{31}&Y_{32}&Y_{33}\end{pmatrix}\equiv\begin{pmatrix}P_{\dot{1}23}&P_{\dot{1}31}&P_{\dot{1}12}\\P_{\dot{2}23}&P_{\dot{2}31}&P_{\dot{2}12}\\P_{\dot{3}23}&P_{\dot{3}31}&P_{\dot{3}12}\end{pmatrix}. \tag{57}$$

With this notation, the quartic polynomial of Eq. (52) is

$$\mathcal{D}(P)=[\eta\xi-\mathrm{Tr}(XY)]^2-4\,\mathrm{Tr}(X^\sharp Y^\sharp)+4\eta\,\mathrm{Det}(X)+4\xi\,\mathrm{Det}(Y), \tag{58}$$

where $X^\sharp$ and $Y^\sharp$ correspond to the regular adjoint matrices for X and Y, hence, for example, $XX^\sharp=X^\sharp X=\mathrm{Det}(X)I$ with I the 3×3 identity matrix.

Now, according to Eq. (45), we can embed a three-qubit state ψ into our system of three-fermions with six single-particle states as the state P_ψ. However, for convenience we chose another form of this embedding [30] which amounts to a permutation (3245) of the basis vectors $e^1,\dots e^6$. One can show that under this permutation, the matrix of embedded SLOCC transformations familiar from Eq. (47) takes a form of a 6×6 matrix consisting of four blocks of 3×3 diagonal matrices. Via this embedding, we keep merely eight complex amplitudes from the 20 ones of P which transform according to the restricted SLOCC group as the amplitudes of a three-qubit system. Let us label the eight amplitudes of P_ψ as

$$\begin{aligned}&(P_{123},P_{12\dot{3}},P_{1\dot{2}3},P_{\dot{1}23},P_{\dot{1}\dot{2}\dot{3}},P_{\dot{1}\dot{2}3},P_{\dot{1}2\dot{3}},P_{1\dot{2}\dot{3}})\\&\quad=(\psi_{000},\psi_{001},\psi_{010},\psi_{100},\psi_{111},\psi_{110},\psi_{101},\psi_{011}),\end{aligned} \tag{59}$$

where unlike in Eq. (45), now we switched to the use of the conventional labeling $\mu_1,\mu_2,\mu_3=0,1$ of basis states. Then, $\mathcal{D}(P_\psi)\equiv D(\psi)$ takes the following form:

$$\begin{aligned}D(\psi)=&[\psi_0\psi_7-\psi_1\psi_6-\psi_2\psi_5-\psi_3\psi_4]^2\\&-4[(\psi_1\psi_6)(\psi_2\psi_5)+(\psi_2\psi_5)(\psi_3\psi_4)\\&+(\psi_3\psi_4)(\psi_1\psi_6)]+4\psi_1\psi_2\psi_4\psi_7+4\psi_0\psi_3\psi_5\psi_6,\end{aligned} \tag{60}$$

where $(\psi_0,\psi_1,\ldots,\psi_7)\equiv(\psi_{000},\psi_{001},\ldots,\psi_{111})$. $D(\psi)$ gives rise to a famous entanglement measure [47] called the *three-tangle* τ_{123} which for normalized states satisfies

$$0\leqslant\tau_{123}=4|D(\psi)|\leqslant 1. \tag{61}$$

Hence, $\mathcal{D}(P)$ with the normalization as given by Eq. (52) is a natural generalization of the three-tangle for three-fermions with six single-particle states. For normalized fermionic states, it can be shown [30] that an analogous quantity $\mathcal{T}_{123}$ formed from $\mathcal{D}(P)$ satisfies

$$0\leqslant\mathcal{T}_{123}=4|\mathcal{D}(P)|\leqslant 1 \tag{62}$$

just like the three-tangle does for three-qubits. We note that the expression for $\mathcal{D}$ as given by Eq. (58) is just the quartic invariant of the Freudenthal triple system over the cubic Jordan algebra $M(3,\mathbb{C})$ of 3×3 complex matrices [30,48].

Let us give yet another form [26,49] of the quartic invariant $\mathcal{D}(P)$. Define a symplectic form on $\wedge^3V^*$,

$$\begin{aligned}&\{\ldots,\ldots\}:\wedge^3V^*\times\wedge^3V^*\to\mathbb{C},\\&(P,Q)\mapsto\frac{1}{3!3!}\varepsilon^{ijklmn}P_{ijk}Q_{lmn},\end{aligned} \tag{63}$$

and a three-form $\tilde{P}$ dual to the original three-form $P\in\wedge^3V^*$ as

$$\begin{aligned}\tilde{P}&=\frac{1}{3!}\tilde{P}_{abc}e^a\wedge e^b\wedge e^c,\\\tilde{P}_{abc}&=\frac{1}{2!3!}\varepsilon^{di_2i_3i_4i_5i_6}P_{bcd}P_{ai_2i_3}P_{i_4i_5i_6}\\&=P_{bcd}\big(K_P^{[1,1]}\big)^d_{\ a}.\end{aligned} \tag{64}$$

Then, the new form of the quartic invariant is

$$\mathcal{D}(P)=\tfrac{1}{2}\{\tilde{P},P\}. \tag{65}$$

In the theory of Freudenthal triple systems, the quantity $\tilde{P}$ which is cubic in the original amplitudes of P is usually defined via the so-called trilinear form [48]. With the help of $\tilde{P}$ for a state with $\mathcal{D}\neq 0$, one can define a *dual fermionic state* as

$$\hat{P}\equiv -i\frac{\tilde{P}}{\sqrt{\mathcal{D}}}. \tag{66}$$

With our convention of defining a factor of $-i$, the expression of $\hat{P}$ boils down to the expression of the so-called *Freudenthal dual* of P defined only for *real* states in the paper [49] of Borsten *et al*. One can check that the dual state satisfies the identities

$$\mathcal{D}(\hat{P})=\mathcal{D}(P),\quad \hat{\hat{P}}=-P. \tag{67}$$

Notice also that according to Eqs. (53) and (66) (unlike the quantity $\tilde{P}$) the one $\hat{P}$ does not pick up a determinant factor under SLOCC transformations.

The classification problem for three-forms in $V=\mathbb{C}^6$ under the group action $\mathrm{GL}(V)$ has been solved long ago by Reichel [14]. In the context of fermionic entanglement, it has recently been rediscovered by physicists [30]. According to this result, the $\mathrm{GL}(V)$ orbits of three-forms correspond to the SLOCC orbits of three-fermions with six single-particle states. We have five SLOCC classes. Using the notation

$$e^{ijk}\equiv e^i\wedge e^j\wedge e^k, \tag{68}$$

the representatives of these classes taken together with the ranks of the basic covariants can be seen in Table I.

TABLE I. Entanglement classes of three-fermions with six single-particle states, and the ranks of the simplest covariants.

Type	Canonical form of P	Rank $P^{(2)}$	Rank $\kappa_P^{(1)}$	Rank $\kappa_P^{(2)}$
Null	0	0	0	0
Sep	e^{123}	3	0	0
Bisep	$e^{123}+e^{156}$	5	1	4
W	$e^{126}+e^{423}+e^{153}$	6	3	6
GHZ	$e^{123}+e^{456}$	6	6	6

The four nontrivial classes are labeled by the states familiar from the classification of three-qubits [23]. Namely, we have the totally separable, biseparable, W, and GHZ (Greenberger-Horne-Zeilinger) classes. Using the language of embedded systems, the notation of Eq. (54), and the mapping of Eq. (59), one obtains the normalized representatives of these classes as $|000\rangle$ for the separable, $(|000\rangle+|011\rangle)/\sqrt{2}$ for the biseparable, $(|001\rangle+|010\rangle+|100\rangle)/\sqrt{3}$ for the W, and $(|000\rangle+|111\rangle)/\sqrt{2}$ for the GHZ class.

An alternative description of the nontrivial SLOCC classes in terms of normalized representatives can also be given using the invariant $\mathcal{D}$ and the covariant $\tilde{P}$ as follows:

$$P_{\rm GHZ}=\frac{1}{2}(e^{123}+e^{156}+e^{264}+e^{345}),\quad \mathcal{D}(P)\neq 0 \tag{69}$$

$$P_{\rm W}=\frac{1}{\sqrt{3}}(e^{123}+e^{156}+e^{264}),\quad \mathcal{D}(P)=0,\quad \tilde{P}\neq 0 \tag{70}$$

$$P_{\rm bisep}=\frac{1}{\sqrt{2}}(e^{123}+e^{156}),\quad \mathcal{D}(P)=0,\quad \tilde{P}=0 \tag{71}$$

$$P_{\rm sep}=e^{123},\quad \mathcal{D}(P)=0,\quad \tilde{P}=0. \tag{72}$$

Here, we have given the representatives of the GHZ and W classes in a form different from those appearing in Table I. In this new form, the number of terms appearing in the representatives is increasing as we proceed from the separable class to the maximally entangled GHZ one. Notice that the difference from the representatives of the GHZ and W classes of Table I and Eqs. (69) and (70) amounts to a SLOCC transformation. The meaning of these transformations can easily be clarified if we reinterpret these states as three-qubit ones according to the prescription of Eq. (59). Indeed, using the new labeling of Eq. (54), the three-qubit states corresponding to those of Eqs. (69)–(72) are

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle), \tag{73}$$

$$\frac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |101\rangle), \tag{74}$$

$$\frac{1}{2}(|000\rangle + |011\rangle), \tag{75}$$

$$|000\rangle. \tag{76}$$

Now, it is easy to show that

$$\begin{aligned} &\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \\ &= (H \otimes H \otimes H)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ &= H \otimes H \otimes H|\,\mathrm{GHZ}\rangle \end{aligned} \tag{77}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |101\rangle) \\ &= (I \otimes I \otimes X)\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \\ &= (I \otimes I \otimes X)|\mathrm{W}\rangle, \end{aligned} \tag{78}$$

where H and X are the usual Hadamard and bit flip gates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{79}$$

Hence, these states are local unitary (hence also SLOCC) equivalent to the usual GHZ and W states [23]. Notice also that since

$$\begin{aligned} D[(g_1 \otimes g_2 \otimes g_3)\psi] &= \mathrm{Det}(g_1)^2\mathrm{Det}^2(g_2)\mathrm{Det}^2(g_3)D(\psi), \\ g_1,g_2,g_3 &\in \mathrm{GL}(2,\mathbf{C}) \end{aligned} \tag{80}$$

none of these transformations change the value of Cayley's hyperdeterminant $D(\psi)$. Moreover, since $\mathcal{D}(P_\psi) = D(\psi)$ after reinterpreting again our three-qubit states as three-fermionic ones via the correspondence $\psi \mapsto P_\psi$, the SLOCC transformations acting on the corresponding fermionic states can be obtained from those of Eqs. (77) and (78) using Eqs. (46) and (47) and the permutation (3245).

Notice also that in order to separate the last two classes with representatives of Eqs. (71) and (72), one has to use the Plücker relations of Eqs. (38) and (39). In our special case, these relations can be described in the following elegant form [50]:

$$\eta X = Y^\sharp, \quad \xi Y = X^\sharp, \quad \eta\xi I = XY, \tag{81}$$

where for the connection between the amplitudes of P and the quantities (η,X,Y,ξ), see Eqs. (55)–(57). These relations hold if and only if the corresponding fermionic state is separable, i.e., can be written in the form of a single Slater determinant.

The GHZ and W classes are the two inequivalent classes for tripartite entangled fermionic systems with six modes. These classes are completely characterized by the relative invariant $\mathcal{D}(P)$ and the dual state $\tilde{P}$ (a covariant). Note that the GHZ class corresponds to a stable SLOCC orbit [32]. Stability means that states in a neighborhood (with respect to the Zariski topology) of a particular one are all SLOCC equivalent ones. More precisely, states of the GHZ class form an open dense orbit within the state space of three-fermions with six single-particle states. This fact is related to the result that our state space of such fermions corresponds to a prehomogeneous vector space which is the class No. 5 in the Sato-Kimura classification [32] of such spaces.

Let us elaborate on this stable class of GHZ states. As we know, the canonical form of a representative from the genuine entangled (GHZ) class is

$$P_0 = e^{123} + e^{456}. \tag{82}$$

For this representative, one can easily check that the matrix of $K^{[1,1]}_{P_0}$ is of the form

$$\left(K^{[1,1]}_{P_0}\right)^a_{\ b} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}. \tag{83}$$

Now, $\mathcal{D}(P_0) = 1$, hence, for the dual state of Eq. (66) we have

$$\hat{P}_0 = -i(e^{123} - e^{456}). \tag{84}$$

Now, the states $P_0 + i\hat{P}_0$ and $P_0 - i\hat{P}_0$ are clearly separable ones. Moreover, since P and $\hat{P}$ both transform covariantly under SLOCC transformations, separability is preserved; hence, for *any* state P with $\mathcal{D}(P) \neq 0$ (i.e., one in the GHZ class) the states

$$U_\pm = P \pm i\hat{P} \tag{85}$$

are separable ones. In other words, for any state in the GHZ class the expression

$$P = \tfrac{1}{2}(U_+ + U_-) \tag{86}$$

provides a canonical decomposition in terms of two Slater determinants.

Let us also discuss the structure of the SLOCC classes for *real states*. In this case, the vector space underlying our three-fermion state space is $V = \mathbb{R}^6$ and the SLOCC group is $\mathrm{GL}(6,\mathbb{R})$. In contrast to the five classes of Table I, now we have *six entanglement classes*. The extra class is coming from a splitting of the usual GHZ class into *two classes*. The two classes are having $\mathcal{D}(P) > 0$ and $\mathcal{D}(P) < 0$ are called GHZ_+ and GHZ_- classes, respectively. The normalized representatives are

$$\begin{aligned} P_{\mathrm{GHZ}_+} &= \tfrac{1}{2}(e^{123} + e^{156} + e^{264} + e^{345}), \\ P_{\mathrm{GHZ}_-} &= \tfrac{1}{2}(e^{123} - e^{156} - e^{264} - e^{345}). \end{aligned} \tag{87}$$

Of course, P_{GHZ_+} is just the state known from Eq. (69) which is real SLOCC equivalent to the GHZ representative of Table I. Let us illustrate this result in the language of embedded three-qubit systems. These fermionic states correspond to the ones

$$\begin{aligned} |\mathrm{GHZ}_+\rangle &= \tfrac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle), \\ |\mathrm{GHZ}_-\rangle &= \tfrac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle). \end{aligned} \tag{88}$$

We already know from Eq. (77) that

$$|\mathrm{GHZ}_+\rangle = H \otimes H \otimes H|\mathrm{GHZ}\rangle. \tag{89}$$

On the other hand, we have

$$|\mathrm{GHZ}_-\rangle = \mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}|\mathrm{GHZ}\rangle, \quad \mathcal{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}. \tag{90}$$

These expressions illustrate the fact that although the states $|\mathrm{GHZ}_\pm\rangle$ are complex SLOCC equivalent, however, they are real SLOCC inequivalent. One can also write these states as

$$\begin{aligned} |\mathrm{GHZ}_+\rangle &= \frac{1}{\sqrt{2}}(|F_+\rangle\otimes|F_+\rangle\otimes|F_+\rangle + |F_-\rangle\otimes|F_-\rangle\otimes|F_-\rangle), \\ |F_\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle), \end{aligned} \tag{91}$$

$$\begin{aligned} |\mathrm{GHZ}_-\rangle &= \frac{1}{\sqrt{2}}(|E\rangle\otimes|E\rangle\otimes|E\rangle + \overline{|E\rangle\otimes|E\rangle\otimes|E\rangle}), \\ |E\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \end{aligned} \tag{92}$$

where the overline means complex conjugation. These expressions illustrate our result of Eq. (86) for decomposing an arbitrary state from the GHZ class into *two* separable states. According to the definition of the dual state $\hat{P}$ of Eq. (66), we see that for real states with $\mathcal{D}(P) > 0$ the separable components are remaining *real*; on the other hand, for $\mathcal{D}(P) < 0$ they are *complex*-conjugate states. In this latter case, one can define the 6×6 matrix [33]

$$J_P \equiv \frac{K_P^{[1,1]}}{\sqrt{-\mathcal{D}(P)}}. \tag{93}$$

One can prove that $(K_P^{[1,1]})^2 = \mathcal{D}(P)I$, where I is the 6×6 identity matrix. Hence, for $\mathcal{D}(P) < 0$ we have

$$J_P^2 = -I. \tag{94}$$

This means that if we start with a *real* three-fermion state P satisfying $\mathcal{D}(P) < 0$, then on its single-particle space $V = \mathbb{R}^6$ this P defines a *complex structure*. For the special state P_{GHZ_-} of Eq. (87), the complex structure in question is just the canonical one giving rise on V to the complex coordinates

$$E^{1,2,3} = e^{1,2,3} + ie^{4,5,6}, \quad E^{\overline{1},\overline{2},\overline{3}} = e^{1,2,3} - ie^{4,5,6}. \tag{95}$$

Since

$$E^{123} + E^{\overline{123}} = 2(e^{123} - e^{156} - e^{426} - e^{345}) \tag{96}$$

in the three-qubit reinterpretation, these complex coordinates correspond to our writing $|\mathrm{GHZ}_-\rangle$ in the (92) form. Notice also that with respect to the complex structure $J_{P_{\mathrm{GHZ}_-}}$, the components E^{123} and $E^{\overline{123}}$ are giving rise to the (3,0) holomorphic and (0,3) antiholomorphic parts of P_{GHZ_-}.

We note in closing that there is an interesting physical application of these complex structures as defined by real three-fermion states. For this one takes a closed oriented six-dimensional real manifold $\mathcal{M}$ equipped with a real differential three-form P with $\mathcal{D}(P) < 0$ everywhere. Notice that locally at each point of $\mathcal{M}$ the tangent space and its dual gives rise to copies of a $V = \mathbb{R}^6$, hence, we can regard the differential three-form P as a collection of three-fermion states parametrized by the points of $\mathcal{M}$. Now, such a P defines an *almost complex structure* J_P on $\mathcal{M}$. One can then show [33] that when P is closed and belonging to a fixed cohomology class, then the critical points of the functional

$$V_H = \int_{\mathcal{M}} \sqrt{-\mathcal{D}(P)}d^6x \tag{97}$$

are satisfying the equation

$$d\hat{P} = 0, \tag{98}$$

meaning that the dual form $\hat{P}$ is also closed. Hence, the *separable* differential form $\Omega = P + i\hat{P}$ is of type (3,0), closed, and the almost complex structure J_P is *integrable*. In this way, one can generate a complex structure to a six-dimensional manifold $\mathcal{M}$ rendering it to a *threefold*. Calabi-Yau threefolds are particularly important actors in string theory. Such spaces describe the structure of extra dimensions. The shapes and volumes of such spaces are subject to quantum fluctuations. It can be shown that the fluctuations in shapes preserving volume correspond to fluctuations in the complex structure of $\mathcal{M}$. Hence, the result briefly discussed above means that the critical points of certain action functionals of three-forms belonging to a fixed cohomology class single out *special* complex structures. Fixing a cohomology class physically means that we fix the wrapping configurations of three-dimensional extended objects, membranes, around the noncontractible three-cycles of the extra dimensions. Under certain conditions, the projections of these higher-dimensional configurations to our four-dimensional space-time look like charged black holes. For an application of these ideas within the interesting field of the so-called black-hole and qubit correspondence [51], see our recent paper on Hitchin functionals related to measures of entanglement [26].

B. Seven single-particle states

In the case of three-fermions with seven single-particle states, an arbitrary unnormalized state is described by the element $\mathcal{P} \in \wedge^3 V^*$ where $V = \mathbb{C}^7$. Such an element can be written as

$$\mathcal{P} = \frac{1}{3!}\mathcal{P}_{I_1I_2I_3}e^{I_1}\wedge e^{I_2}\wedge e^{I_3} \tag{99}$$

with $\{e^I\}_{I=1}^7$ a basis of V^*. Now, the SLOCC group is $\mathrm{GL}(V) = \mathrm{GL}(7,\mathbb{C})$ with the same kind of action as in (22).

Let us first consider the covariants $\kappa_P^{(1)} \equiv \kappa_P^{[2,1]}$ and $\kappa_P^{(1,1)} \equiv \kappa_P^{[0,2]}$. For simplicity, we introduce the notation

$$(M^A)^B_C \equiv \left(K_P^{[2,1]}\right)^{AB}_C, \tag{100}$$

$$N_{AB} \equiv \left(K_P^{[0,2]}\right)_{AB}, \tag{101}$$

where their explicit form according to Eq. (44) is

$$(M^A)^B_C = \frac{1}{12}\varepsilon^{ABI_1I_2I_3I_4I_5}\mathcal{P}_{CI_1I_2}\mathcal{P}_{I_3I_4I_5}, \tag{102}$$

$$N_{AB} = \frac{1}{24}\varepsilon^{I_1I_2I_3I_4I_5I_6I_7}\mathcal{P}_{AI_1I_2}\mathcal{P}_{BI_3I_4}\mathcal{P}_{I_5I_6I_7}. \tag{103}$$

Note that for later use we have regarded M as a collection of seven 7×7 matrices. Also note that in the real case $V=\mathbb{R}^7$ used in the literature on manifolds of special holonomy [33,35,52], a suitable scalar multiple of the latter covariant shows up as

$$\mathcal{B}_{AB} = -\tfrac{1}{6}N_{AB}. \tag{104}$$

It is arising from the map $\mathcal{B}_\mathcal{P}: V\otimes V \to \wedge^7 V^*$ which gives rise to a seven-form when acting on the pair of vectors v and u as

$$\mathcal{B}_\mathcal{P}(v,u) = -\tfrac{1}{6}\iota_v\mathcal{P}\wedge\iota_u\mathcal{P}\wedge\mathcal{P}. \tag{105}$$

The transformation properties of these covariants are

$$(M^A)^B_C \mapsto (\mathrm{Det}g')g^A_D\, g^B_E\, g'^F_C\,(M^D)^E_F, \tag{106}$$

$$N_{AB} \mapsto (\mathrm{Det}g')g'^C_A\, g'^D_B N_{CD}. \tag{107}$$

It is convenient to study the case of three-fermions with seven single-particle states as the case of adding an extra mode to the six original ones of three-fermions discussed in the previous section. For this purpose, we split our seven-dimensional vector space V to the direct sum of a six- and a one-dimensional vector space spanned by the extra basis vector e^7. Then, we write

$$\mathcal{P} = P + \omega\wedge e^7, \tag{108}$$

where P is given by Eq. (48) and ω is a two-form

$$\omega = \tfrac{1}{2}\omega_{ij}e^i\wedge e^j. \tag{109}$$

In the following, we adopt the convention for the indices such as $A,B,\ldots I,J,\ldots$ running from 1 to 7, and on the other hand indices such as $a,b,\ldots i,j,\ldots$ are running from 1 to 6. Hence, we have

$$\mathcal{P}_{abc}\equiv P_{abc}, \quad \mathcal{P}_{ab7}=\omega_{ab}. \tag{110}$$

Now, for the components of our covariants, a straightforward calculation yields the following results:

$$\begin{aligned}&(M^7)^7_7 = 0, \quad (M^7)^7_c = 0,\\ &(M^7)^b_7 = \tfrac{1}{12}\varepsilon^{bijklm}\omega_{ij}P_{klm}, \quad (M^7)^b_c = K^b_c,\end{aligned} \tag{111}$$

$$\begin{aligned}&(M^a)^7_7 = -\tfrac{1}{12}\varepsilon^{aijklm}\omega_{ij}P_{klm}, \quad (M^a)^7_c = -K^a_c,\\ &(M^a)^b_7 = \tfrac{1}{4}\varepsilon^{abijkl}\omega_{ij}\omega_{kl},\end{aligned} \tag{112}$$

$$(M^a)^b_c = \tfrac{1}{4}\varepsilon^{abijkl}\left(P_{cij}\omega_{kl} - \tfrac{2}{3}\omega_{ci}P_{jkl}\right), \tag{113}$$

$$\begin{aligned}&N_{77} = 6\mathrm{Pf}(\omega),\\ &N_{a7} = N_{7a} = -\tfrac{1}{12}\varepsilon^{ijklmn}\left(\omega_{ai}\omega_{jk}P_{lmn} + \tfrac{2}{3}P_{aij}\omega_{kl}\omega_{mn}\right),\end{aligned} \tag{114}$$

$$N_{ab} = \tfrac{1}{8}\varepsilon^{ijklmn}P_{aij}P_{bkl}\omega_{mn} + K^c_a\omega_{cb} + K^c_b\omega_{ac}. \tag{115}$$

Here, by an abuse of notation for the covariant of Eq. (50) we have used the shorthand K^a_b and $\mathrm{Pf}(\omega)$ is the Pfaffian of ω defined as

$$\mathrm{Pf}(\omega) = \frac{1}{2^3 3!}\varepsilon^{ijklmn}\omega_{ij}\omega_{kl}\omega_{mn}. \tag{116}$$

These expressions can be further simplified in the special case when

$$P\wedge\omega = 0. \tag{117}$$

The meaning of this condition is as follows. Due to the split of Eq. (108), one can understand the structure of three-fermions with seven single-particle states with 35 amplitudes via looking at the simpler structure of three-fermions with six single-particle states having merely 20 ones. In this perspective, the two-form ω giving rise to the extra 15 amplitudes can be regarded as an extra structure living on the six-dimensional vector space: a symplectic form. The condition $P\wedge\omega=0$ encapsulates a compatibility condition between the symplectic form ω and the three-form P. In the mathematical literature, this condition means that the three-form P is *primitive* with respect to ω. When P is primitive, using the identity $\iota_{e^i}(P\wedge\omega) = \iota_{e^i}P\wedge\omega - P\wedge\iota_{e^i}\omega$ one can show that

$$\left(P_{aij}\omega_{kl} + \tfrac{2}{3}\omega_{ai}P_{jkl}\right)e^i\wedge e^j\wedge e^k\wedge e^l = 0. \tag{118}$$

For our covariants, this result yields the much simpler looking expressions

$$(M^7)^7_7 = (M^7)^7_c = (M^7)^b_7 = 0, \quad (M^7)^b_c = K^b_c, \tag{119}$$

$$\begin{aligned}&(M^a)^7_7 = 0, \quad (M^a)^7_c = -K^a_c, \quad (M^a)^b_7 = \tfrac{1}{4}\varepsilon^{abijkl}\omega_{ij}\omega_{kl},\\ &(M^a)^b_c = \tfrac{1}{2}\varepsilon^{abijkl}P_{cij}\omega_{kl},\end{aligned} \tag{120}$$

$$\begin{aligned}&N_{77} = 6\mathrm{Pf}(\omega), \quad N_{a7} = N_{7a} = 0,\\ &N_{ab} = 3K^c_a\omega_{cb} = 3K^c_b\omega_{ca}.\end{aligned} \tag{121}$$

Notice that by virtue of Eq. (121), the 7×7 matrix $\boldsymbol{N}$ can be written in the factorized form

$$\boldsymbol{N} = \left(\begin{array}{c|c} -3\boldsymbol{\omega} & 0 \\ \hline 0 & 6\mathrm{Pf}(\boldsymbol{\omega})\end{array}\right)\left(\begin{array}{c|c} \boldsymbol{K} & 0 \\ \hline 0 & 1\end{array}\right), \tag{122}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{K}$ are the 6×6 matrices corresponding to the coefficient matrix of the two-form ω and the matrix of

Eq. (50). We see that if the first matrix has full rank, then rank $\boldsymbol{N}$ =rank$\boldsymbol{K}+1$. Also, because the matrix $\boldsymbol{N}$ has to be symmetric, $\boldsymbol{\omega}$ and $\boldsymbol{K}$ must anticommute.

It is convenient to define another symmetric 7×7 matrix

$$L^{AB}\equiv (M^A)_D^{\ C}(M^B)_C^{\ D}. \tag{123}$$

This matrix is a covariant quartic in the original amplitudes and under SLOCC transformations transforms as

$$L^{AB}\mapsto (\mathrm{Det}g')^2 g_C^{\ A}g_D^{\ B}L^{CD}. \tag{124}$$

The 7×7 matrix $\boldsymbol{L}$ can be regarded as the dual of the one $\boldsymbol{N}$. In Appendix A, it is shown that for $P\wedge\omega=0$, this matrix can also be written in the factorized form

$$\boldsymbol{L}=\left(\begin{array}{c|c}-12\tilde{\boldsymbol{\omega}} & 0\\ \hline 0 & 6\end{array}\right)\left(\begin{array}{c|c}\boldsymbol{K} & 0\\ \hline 0 & \mathcal{D}(P)\end{array}\right), \tag{125}$$

where

$$\tilde{\omega}^{ij}=\tfrac{1}{8}\varepsilon^{ijklmn}\omega_{kl}\omega_{mn}. \tag{126}$$

From the covariants $\boldsymbol{N}$ and $\boldsymbol{L}$ one can form a relative invariant $\mathcal{J}(\mathcal{P})$ homogeneous of degree 7:

$$\mathcal{J}(\mathcal{P})\equiv\frac{1}{2^4 3^2 7}\mathrm{Tr}(\boldsymbol{L}\boldsymbol{N})=\frac{1}{2^4 3^2 7}L^{AB}N_{AB}, \tag{127}$$

where the normalization was chosen for future convenience. Under SLOCC transformations we have

$$\mathcal{J}(\mathcal{P})\mapsto(\mathrm{Det}g')^3\mathcal{J}(\mathcal{P}). \tag{128}$$

Clearly, when written in terms of the components of P and ω, the relative invariant $\mathcal{J}(\mathcal{P})$ has a complicated expression. However, by virtue of Eqs. (122) and (125) in the special case when $P\wedge\omega=0$, it has a factorized form

$$\mathcal{J}(\mathcal{P})=\tfrac{1}{4}\mathrm{Pf}(\omega)\mathcal{D}(P). \tag{129}$$

As a useful relative invariant, one can also define either Det($\boldsymbol{N}$) or Det($\boldsymbol{L}$). However, it is easy to see that none of them are independent from $\mathcal{J}(\mathcal{P})$. Indeed, using, e.g., Eq. (122) for calculating Det($\boldsymbol{N}$), one obtains

$$\mathrm{Det}(\boldsymbol{N})=-6\times[9\mathrm{Pf}(\omega)\mathcal{D}(P)]^3. \tag{130}$$

Here, we have used $\mathrm{Det}(K)=-\mathcal{D}^3$ which follows from $K^2=\mathcal{D}\mathbf{1}$, $\mathrm{Tr}(K)=0$, and the Newton identities, moreover, we have also used that $\mathrm{Det}(\omega)=[\mathrm{Pf}(\omega)]^2$. The case of real states is important in the string theory literature where the determinant of the matrix $\mathcal{B}_{\mathcal{P}}$ of Eq. (104) is used [35] as a relative invariant. With our normalization as used in Eq. (127), we have

$$\mathrm{Det}\mathcal{B}_{\mathcal{P}}=[\mathcal{J}(\mathcal{P})]^3. \tag{131}$$

In order to present the SLOCC classification of three-fermions with seven single-particle states, let us consider again our seven-dimensional complex vector space V with its canonical basis vectors by e_A. Let us denote as usual the basis vectors of its six-dimensional subspace by e_a, $a=1,\dots,6$. As a complex basis of the dual of this subspace we define

$$E^{1,2,3}=e^{1,2,3}+ie^{4,5,6},\quad E^{\overline{1},\overline{2},\overline{3}}=e^{1,2,3}-ie^{4,5,6},\quad E^7=ie^7. \tag{132}$$

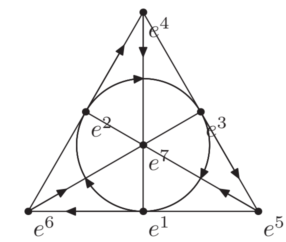

FIG. 1. The oriented Fano plane. The points of the plane correspond to the basis vectors of the seven-dimensional single-particle space. The lines of the plane represent three-fermion basis vectors with the arrows indicating the order of single-particle states in them to get a plus sign.

Then, a GHZ-type state in the six-dimensional subspace can be written as

$$E^{123}+E^{\overline{123}}=2(e^{123}-e^{156}+e^{246}-e^{345}). \tag{133}$$

With the usual relabeling $4,5,6\mapsto\overline{1},\overline{2},\overline{3}$ and up to normalization, the state on the right-hand side is just the one of Eq. (96). Let us add to this state the one $(E^{1\overline{1}}+E^{2\overline{2}}+E^{3\overline{3}})\wedge E^7$. This contains a full rank symplectic form of that six-dimensional subspace in complex form. Then, as our basic three-fermion state with seven single-particle states we chose

$$\begin{aligned}\mathcal{P}_0&\equiv\tfrac{1}{2}[E^{123}+E^{\overline{123}}+(E^{1\overline{1}}+E^{2\overline{2}}+E^{3\overline{3}})\wedge E^7]\\ &=e^{123}-e^{156}+e^{246}-e^{345}+e^{147}+e^{257}+e^{367}.\end{aligned} \tag{134}$$

Notice that the structure of our tripartite state $\mathcal{P}_0$ is encoded in the incidence structure of the *lines* of the *oriented* Fano plane which is also encoding the multiplication table of the octonions (see Fig. 1). As a *complex* three-form it can be shown [15,52] that the subgroup Stab($\mathcal{P}_0$) of the SLOCC group GL(7,$\mathbb{C}$) that fixes $\mathcal{P}_0$ is the exceptional group $G_2^{\mathbb{C}}\times\{\omega\mathbf{1}|\omega^3=1\}$ where $\mathbf{1}$ is the 7×7 identity matrix. From the theory of prehomogeneous vector spaces [31,32] it is known that the three-form $\mathcal{P}_0$ can be regarded as the representative of the Zariski-open SLOCC orbit of the prehomogeneous vector space $[\wedge^3 V,\mathrm{GL}(7,\mathbb{C}),\varrho]$. Here, ϱ refers to the representation of $G\equiv\mathrm{GL}(7,\mathbb{C})$ on $W\equiv\wedge^3 V$ of the (22) form, i.e., the one induced by the canonical representation of G on V. The orbit determined by $\mathcal{P}_0$ is dense, meaning its Zariski closure gives the full space $\wedge^3 V$. Notice that $\dim\wedge^3 V=35$, $\dim G=49$, and $\dim\mathrm{Stab}(\mathcal{P}_0)=\dim G_2=14$, hence, $\dim G=\dim W+\dim\mathrm{Stab}(\mathcal{P}_0)$.

A comment here is in order. Rather than using $\mathcal{P}_0$ as an entangled state, in string theory it is used as a *real* differential form on a seven-dimensional real *manifold*. In this context, instead of the complex SLOCC group, the real one, i.e., GL(7,$\mathbb{R}$), is used. The stabilizer of $\mathcal{P}_0$ as a real three-form is the compact real form G_2 which is the automorphism group of the octonions. In the theory of special holonomy manifolds, invariant forms such as $\mathcal{P}_0$ are called calibrations. Note that after the permutation $e^5\leftrightarrow e^7$ we obtain the form for $\mathcal{P}_0$ usually used in the literature on such manifolds [52,53].

The orbit structure of three-fermions with seven single-particle states is available in the mathematical literature [14,41] and summarized in Table II with the ranks of the basic covariants computed. Here, we would like to point out an important fact not mentioned in the literature that the structure

TABLE II. Entanglement classes of three-fermions with seven single-particle states.

Name	Type	Canonical form of $\mathcal{P}$	Rank $\kappa_{\mathcal{P}}^{(1,1)}$	Rank $\mathcal{P}^{(2)}$	Rank $\kappa_{\mathcal{P}}^{(1)}$
I	Null	0	0	0	0
II	Sep	E^{123}	0	3	0
III	Bisep	$E^{1}\wedge(E^{23}+E^{\bar{2}\bar{3}})$	0	5	1
IV	W	$E^{12\bar{3}}+E^{1\bar{2}3}+E^{\bar{1}23}$	0	6	3
V	GHZ	$E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	0	6	6
VI	Sympl3-null	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^{7}$	1	7	1
VII	Sympl3-sep	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^{7}+E^{123}$	1	7	4
VIII	Sympl3-bisep	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^{7}+E^{1}\wedge(E^{23}+E^{\bar{2}\bar{3}})$	2	7	6
IX	Sympl3-W	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^{7}+E^{12\bar{3}}+E^{1\bar{2}3}+E^{\bar{1}23}$	4	7	7
X	Sympl3-GHZ	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^{7}+E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	7	7	7

of these SLOCC classes can elegantly be described using the Fano plane as follows (see Fig. 2).

Class I (null). This is the null class consisting of the trivial zero state.

Class II (sep). This is the class of separable states consisting of a single Slater determinant. As a representative of this class we choose the e^{367} part of the state $\mathcal{P}_0$ of Eq. (134). We can choose a graphical representation for this state as an oriented circle with three distinguished points 3, 6, and 7. Alternatively, after remembering that the numbers 367 can be cyclically permuted without introducing a sign, we can represent this state as an oriented line (673) of the Fano plane starting from the point 6 and ending at the point 3.

Class III (bisep). This is the class of biseparable states. As a representative of this class, we choose the $e^{257}+e^{367}$ part of the state $\mathcal{P}_0$. Graphically, we can refer to this class as two oriented circles touching each other at the point 7. Alternatively, using a cyclic rearrangement, one can depict this class as the two oriented lines 572 and 673 of the Fano plane intersecting at the point 7.

Class IV (W). This is the SLOCC class of W states with representative taken to be the $e^{246}-e^{345}-e^{156}$ part of $\mathcal{P}_0$. Notice that in the notation of Eq. (54) using a cyclic rearrangement, this state is of the $-e^{\dot{1}23}-e^{1\dot{2}3}-e^{12\dot{3}}$ form which is the negative of a state reminiscent of the three-qubit W state, hence the name. For a graphical representation, one can imagine three oriented circles touching each other in the points 4, 5, and 6. Alternatively, one can take the three oriented lines (435), (516), and (624) of the Fano plane forming a clockwise-oriented triangle with its vertices taken as the points 4, 5, and 6.

Class V (GHZ). This is the SLOCC class of GHZ states with representative taken to be the $e^{123}-e^{156}+e^{246}-e^{345}$ part of the state $\mathcal{P}_0$. To see that this state is SLOCC equivalent to the usual two-term GHZ state, just refer to Eq. (133). For a graphical representation, one can envisage the equilateral triangle of class IV with a clockwise-oriented circle inserted in the middle touching the triangle in the three points: 1, 2, and 3. Clearly, the resulting picture is just that part of the Fano plane that we obtain after omitting the three lines intersecting in the point 7.

Class VI (sympl3-null). This class is the one whose representative is based on a symplectic form with rank 3 defined on a six-dimensional subspace of V. The representative we chose is just the one based on the symplectic form $e^{14}+e^{25}+e^{36}$. This gives rise to the $e^{147}+e^{257}+e^{367}$ part of $\mathcal{P}_0$. As a graphical representation for this class we chose the three lines 572, 673, and 471 of the Fano plane intersecting in the point 7. Notice that the arising diagram is just the complement of that part of the Fano plane which represents the GHZ class.

Class VII (sympl3-sep). This class is represented by a full rank symplectic form plus a separable state. The corresponding representative is the $e^{123}+e^{147}+e^{257}+e^{367}$ part of $\mathcal{P}_0$. The graphical picture we can attach to this case is that part of the Fano plane which consists of three lines intersecting in the point 7 and a circle 123. This picture is just the complement of the triangle part of the Fano plane corresponding to the W class.

Class VIII (sympl1-GHZ). This class is represented by the four-term GHZ state plus a term containing a rank-one part from the symplectic form. This means that we keep the following five terms from $\mathcal{P}_0$: $e^{123}-e^{156}+e^{246}-e^{345}+e^{147}$. The resulting diagram is containing five lines of the Fano

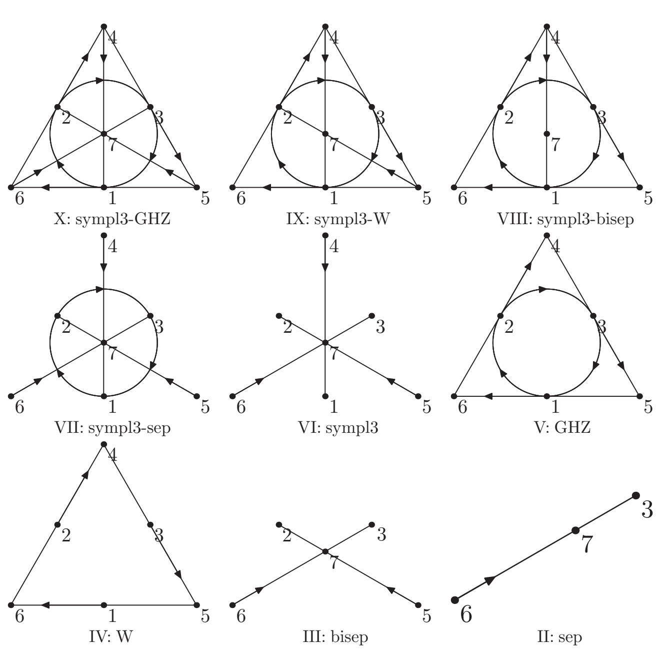

FIG. 2. Graphical representation of the nine entanglement classes of three-fermions with seven single-particle states with the use of the Fano plane.

plane. These are the ones that are the complements of the two lines that show up in the class of biseparable states.

Class IX (sympl2-GHZ). This class is represented by the four-term GHZ state plus two terms containing the rank-two part of the symplectic form. In this case, we keep the following six terms from $\mathcal{P}_0$: $e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{257}$. The corresponding diagram contains six lines of the Fano plane. These are the ones that form the complement of the single line (673) showing up in the separable class.

Class X (sympl3-GHZ). This is the class which corresponds to the Zariski dense orbit in the space of three-forms. It is represented by the state P_0 itself. Clearly, since now we keep all seven Slater determinants: the four comprising the GHZ state and the three giving rise to the full rank symplectic form. The graphical representation of this class is just the Fano plane itself. For the sake of completeness, we mention that this case can be regarded as the complement of the null class represented by the zero state.

Using this graphical representation based on the Fano plane, one can obtain an alternative description of the SLOCC classes. First, observe that one can organize the 10 classes into dual pairs [41]. These pairs are as follows (I,X), (II,IX), (III,VIII), (IV,VII), (V,VI). The pairs exhibit complementary sets of lines of the Fano plane. The five dual pairs can be labeled by the classes I–V that are just the well-known five classes of three-fermions with six single-particle states. Some of the remaining five classes, namely classes VI, VII, and X, can be labeled by a full rank symplectic form (sympl3) plus representatives from the classes I–V (null, sep, GHZ). However, using the finite geometry of the Fano plane, one can easily see that even the exceptional classes, i.e., VIII and IX, can be given this interpretation based on a full rank symplectic form. Indeed, class VIII which is sympl1-GHZ can be reinterpreted as sympl3-bisep; on the other hand, class IX which is sympl2-GHZ can be reinterpreted as sympl3-W. In order to see this, just look at the diagram representing class IX with a representative state having six Slater determinants. Take the triple of lines (624), (354), and (714). Taken together, they form a three-term state $(e^{62} + e^{35} + e^{71}) \wedge e^4$ which is based on a the full rank symplectic form $e^{62} + e^{35} + e^{71}$ in the six-dimensional subspace spanned by $\{e^1, e^2, e^3, e^5, e^6, e^7\}$. Take now the remaining three oriented lines (231), (165), and (572). It is easy to see that in our new six-dimensional subspace, the corresponding states form a W state. Indeed, the oriented triangle graphically representing such a W state has now vertices the points 1, 5, and 2. Taking the permutation (16) (2473), this new W state and symplectic form is transformed back to the one familiar from classes IV and VI. Similar reasoning gives the desired reinterpretation for the class VIII. Now, in this new interpretation apart from the presence of a full rank symplectic form, the extra five classes VI–X are having the same structure as the classes I–V. The upshot of these considerations is summarized in Table II.

C. Eight single-particle states

As usual, define the three-fermion state $P \in \wedge^3 V^*$, $V = \mathbb{C}^8$ as

$$P = \frac{1}{3!} P_{i_1 i_2 i_3} e^{i_1} \wedge e^{i_2} \wedge e^{i_3}, \tag{135}$$

with $\{e^i\}_{i=1}^8$ being a basis of V^*. We have $\mathrm{GL}(V) = \mathrm{GL}(8,\mathbb{C})$ as the SLOCC group with action identical as of Eq. (49).

According to Eqs. (43) and (44) for three-fermions with eight single-particle states, one can define the covariants $\kappa_P{}^{[m,n]}$ with $3 + 2n + m = N = 8$ with the corresponding matrix elements $(K_P{}^{[m,n]})^{a_1 \ldots a_m}{}_{b_1 \ldots b_n}$. We need two such covariants based on

$$(F^a)_{b_1 b_2} \equiv \left(K_P^{[1,2]}\right)^a{}_{b_1 b_2} = \frac{1}{24} \varepsilon^{a i_1 i_2 i_3 i_4 i_5 i_6 i_7} P_{b_1 i_1 i_2} P_{b_2 i_3 i_4} P_{i_5 i_6 i_7} \tag{136}$$

and

$$\left(E^{a_1 a_2 a_3}\right)_b \equiv \left(K_P^{[3,1]}\right)^{a_1 a_2 a_3}{}_b = \frac{1}{12} \varepsilon^{a_1 a_2 a_3 i_1 i_2 i_3 i_4 i_5} P_{b i_1 i_2} P_{i_3 i_4 i_5}. \tag{137}$$

From one of these one can form the 8×8 symmetric matrix $\boldsymbol{G}$ which is of degree six in P and transforming just as N_{AB} of Eq. (107):

$$G_{ab} \equiv (F^c)_{ad} (F^d)_{bc}. \tag{138}$$

Alternatively, one can define an 8×8 symmetric matrix of degree 10 in P as follows:

$$H^{ab} = (F^a)_{ci} (E^{ckl})_j (F^b)_{dk} (E^{dij})_l. \tag{139}$$

This quantity transforms as

$$H^{ab} \mapsto (\mathrm{Det} g')^4 g^a{}_c g^b{}_d H^{cd}. \tag{140}$$

Now, using the matrices $\boldsymbol{G}$ and $\boldsymbol{H}$, one can form the relative invariant of degree 16

$$\mathcal{I}(P) = \mathrm{Tr}(\boldsymbol{GH}) \tag{141}$$

transforming as

$$\mathcal{I}(P) \mapsto [\mathrm{Det}(g')]^6 \mathcal{I}(P). \tag{142}$$

The orbit structure of $\wedge^3 V^*$ is available in the mathematical literature [41]. It turns out that in addition to the 10 classes of the previous section, we have 13 more classes. From the above and considerations of Sec. III, we have the noncontinuous independent invariants: rank$G$, rank$F$=rank$\kappa_P^{(11)}$, rank$\kappa_P^{(1)}$, and rank$P^{(1)}$ so far to classify these. It turns out that this is not sufficient for full classification, we need to use the degree five map $(F \bullet E)^{akl}_{ij} \equiv (F^a)_{ci} (E^{ckl})_j$. The rank of this map is now sufficient for the full SLOCC classification.

The entanglement classes and the corresponding ranks are shown in Table III. The representative states are encoded as

$$\Lambda = \alpha E^{123} + \beta E^{567} + \gamma E^{154} + \delta E^{264} + \epsilon E^{374} + \lambda E^{278} + \mu E^{368}, \tag{143}$$

where as usual $E^{ijk} = e^i \wedge e^j \wedge e^k$. The continuous invariant of Eq. (141) is only nonzero for the class XXIII which is a Zariski-open orbit of the prehomogeneous vector space [31,32] $[\wedge^3 V, \mathrm{GL}(8,\mathbb{C}), \varrho]$. Here, again ϱ is the representation (22) of $\mathrm{GL}(8,\mathbb{C})$ on $\wedge^3 V$.

D. Nine single-particle states

Again, write a three-fermion state as

$$P = \frac{1}{3!} P_{i_1 i_2 i_3} e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \in \wedge^3 V^*, \tag{144}$$

TABLE III. Entanglement classes of three-fermions with eight single-particle states. The classes I,...,X of Table II are omitted here.

Name	α	β	γ	δ	ϵ	λ	μ	Rank G	Rank F	Rank $\kappa_P^{(1)}$	Rank $F \bullet E$
XI	0	0	1	1	1	0	1	0	3	6	0
XII	0	1	1	1	1	0	1	0	4	7	0
XIII	1	1	1	0	0	0	1	0	4	8	0
XIV	1	1	1	1	0	0	1	0	5	8	1
XV	1	1	1	1	1	0	1	0	6	8	2
XVI	0	0	1	0	0	1	1	1	8	8	1
XVII	0	0	1	1	0	1	1	1	8	8	2
XVIII	0	1	1	1	0	1	1	1	8	8	4
XIX	0	0	1	1	1	0	1	2	8	8	2
XX	0	1	1	1	1	1	1	2	8	8	5
XXI	1	1	1	0	0	1	1	3	8	8	7
XXII	1	1	1	1	0	1	1	5	8	8	8
XXIII	1	1	1	1	1	1	1	8	8	8	8

where $\{e^i\}_{i=1}^9$ is a basis of V^* and now $V = \mathbb{C}^9$. The SLOCC group is $\mathrm{GL}(V) = \mathrm{GL}(9,\mathbb{C})$ and the action is still the one of (49).

The orbit structure is available due to the work of Vinberg and Élashvili [21]. It turned out that there are a total of 164 entanglement classes. The classification is based on the unique decomposition of P:

$$P = Q + R, \tag{145}$$

where Q is semisimple and R is nilpotent. States which have closed orbits under the unimodular group $\mathrm{SL}(V)$ are called semisimple and states that have $\mathrm{SL}(V)$ orbits whose closure contains the zero vector are called nilpotent. The 164 orbits can be grouped into seven families according to the type of their semisimple part.

The case of nine dimensions is a bit different from the previously discussed ones. Recall that in the cases discussed so far we had only one relative invariant nonvanishing only on one particular orbit. It follows that these orbits were dense open subsets of the whole three-fermion state space. We call these kinds of orbits stable. The vector spaces admitting a stable orbit when considered as a representation of a particular algebraic group are called prehomogeneous vector spaces [31,32]. In the case of nine dimensions, we have $\dim \wedge^9 V^* = 84$ and $\dim \mathrm{GL}(9) = 81$. It follows that the highest value of the local dimension of an orbit can be at most 81, thus *there are no stable orbits*. It turns out [21] that there are seven orbits with a zero-dimensional stabilizer subgroup, thus with a maximal local dimension of 81, and there is precisely one in every family.

Now recall two standard textbook results:

(1) If ϕ_i, $i = 1,\dots,m$, are differentiable functions on a vector space W, then the existence of a nontrivial relation $\Omega(\phi_1,\dots,\phi_m) = 0$ holding on an open subset U of W implies that the system of gradients $\{\mathrm{grad}\phi_1,\dots,\mathrm{grad}\phi_m\}$ is linearly dependent. Equivalently, if this system is linearly independent, such a relation does not exist. In this latter case, we say these functions are algebraically independent on U (see Proposition 4 of Appendix A).

(2) If a Lie group G acts on W, then there are at most $\dim W - \dim G - \dim \mathrm{Stab}(v)$ algebraically independent invariant differentiable functions satisfying $\phi(v) = \phi(gv)$, $v \in W$, $\forall\, g \in G$ on W. Here, $\mathrm{Stab}(v)$ is the stabilizer of $v \in W$ (see Proposition 5 of Appendix A).

We apply this to $W = \wedge^3 V^*$ and $G = \mathrm{SL}(9,\mathbb{C})$. The 80 dimensional orbits obviously have $\dim \mathrm{Stab} = 0$, thus we have $84 - 80 = 4$ independent invariants w.r.t. the action of the unimodular group $\mathrm{SL}(9,\mathbb{C})$. These are relative invariants w.r.t. the SLOCC group $\mathrm{GL}(9,\mathbb{C})$ picking up determinant factors. Indeed, as shown previously by Vinberg [54] for this particular group and representation, the algebra of invariants is freely generated by four polynomial invariants. They were first found by Egorov [55] with a different method from the one described here. Now, we construct these invariants with the methods described in Sec. III. It turned out that this method was first used by Katanova [56]. Consider the covariant $\kappa_P^{(2,1)}$ with matrix elements

$$\left(K_P^{(2,1)}\right)^{abc}_{def} \equiv T^{abc}_{def} = \frac{1}{2!3!}\epsilon^{abcpqrstu} P_{dep}P_{fqr}P_{stu}. \tag{146}$$

Now, because T has three upper and three lower indices, one can take its powers which will have the same index structure:

$$\begin{aligned}(T^2)^{a_1b_1c_1}_{a_3b_3c_3} &= T^{a_1b_1c_1}_{a_2b_2c_2}T^{a_2b_2c_2}_{a_3b_3c_3},\\ &\vdots\\ (T^m)^{a_1b_1c_1}_{a_{m+1}b_{m+1}c_{m+1}} &= T^{a_1b_1c_1}_{a_2b_2c_2}\dots T^{a_mb_mc_m}_{a_{m+1}b_{m+1}c_{m+1}}.\end{aligned} \tag{147}$$

More strictly speaking, by antisymmetrization of its lower indices, T can be regarded as a linear map $T : \wedge^3 V \to \wedge^3 V$, hence one can compose T with itself and form $T^2 = T \circ T$,...,$T^m = T \circ \dots \circ T$. Now, define a set of relative invariants by

$$\phi_{3n} = \mathrm{Tr}T^n = (T^n)^{abc}_{abc}. \tag{148}$$

The subscript $3n$ denotes the homogeneous degree of ϕ_{3n} in the amplitudes P_{ijk}. Note that $\phi_{3n} = 0$ for n odd and $n = 2$. Let us introduce the notation

$$\begin{aligned}J_{12} &= \frac{1}{2^7\times 3^3\times 7}\phi_{12}, \quad J_{18} = -\frac{1}{2^{10}\times 3^3\times 7\times 13}\phi_{18},\\ J_{24} &= \frac{1}{2^{11}\times 3^2\times 7\times 19}\phi_{24}, \quad J_{30} = -\frac{1}{2^{12}\times 3^3\times 5\times 7\times 13}\phi_{30},\end{aligned} \tag{149}$$

for later convenience. Before proceeding with the review of the seven families and describing the properties of these invariants in each class, we make an important observation.

Proposition 2. The value of any continuous $\mathrm{SL}(V)$ invariant function is independent of the nilpotent part.

Proof. A nilpotent state by definition has the zero state in the closure of its $\mathrm{SL}(V)$ orbit, hence for R nilpotent there exists a sequence $\{s_n\}$ in $\mathrm{SL}(V)$ such that $s_n^* R \to 0$. Let ϕ be a continuous $\mathrm{SL}(V)$ invariant and $P = Q + R$ an arbitrary state. We have

$$\phi(P) = \phi(Q + R) = \phi(s_n^* Q + s_n^* R). \tag{150}$$

Taking the limit and using continuity of ϕ gives $\phi(P) = \phi(Q)$. ■

Now, any semisimple state Q can be brought by an SL(V) transformation to the following form [21]:

$$Q_0 = aq_1 + bq_2 + cq_3 + dq_4, \tag{151}$$

where for simplicity we use the notation

$$\begin{aligned} q_1 &= E^{123} + E^{456} + E^{789}, \quad q_2 = E^{147} + E^{258} + E^{369}, \\ q_3 &= E^{159} + E^{267} + E^{348}, \quad q_4 = E^{168} + E^{249} + E^{357}. \end{aligned} \tag{152}$$

We give explicit expressions for the invariants J_{12}, J_{18}, J_{24}, J_{30} evaluated at Q_0 in Appendix B. Define the matrix

$$M = \begin{pmatrix} \partial_a J_{12} & \partial_a J_{18} & \partial_a J_{24} & \partial_a J_{30} \\ \partial_b J_{12} & \partial_b J_{18} & \partial_b J_{24} & \partial_b J_{30} \\ \partial_c J_{12} & \partial_c J_{18} & \partial_c J_{24} & \partial_c J_{30} \\ \partial_d J_{12} & \partial_d J_{18} & \partial_d J_{24} & \partial_d J_{30} \end{pmatrix}. \tag{153}$$

Now, one can check with any computer algebra system that the determinant of M is a not identically zero-degree 80 polynomial expression in the coefficients a,b,c,d:

$$\begin{aligned} &\frac{1}{2^{14}3^4 5^7 11^2 \times 6 \times 199} \det M \\ &\quad = a^2b^2c^2d^2[(a^3+b^3-c^3)^3+(3abc)^3]^2[(a^3-b^3+d^3)^3 \\ &\qquad +(3abd)^3]^2[(c^3+b^3+d^3)^3-(3cbd)^3]^2 \\ &\qquad \times [(c^3+a^3-d^3)^3+(3cad)^3]^2. \end{aligned} \tag{154}$$

As a consequence, the invariants J_{12}, J_{18}, J_{24}, J_{30} are algebraically independent on any open subset of semisimple states and hence by Proposition 2 on any open subset of $\wedge^3 V$. We note here that the rank of the linear map $T: \wedge^3 V \to \wedge^3 V$ is also a SLOCC invariant, but it is not continuous so Proposition 2 does not apply to it. We now review the seven families of states based on where their semisimple part belongs to.

First family. This family contains only semisimple states with no possible nilpotent part. According to the work of Vinberg and Élashvili [21], the coefficients of the canonical form (151) satisfy

$$\begin{aligned} abcd &\neq 0, \\ (b^3+c^3+d^3)^3-(3bcd)^3 &\neq 0, \\ (a^3+c^3-d^3)^3+(3acd)^3 &\neq 0, \\ (a^3-b^3+d^3)^3+(3abd)^3 &\neq 0, \\ (a^3+b^3-c^3)^3+(3abc)^3 &\neq 0. \end{aligned} \tag{155}$$

Notice that this is equivalent with $\det M \neq 0$. Indeed, this is the only orbit with rank$M = 4$. This orbit has a discrete thus zero-dimensional stabilizer.

Second family. The semisimple part has the canonical form

$$aq_1 - bq_2 + dq_4. \tag{156}$$

Formally, we can obtain this by setting $c = 0$ and $b \to -b$ in (151). The amplitudes satisfy [21]

$$abd(a^3-b^3)(a^3-d^3)(b^3-d^3)[(a^3+b^3+d^3)^3-(3abd)^3] \neq 0. \tag{157}$$

Now, one can check that this is equivalent to dropping the third row of M, setting $c = 0$ and $b \to -b$ in it and requiring any of the 3×3 subdeterminant of the resulting 3×4 matrix to be nonzero. Indeed, we have rank$M = 3$ for these semisimple states. The vanishing of $\det M$ means that it is possible that a function of the four invariants exists which equals zero. It turns out that indeed there exists an invariant of degree 132 which vanishes for this family:

$$\begin{aligned} \Delta_{132} = J_{12}^{11} &- \frac{449\,402\,187\,651\,722\,704\,63}{223\,219\,999\,424\,885\,5116} J_{12}^8 J_{18}^2 + \frac{113\,325\,967\,730\,636\,958\,495\,085\,217}{100\,918\,096\,569\,989\,877\,122\,6274} J_{12}^5 J_{18}^4 \\ &- \frac{115\,188\,459\,017\,686\,510\,39}{329\,340\,982\,758\,027\,804} J_{12}^2 J_{18}^6 - \frac{188\,875}{152\,6823} J_{12}^9 J_{24} + \frac{209\,558\,437\,596\,771\,340\,00}{150\,673\,499\,611\,797\,720\,33} J_{12}^6 J_{18}^2 J_{24} \\ &- \frac{480\,987\,578\,992\,750\,926\,25}{150\,673\,499\,611\,797\,720\,33} J_{12}^3 J_{18}^4 J_{24} + \frac{156\,259\,946\,875}{279\,742\,616\,799\,48} J_{12}^7 J_{24}^2 \\ &- \frac{433\,810\,987\,242\,942\,718\,75}{244\,091\,069\,371\,112\,306\,9346} J_{12}^4 J_{18}^2 J_{24}^2 - \frac{327\,783\,664\,656\,25}{485\,912\,925\,380\,696\,76} J_{12} J_{18}^4 J_{24}^2 \\ &- \frac{373\,398\,260\,937\,50}{327\,991\,224\,631\,970\,313} J_{12}^5 J_{24}^3 - \frac{198\,339\,133\,437\,500}{741\,017\,211\,205\,562\,559} J_{12}^2 J_{18}^2 J_{24}^3 \\ &+ \frac{351\,718\,750\,000}{327\,991\,224\,631\,970\,313} J_{12}^3 J_{24}^4 - \frac{125\,000\,0000}{327\,991\,224\,631\,970\,313} J_{12} J_{24}^5 \\ &+ \frac{522\,717\,082\,571\,600\,510}{502\,244\,998\,705\,992\,4011} J_{12}^7 J_{18} J_{30} - \frac{463\,179\,817\,627\,822\,843\,297\,4860}{454\,131\,434\,564\,954\,447\,051\,8233} J_{12}^4 J_{18}^3 J_{30} \\ &+ \frac{456\,915\,743\,822\,635\,90}{741\,017\,211\,205\,562\,559} J_{12} J_{18}^5 J_{30} - \frac{951\,594\,557\,840\,795\,000}{135\,606\,149\,650\,617\,948\,297} J_{12}^5 J_{18} J_{24} J_{30} \\ &+ \frac{213\,381\,682\,764\,464\,5000}{135\,606\,149\,650\,617\,948\,297} J_{12}^2 J_{18}^3 J_{24} J_{30} + \frac{140\,973\,248\,590\,625\,000}{122\,045\,534\,685\,556\,153\,4673} J_{12}^3 J_{18} J_{24}^2 J_{30} \\ &+ \frac{108\,902\,750\,000\,00}{200\,074\,647\,025\,501\,890\,93} J_{12} J_{18} J_{24}^3 J_{30} - \frac{800\,769\,966\,485\,1700}{452\,020\,498\,835\,393\,160\,99} J_{12}^6 J_{30}^2 \end{aligned}$$

$$+\frac{668\,635\,746\,252\,714\,792\,5300}{151\,377\,144\,854\,984\,815\,683\,9411}J_{12}^3J_{18}^2J_{30}^2+\frac{139\,240\,333\,581\,2500}{135\,606\,149\,650\,617\,948\,297}J_{12}^4J_{24}J_{30}^2$$
$$-\frac{237\,196\,179\,151\,2500}{135\,606\,149\,650\,617\,948\,297}J_{12}J_{18}^2J_{24}J_{30}^2-\frac{216\,716\,472\,500\,000}{122\,045\,534\,685\,556\,153\,4673}J_{12}^2J_{24}^2J_{30}^2$$
$$-\frac{144\,455\,405\,710\,417\,120\,00}{151\,377\,144\,854\,984\,815\,683\,9411}J_{12}^2J_{18}J_{30}^3+\frac{343\,287\,561\,098\,900\,00}{454\,131\,434\,564\,954\,447\,051\,8233}J_{12}J_{30}^4. \quad (158)$$

We have $\Delta_{132}=0$ for the second and $\Delta_{132}\neq 0$ for the first family. There are three types of possible nilpotent parts in this family. These can be found in the work of Vinberg [21]. Semisimple states of this family have a two-dimensional T_2-type stabilizer subgroup. We note here that the second family contains general three-qutrit states via the embedding of Eq. (45). We will discuss this in more detail in Sec. IV D 1.

Third family. The canonical form of the semisimple part is

$$aq_1+dq_4. \quad (159)$$

We can obtain this by setting $b=0$ in the canonical form of the second family. The coefficients satisfy $ad(a^6-d^6)\neq 0$. There are nine types of possible semisimple parts. Semisimple states in this family have four-dimensional stabilizer subgroups of type T_4. We have $\Delta_{132}=0$ in this family. The rank of the matrix M is 2 and as one expects there exists one more function of the invariants which is identically zero in this family. Define an invariant of homogeneous degree 48 by

$$\Delta_{48}=J_{24}^2+\frac{13\times 23^2\times 293}{2^25^4}J_{12}^4+\frac{3^2\times 11\times 127\times 199^2}{2^35^4\times 61}J_{12}J_{18}^2-\frac{257\times 3^2}{5\times 2^3}J_{12}^2J_{24}-\frac{11\times 199^2}{2^25^3\times 61}J_{18}J_{30}. \quad (160)$$

As we will explain soon, Δ_{48} is the generalization of the hyperdeterminant for $3\times 3\times 3$ arrays. We have $\Delta_{48}=0$ for states in the third family but $\Delta_{48}\neq 0$ for the first and the second families.

Fourth family. The canonical form of the semisimple part is

$$aq_1+bq_2-bq_3. \quad (161)$$

Formally, one obtains this by setting $d=0$ and $c=-b$ in (151). The coefficients must satisfy $ab(a^3-b^3)(a^3+8b^3)\neq 0$. The matrix M has rank 2. We have

$$\Delta_{48}=2^2\times 5\times 11^2\times 199^2\times b^9(a^3-b^3)^9(a^4+8ab^3)^3 \quad (162)$$

for this family. The condition $\Delta_{48}\neq 0$ is obviously equivalent with the previous one. Of course, we have $\Delta_{132}=0$ but we have another invariant of degree 48 which vanishes here:

$$\Delta'_{48}=113\times 193J_{12}^4-\frac{11\times 199^2\times 21347}{3^561}J_{12}J_{18}^2+\frac{2\times 5^3\times 257}{3^4}J_{12}^2J_{24}-\frac{2^45^4}{3^6}J_{24}^2+\frac{2^3\times 5\times 11\times 199^2}{3^5\times 61}J_{18}J_{30}. \quad (163)$$

We have $\Delta'_{48}=0$ for the fourth family, but $\Delta'_{48}\neq 0$ for the first, second, and third families. There are six types of possible nilpotent parts. The stabilizer subgroup of semisimple states is of type A_2 and has dimension 8.

Fifth family. The canonical form of the semisimple part is

$$-cq_2+cq_3. \quad (164)$$

This is just the canonical form of the fourth family with $a=0$. We require $c\neq 0$. The matrix M has rank 1. We have $\Delta_{132}=\Delta_{48}=\Delta'_{48}=0$. There are 18 different types of possible nilpotent parts. The stabilizer subgroup is 10 dimensional and of type A_2+T_2 for semisimple states.

Sixth family. The canonical form of the semisimple part is

$$aq_1, \quad (165)$$

with $a\neq 0$. This is just the state (151) with $b=c=d=0$. The matrix M has rank 1 and $\Delta_{132}=\Delta_{48}=\Delta'_{48}=0$. Moreover, it is easy to see that the degree 24 invariant

$$\Delta_{24}=J_{12}^2-\tfrac{1}{111}J_{24} \quad (166)$$

is zero for this family while it is nonzero for families 1–5. There are 25 different types of possible nilpotent parts. The semisimple states in this family have a 24-dimensional stabilizer subgroup of type $3A_2$.

Seventh family. The semisimple part is zero here, thus this family is the family of nilpotent states. By Proposition 2, all the continuous invariants are zero here. There are 102 different types of nilpotent states listed in the work of Vinberg and Élashvili [21]. When considered as states of the nine-dimensional system, all states with a lower-dimensional single-particle Hilbert space discussed in the previous sections are in this family.

A summary of the families and their resolution with the invariants $\Delta_{132},\Delta_{48},\Delta'_{48},\Delta_{24}$ can be found in Table IV.

Recall that every previously discussed case had a stable, "GHZ-type" orbit with nonzero value of an invariant.

TABLE IV. Values of the new continuous invariants on families of three-fermions with nine single-particle states. The last column is the rank of the map $T=\kappa^{(2,1)}$ on semisimple states with zero nilpotent part.

Family	Δ_{132}	Δ_{48}	Δ'_{48}	Δ_{24}	Rank T
First	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	80
Second	0	$\neq 0$	$\neq 0$	$\neq 0$	78
Third	0	0	$\neq 0$	$\neq 0$	76
Fourth	0	$\neq 0$	0	$\neq 0$	72
Fifth	0	0	0	$\neq 0$	70
Sixth	0	0	0	0	56

Intuitively speaking, we have more than one "GHZ-type" orbit in the present case. These are families 1–6 which have at least one invariant with a nonzero value. The nilpotent orbits of the seventh family can be thought of as non-GHZ-type orbits where all invariants vanish. In the first family, there is no possible way of combining zero out of the four invariants. By this property, one can think of the first family as the "most GHZ-type" orbit of the "GHZ-type" orbits.

Before moving on to the discussion of the embedded three-qutrit system, we would like to make an interesting observation. The rank of the linear map $T : \wedge^3 V \to \wedge^3 V$ defined in (146) is just

$$\mathrm{rank} T = 80 - \dim \mathrm{stab}(Q) \qquad (167)$$

for semisimple states Q with zero nilpotent part.

1. Entanglement of three-qutrits

A qutrit is a three-state quantum system with Hilbert space $\mathcal{H} = \mathbb{C}^3$. The Hilbert space of three distinguishable qutrits is just $\mathcal{H}^{\otimes 3} \cong \mathbb{C}^9$. With $\{|1\rangle, |2\rangle, |3\rangle\}$ being a basis of $\mathcal{H}$ a general three-qutrit state can be written as

$$|\psi\rangle = \sum_{\mu_1,\mu_2\mu_3=1}^{3} \psi_{\mu_1\mu_2\mu_3}|\mu_1\mu_2\mu_3\rangle. \qquad (168)$$

The SLOCC group is $\mathrm{GL}(3,\mathbb{C})^{\times 3}$ and it acts on the nine complex amplitudes as

$$\psi_{\mu_1\mu_2\mu_3} \mapsto (S_1)_{\mu_1}^{\nu_1}(S_2)_{\mu_2}^{\nu_2}(S_3)_{\mu_3}^{\nu_3}\psi_{\nu_1\nu_2\nu_3},$$
$$S_1 \otimes S_2 \otimes S_3 \in GL(3,\mathbb{C})^{\times 3}. \qquad (169)$$

The mathematical problem of finding the SLOCC classes was solved by Nurmiev [57,58]. Explicit expressions for the three continuous invariants generating the invariant algebra of this system was found by Briand *et al.* [59] where the problem was also recognized as the problem of SLOCC classification of three-qutrits. Later, Bremner and Hu managed to express the hyperdeterminant [29] of a $3 \times 3 \times 3$ array with these three invariants [60,61]. In the following, we identify the problem of three-qutrit entanglement as a special case of entanglement of three-fermions with nine single-particle states. We relate the invariants I_6, I_9, I_{12} and the hyperdeterminant Δ_{333} of Bremner and Hu with the invariants of Eq. (149).

According to Nurmiev [57,58], any $3 \times 3 \times 3$ array can be uniquely written as the sum of a semisimple and a nilpotent part. Just like in the case of three-fermions, a semisimple state is defined to have a closed $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ orbit while a nilpotent state has the zero vector in the closure of its orbit. Now, any semisimple state can be brought to a so-called normal form

$$|\psi_0\rangle = a|X_1\rangle - b|X_2\rangle + c|X_3\rangle, \qquad (170)$$

where

$$|X_1\rangle = |111\rangle + |222\rangle + |333\rangle,$$
$$|X_2\rangle = |123\rangle + |231\rangle + |312\rangle, \qquad (171)$$
$$|X_3\rangle = |132\rangle + |213\rangle + |321\rangle.$$

There are a total of 43 orbits under the action of $\mathrm{GL}(3,\mathbb{C})^{\times 3}$ and these can be grouped into five families according to the type of their semisimple part. The three fundamental invariants evaluated at the normal form $|\psi_0\rangle$ are [61]

$$I_6 = a^6 + 10a^3b^3 + b^6 - 10a^3c^3 + 10b^3c^3 + c^6,$$
$$I_9 = (a+b)(a-c)(b+c)(a^2 - ab + b^2)(a^2 + ac + c^2) \times (b^2 - bc + c^2),$$
$$I_{12} = -a^9b^3 - 4a^6b^6 - a^3b^9 + a^9c^3 - 2a^6b^3c^3 + 2a^3b^6c^3 - b^9c^3 - 4a^6c^6 - 2a^3b^3c^6 - 4b^6c^6 + a^3c^9 - b^3c^9. \qquad (172)$$

The hyperdeterminant for $3 \times 3 \times 3$ arrays has degree 36 and expressed with these invariants as [61]

$$\Delta_{333} = I_6^3 I_9^2 - I_{12}^2 I_6^2 - 32 I_{12}^3 + 36 I_{12} I_6 I_9^2 + 108 I_9^4. \qquad (173)$$

It has the property that it is zero for all families except the first one. Now, consider the map defined in (45) for $d = k = 3$ and denote it by $\chi : \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \to \wedge^3(\mathbb{C}^9)^*$:

$$\chi : |\psi\rangle \mapsto P_\psi = \sum_{\mu_1,\mu_2,\mu_3=1}^{3} \psi_{\mu_1\mu_2\mu_3} e^{\mu_1} \wedge e^{3+\mu_2} \wedge e^{6+\mu_3}. \qquad (174)$$

Now, it is very easy to check that

$$\chi(|X_1\rangle) = P_{X_1} = q_2, \quad \chi(|X_2\rangle) = P_{X_2} = q_3,$$
$$\chi(|X_3\rangle) = P_{X_3} = q_4, \qquad (175)$$

where $q_1, \ldots, q_4$ are defined in Eq. (152).

Proposition 3. On $\mathrm{Im}\chi \subset \wedge^3 V$, the invariants of (149) can be expressed with the fundamental invariants of three-qutrits as

$$J_{12} = I_6^2 + 20 I_{12},$$
$$J_{18} = I_6^3 + 30 I_{12} I_6 + 100 I_9^2,$$
$$J_{24} = 111 I_6^4 + 4440 I_6^2 I_{12} + 2 \times 3^4 \times 193 I_{12}^2 + 2^2 \times 11 \times 199 I_6 I_9^2,$$
$$J_{30} = 2 \times 3^2 \times 5^2 \times 2521 I_9^2 I_{12} + 3^3 \times 5 \times 2521 I_6 I_{12}^2 + 2 \times 5 \times 17 \times 383 I_6^2 I_9^2 + 2^4 \times 5^2 \times 73 I_6^3 I_{12} + 2^3 \times 73 I_6^5. \qquad (176)$$

Moreover, the invariant Δ_{48} is expressed with the hyperdeterminant as

$$\Delta_{48} = -\frac{5 \times 11^2 \times 199^2}{2} \Delta_{333} I_{12}. \qquad (177)$$

For the other Δ invariants, we have

$$\Delta_{138} = 0,$$
$$\Delta'_{48} = \frac{2^4 5^5 11^2 199^2}{3^5}\left(2^3 I_{12} + \frac{1}{3} I_6^2\right) I_9^4, \qquad (178)$$
$$\Delta_{24} = \frac{2 \times 11 \times 199}{37}\left(I_{12}^2 - \frac{2}{3} I_6 I_9^2\right).$$

Proof. The relations can be checked with any computer algebra system for the image of the normal form P_{ψ_0} which is actually the canonical form (156) of the second family. By invariance,

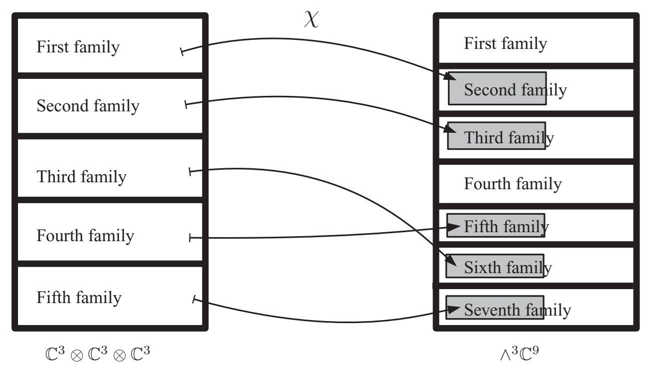

FIG. 3. A sketch showing how the embedding $\chi : \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \to \wedge^3(\mathbb{C}^9)^*$ defined in (174) works. The gray rectangles on the right side represent the image of the three-qutrit families under χ. Different families are mapped into different families.

they are true for any semisimple state. By Proposition 2, they remain true by adding any nilpotent state. ∎

Define

$$D_{36} = \Delta_{333}, \quad D_{24} = I_{12}^2 - \tfrac{2}{3} I_6 I_9^2, \qquad D_{21} = \left(2^3 I_{12} + \tfrac{1}{3} I_6^2\right) I_9. \tag{179}$$

As a consequence of Proposition 3, the invariants D_{36}, D_{24}, D_{21} completely separate the five families of three-qutrits. One can find representatives of these five families, e.g., in the work of Bremner *et al.* [61]. We followed the enumeration of the families used there. The first family has $D_{36} \neq 0$, the second family has $D_{36} = 0, D_{24} \neq 0, D_{21} \neq 0$, the third family has $D_{36} = D_{24} = D_{21} = 0$, and finally the fourth family has $D_{36} = D_{21} = 0, D_{24} \neq 0$. For the nilpotent orbits of the fifth family, every fundamental invariant vanishes. On Fig. 3, we sketched how the embedding χ works. The images of different families are disjoint.

V. PINNING OF OCCUPATION NUMBERS

As a possibly relevant physical application we would like to comment on a connection of the above SLOCC classification of fermionic quantum states with the Klyachko constraints [13] on the eigenvalues of the one-particle reduced density matrix (or one-matrix). These constraints define a polytope in the space of possible eigenvalues of the one-matrix. An important concept is the so-called pinning of occupation numbers which is the saturation of these Klyachko constraints [11,62]. It is widely believed that energy minima of many fermion systems usually do not lie in the Klyachko polytope, thus, the ground state will be on the boundary of the polytope and hence it will be pinned. Indeed, there are both analytical [11] and numerical [63] results that such a pinning occurs in ground states of realistic systems. As shown by Klyachko [62], pinning of a state imposes selection rules on it reducing the number of separable states or Slater determinants that it contains. This is particulary useful in molecular physics since it simplifies the form of the ansatz one must use in variational methods to find the ground state.

Consider first the case of three-fermions with six single-particle states discussed in Sec. IV A. The classical Borland-Dennis result [9] is that if one orders the eigenvalues of the one-matrix as $\lambda_{i+1} \geqslant \lambda_i$, then one has a nontrivial inequality

$$\lambda_5 + \lambda_6 \geqslant \lambda_4. \tag{180}$$

Note that this inequality is independent of the normalization of the original pure state. Now, if (180) is saturated for a state P then it must have the form [11,62]

$$P = \alpha e^1 \wedge e^2 \wedge e^3 + \beta e^1 \wedge e^4 \wedge e^5 + \gamma e^2 \wedge e^4 \wedge e^6 \tag{181}$$

in the basis of natural orbitals. Natural orbitals are the eigenvectors of the one-particle reduced density matrix ρ_P, thus, we have $\rho_P e^i = \lambda_i e^i$. It is clear that transforming an arbitrary state to its natural orbital form amounts to a local unitary transformation, hence, it does not change the SLOCC class of it. Now, if we calculate our covariant $K_P^{[1,1]}$ for the state (181) we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha\gamma & 0 & 0 & 0 & 0 \\ -2\alpha\beta & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{182}$$

This matrix has rank 3, 1, or 0 depending on the value of the coefficients. Looking at Table I, we already conclude that *pinning is impossible for states in the GHZ class* or otherwise stated pinning is impossible for states with $\mathcal{D}(P) = \frac{1}{6}\mathrm{Tr}(K_P^{[1,1]})^2 \neq 0$ [see Eq. (52)]. One might think that this means that all states with $\mathcal{D}(P) = 0$ are pinned, but this is not the case since the spectrum of the one-matrix is not invariant under general SLOCC transformations, thus, pinning is not a SLOCC invariant concept. Indeed, one can easily find both pinned and unpinned states in the W class. Note that these observations are in perfect agreement with the numerical work done by Benavides-Riveros *et al.* [63] where pinning was studied in finite-rank variational approximations of the ground state of lithium. It was observed there that pinning in the rank-six approximation can only occur if the invariant $\mathcal{D}(P)$ is zero.

Consider now the case of seven single-particle states of Sec. IV B. Pinning for this system is investigated by Klyachko as it is important in studying the first excited state of beryllium [62]. Moreover, it is used as the rank-seven approximation of lithium orbitals where pinning was also observed [63]. For seven single-particle states we have four nontrivial Klyachko constraints:

$$\begin{aligned} &\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leqslant 2, \\ &\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leqslant 2, \\ &\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leqslant 2, \\ &\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leqslant 2. \end{aligned} \tag{183}$$

Suppose we saturate the first one: $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 = 2$ for a normalized state $\mathcal{P}$. Then, the arising selection rules imply [62] that $\mathcal{P} \in \wedge^2\mathbb{C}^4 \otimes \mathbb{C}^3 \subset \wedge^3\mathbb{C}^7$. In particular, in the basis of natural orbitals $\mathcal{P}$ must be a linear combination of separable states with two indices from the set {1,2,4,7} and one index from the set {3,5,6}. One can easily calculate the covariants N^{AB} and $(M^{AB})_C$ of Eqs. (103) and (102) for such states and conclude that rank $\kappa_{\mathcal{P}}^{(1,1)} = 4$ and rank $\kappa_{\mathcal{P}}^{(1)} = 7$. Looking at Table II, we already deduce that there is no pinning for states in class X or equivalently for states with a nonvanishing $\mathcal{J}(\mathcal{P})$ invariant [see Eq. (127)].

Now suppose we saturate three [the first two and the last one in Eq. (183)] of the constraints. In this case, $\mathcal{P}$ must have the form [62]

$$\mathcal{P} = \alpha e^1 \wedge e^2 \wedge e^3 + \beta e^1 \wedge e^4 \wedge e^5 + \gamma e^1 \wedge e^6 \wedge e^7 + \delta e^2 \wedge e^4 \wedge e^6, \quad (184)$$

when expanded on its natural orbitals. Calculating the relevant ranks for this state gives rank $\kappa_{\mathcal{P}}^{(1,1)} = 1$ and rank $\kappa_{\mathcal{P}}^{(1)} = 4$ which identifies class VII of Table II. Moreover, one can check that one can not increase the rank of $\kappa_{\mathcal{P}}^{(1)}$ by setting any of the coefficients to zero. This means that states of the form (184) can not be in class V. However, they do cross classes VII, VI, IV, III, II, and I of Table II so we deduce that pinning of three Klyachko constraints is only possible for states in these classes and impossible in classes V, VIII, IX, and X. If we require the saturation of all four constraints, then we have to set $\gamma = 0$ in (184) and we get back to a state of the form (181). Thus, pinning of all four constraints is only possible in a six single-particle subspace and only in the classes I–IV.

VI. CONCLUSIONS

In this work, we presented a method to generate SLOCC covariants and invariants for multifermion systems. Based on results taken from the mathematical literature, we have presented the SLOCC classification of three fermions with six, seven, eight, and nine single-particle states. We also discussed how this classification can be understood with the help of covariants and invariants. In the special cases of six and seven dimensions, we managed to characterize the SLOCC entanglement classes geometrically via mapping the canonical forms of the classes to special subconfigurations of the Fano plane. We have also revealed that in the six-, seven-, and eight-dimensional cases the classes giving rise to stable orbits are examples of prehomogeneous vector spaces [31,32]. For these classes, there is a characteristic relative invariant which is nonvanishing. In all of these cases, these classes are giving rise to dense, Zariski-open orbits with representatives playing a role similar to the classical three-qubit GHZ state. In the cases of six and seven single-particle states, we outlined some connections between the discussed SLOCC classification and the celebrated Klyachko constraints on the spectra of one-particle reduced density matrices. In particular, we observed that saturation (or pinning) of the constraints is not possible in every SLOCC class.

In the case of nine dimensions, there is no stable orbit and there are four algebraically independent polynomial invariants. The SLOCC orbits can be organized into seven families. The seventh family contains nilpotent orbits where all of the four invariants vanish. These can be considered as a generalization of the non-GHZ classes. The rest of the families have at least one invariant with a nonzero value, thus, these can be thought of as a generalization of GHZ-type orbits. We have shown that these families can be distinguished via a calculation of an order 132, two order 48, and an order 24 combination of the fundamental invariants. We have also shown that the entanglement classification of three-qutrits and the corresponding invariant algebra can be recovered from the embedding of the system into the one of three fermions with nine single-particle states. In particular, the $3 \times 3 \times 3$ hyperdeterminant arises as a factorization of one of the invariants of order 48.

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APPENDIX A: PROOF OF SOME THEOREMS

Here, we present some textbook theorems for Sec. IV D and some explicit calculations leading to some of the results of Sec. IV B.

Proposition 4. Let M be an n-dimensional manifold. There are at most n algebraically independent functions on V.

Proof. Let $\phi_i : M \to \mathbb{C}$, $i = 1, \ldots, m$, functions on M. Suppose that there exists $\Omega : \mathbb{C}^m \to \mathbb{C}$ such that

$$\Omega(\phi_1(x), \ldots, \phi_m(x)) = 0 \quad (A1)$$

on every point $x \in U$ of an open subset $U \subset M$. Taking the exterior derivative with coordinates x^a, $a = 1, \ldots, n$, yields

$$d\Omega = \left(\frac{\partial \Omega}{\partial \phi_1} \frac{\partial \phi_1(x)}{\partial x^a} + \ldots + \frac{\partial \Omega}{\partial \phi_m} \frac{\partial \phi_m(x)}{\partial x^a} \right) dx^a = 0, \quad (A2)$$

hence if Ω has nonvanishing derivatives at $\phi_i(x)$, then the system of n component vectors $\{\frac{\partial \phi_1}{\partial x^a}, \ldots, \frac{\partial \phi_m}{\partial x^a}\}$ is linearly dependent. The negation of the above result reads as if the system $\{\frac{\partial \phi_1}{\partial x^a}, \ldots, \frac{\partial \phi_m}{\partial x^a}\}$ is linearly independent on U, then $\Omega(\phi_1, \ldots, \phi_m) = 0$ possible only if Ω is constant zero on U. ■

Proposition 5. Let M be an n-dimensional manifold, G a Lie group with a group action ρ on M, and $\phi : M \to \mathbb{C}$ a differentiable G-invariant function, i.e., $\phi(x) = \phi(\rho(g)x)$, $\forall\, x \in M, \forall\, g \in G$. There are at most $n - \dim G + \dim H_x$ algebraically independent such functions at a point $x \in M$, where H_x is the stabilizer of x.

Proof. Take $g = \exp(\epsilon t)$, $t \in \mathfrak{g}$ in $\phi(x) = \phi(\rho(g)x)$. Then, take the derivative w.r.t ϵ and set $\epsilon = 0$ to obtain

$$d\phi(x)(V_t) = \frac{\partial \phi(x)}{\partial x^a}(V_t)^a = 0, \quad (A3)$$

where $(V_t)^a$ are the components of the tangent vector $V_t = \frac{d}{d\epsilon}\rho[\exp(\epsilon t)]x|_{\epsilon=0} \in T_x M$. Now, since H_x is a subgroup its Lie algebra $\mathfrak{h}_x$ is a linear subspace in $\mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x$ as vector spaces. If $t \in \mathfrak{h}_x$, then the above is automatically satisfied, but if $t \in \mathfrak{m}_x$, then $(V_t)^a \neq 0$ and the above means that $\frac{\partial\phi(x)}{\partial x^a}$ is in the orthogonal complement of the space $M_x = \mathrm{Span}\{V_t | t \in \mathfrak{m}_x\} \subset T_x M$. It is easy to see that $\dim M_x = \dim \mathfrak{m}_x = \dim G - \dim H_x$. Taken together with Proposition 4, the claim follows. ■

Some detailed calculations for Sec. IV B.:

Proposition 6. Let $\tilde{P} = \frac{1}{3!}K_i^c P_{cjk} e^i \wedge e^j \wedge e^k$. Then, $P \wedge \omega = 0$ implies $\tilde{P} \wedge \omega = 0$.

Proof. We know that $\mathrm{Tr}(K) = 0$, hence K is an element of the Lie algebra of SL(6,$\mathbb{C}$). The action of K on the five-form $P \wedge \omega$ is [64]

$$K(P \wedge \omega) = -\tfrac{1}{2}\mathrm{Tr}(K) P \wedge \omega + K^a{}_b e^b \wedge \iota_{e_a}(P \wedge \omega)$$
$$= K^a{}_b e^b \wedge \iota_{e_a} P \wedge \omega - K^a{}_b e^b \wedge P \wedge \iota_{e_a}\omega. \quad \text{(A4)}$$

Now,

$$\iota_{e_a} P \wedge \omega = \tfrac{1}{4} P_{aij}\omega_{kl} e^i \wedge e^j \wedge e^k \wedge e^k, \quad \iota_{e_a}\omega = \omega_{al} e^l \quad \text{(A5)}$$

hence,

$$0 = K(P \wedge \omega)$$
$$= \big(\tfrac{1}{4} K_b^a P_{aij}\omega_{kl} - (K_b^a \omega_{al}) P_{ijk}\big) e^b \wedge e^i \wedge e^j \wedge e^k \wedge e^l. \quad \text{(A6)}$$

According to Eq. (121), $K^a{}_b \omega_{al} = \frac{1}{3}N_{bl} = \frac{1}{3}N_{lb}$. Since N_{bl} is symmetric and $e^b \wedge e^l$ is antisymmetric, the second term gives zero. Hence, the first term which is proportional to $\tilde{P} \wedge \omega$ vanishes as claimed. ■

Proposition 7. If $P \wedge \omega = 0$, then

$$\big(P_{abi}\omega_{jk} + \tfrac{1}{3}\omega_{ab} P_{ijk} + P_{aij}\omega_{bk} - P_{bij}\omega_{ak}\big) e^{ijk} = 0. \quad \text{(A7)}$$

Proof. The result immediately follows from the identity

$$\iota_{e_a}\iota_{e_b}(P \wedge \omega) = \iota_{e_a}\iota_{e_b} P \wedge \omega + \iota_{e_b} P \wedge \iota_{e_a}\omega - \iota_{e_a} P \wedge \iota_{e_b}\omega$$
$$+ P \wedge \iota_{e_a}\iota_{e_b}\omega = 0. \quad \text{(A8)}$$

■

Proposition 8. Let $L^{AB} \equiv (M^A)^C{}_D (M^B)^D{}_C$ and $P \wedge \omega = 0$. Then, we have $L^{77} = 6\mathcal{D}(P)$ and $L^{7a} = L^{a7} = 0$.

Proof. By virtue of Eqs. (119) and (52), we have

$$L^{77} = (M^7)_D^C (M^7)_C^D = (M^7)_d^c (M^7)_c^d$$
$$= K_d^c K_c^d = 6\mathcal{D}(P). \quad \text{(A9)}$$

On the other hand, using Eq. (120) one gets

$$L^{7a} = L^{a7} = (M^7)_d^c (M^a)_c^d = K_d^c \tfrac{1}{2}\varepsilon^{adijkl} P_{cij}\omega_{kl}$$
$$= \tfrac{1}{2}\varepsilon^{adijkl}\tilde{P}_{dij}\omega_{kl}. \quad \text{(A10)}$$

Using Proposition 6, the latter expression is zero. ■

Proposition 9. If $P \wedge \omega = 0$, then $L^{ab} = \frac{3}{2}K^a{}_c \varepsilon^{cbijkl}\omega_{ij}\omega_{kl} = \frac{3}{2}K^b{}_c \varepsilon^{caijkl}\omega_{ij}\omega_{kl}$.

Proof:

$$L^{ab} = (M^a)_d^7 (M^b)_7^d + (M^a)_7^c (M^b)_c^7 + (M^a)_d^c (M^b)_c^d, \quad \text{(A11)}$$

$$(M^a)_d^7 (M^b)_7^d = \tfrac{1}{4}K_d^a \varepsilon^{dbijkl}\omega_{ij}\omega_{kl},$$
$$(M^a)_7^c (M^b)_c^7 = \tfrac{1}{4}K_d^b \varepsilon^{daijkl}\omega_{ij}\omega_{kl}, \quad \text{(A12)}$$

$$(M^a)_d^c (M^b)_c^d = \tfrac{1}{4}\varepsilon^{acijkl}\varepsilon^{bdmnrs} P_{dij} P_{cmn}\omega_{kl}\omega_{rs}. \quad \text{(A13)}$$

Now using Proposition 7, in the last term one can write

$$\varepsilon^{bmndrs} P_{ijd}\omega_{rs}$$
$$= \big(-\tfrac{1}{3}\omega_{ij} P_{drs} + P_{idr}\omega_{js} - P_{jdr}\omega_{is}\big)\varepsilon^{bmndrs}. \quad \text{(A14)}$$

Using this, one can write

$$(M^a)_d^c (M^b)_c^d = K_d^b \varepsilon^{daijkl}\omega_{ij}\omega_{kl}$$
$$+ \tfrac{1}{2}\varepsilon^{acijkl}\varepsilon^{bmndrs}\omega_{kl}\omega_{js} P_{cmn} P_{idr}. \quad \text{(A15)}$$

Now, since in the first Levi-Civitá symbol we have antisymmetry in the indices (c,i) and in the second Levi-Civitá symbol we have symmetry in the pair of indices (mn,dr), the last term is zero. Using the symmetry of G^{ab}, the three different terms of Eq. (A11) give the same type of terms with a prefactor of $\frac{3}{2}$. Notice that using the definition of $\tilde{\omega}$ of Eq. (126) one can write

$$L^{ab} = -12\tilde{\omega}^{ac} K_c^b = -12\tilde{\omega}^{bc} K_c^a. \quad \text{(A16)}$$

This result, taken together with the ones of Proposition 8, yields the factorized form for $\boldsymbol{L}$ of Eq. (125). ■

APPENDIX B: EXPLICIT EXPRESSIONS FOR THE FOUR INDEPENDENT INVARIANTS OF THREE-FERMIONS WITH NINE SINGLE-PARTICLE STATES

Here, we list explicit expressions for the invariants of Eq. (149) of Sec. IV D evaluated on the canonical form (151) of semisimple states

$$J_{12} = a^{12} + b^{12} + c^{12} + 22c^6 d^6 + d^{12} - 220a^3(b^3 - c^3)(b^3 - d^3)(c^3 - d^3) + 220b^3c^3d^3(c^3 + d^3) + 22b^6(c^6 + 10c^3d^3 + d^6)$$
$$+ 22a^6[b^6 + c^6 - 10c^3d^3 + d^6 - 10b^3(c^3 + d^3)], \quad \text{(B1)}$$

$$J_{18} = a^{18} + b^{18} + c^{18} - 17c^{12}d^6 - 17c^6d^{12} + d^{18} + 1870a^9(b^3 - c^3)(b^3 - d^3)(c^3 - d^3)$$
$$- 1870b^9c^3d^3(c^3 + d^3) - 17b^{12}(c^6 + 10c^3d^3 + d^6)$$
$$- 170b^3c^3d^3(c^9 + 11c^6d^3 + 11c^3d^6 + d^9) - 17b^6(c^{12} + 110c^9d^3 + 462c^6d^6 + 110c^3d^9 + d^{12})$$
$$- 17a^{12}[b^6 + c^6 - 10c^3d^3 + d^6 - 10b^3(c^3 + d^3)] - 17a^6[b^{12} + c^{12}$$

$$-110c^9d^3+462c^6d^6-110c^3d^9+d^{12}-110b^9(c^3+d^3)+462b^6(c^6+d^6)-110b^3(c^9+d^9)]+170a^3[b^{12}(c^3-d^3)-11b^9(c^6-d^6)+11b^6(c^9-d^9)+c^3d^3(c^9-11c^6d^3+11c^3d^6-d^9)+b^3(-c^{12}+d^{12})], \quad \text{(B2)}$$

$$J_{24}=111a^{24}+111b^{24}+111c^{24}+506c^{18}d^6+10166c^{12}d^{12}+506c^6d^{18}+111d^{24}-206448a^{15}(b^3-c^3)(b^3-d^3)(c^3-d^3)+206448b^{15}c^3d^3(c^3+d^3)+506b^{18}(c^6+10c^3d^3+d^6)+1118260b^9c^3d^3(c^9+11c^6d^3+11c^3d^6+d^9)+10166b^{12}(c^{12}+110c^9d^3+462c^6d^6+110c^3d^9+d^{12})+1012b^3c^3d^3(5c^{15}+204c^{12}d^3+1105c^9d^6+1105c^6d^9+204c^3d^{12}+5d^{15})+506b^6(c^{18}+408c^{15}d^3+9282c^{12}d^6+24310c^9d^9+9282c^6d^{12}+408c^3d^{15}+d^{18})+506a^{18}[b^6+c^6-10c^3d^3+d^6-10b^3(c^3+d^3)]+10166a^{12}[b^{12}+c^{12}-110c^9d^3+462c^6d^6-110c^3d^9+d^{12}-110b^9(c^3+d^3)+462b^6(c^6+d^6)-110b^3(c^9+d^9)]-1118260a^9[b^{12}(c^3-d^3)-11b^9(c^6-d^6)+11b^6(c^9-d^9)+c^3d^3(c^9-11c^6d^3+11c^3d^6-d^9)+b^3(-c^{12}+d^{12})]+506a^6[b^{18}+c^{18}-408c^{15}d^3+9282c^{12}d^6-24310c^9d^9+9282c^6d^{12}-408c^3d^{15}+d^{18}-408b^{15}(c^3+d^3)+9282b^{12}(c^6+d^6)-24310b^9(c^9+d^9)+9282b^6(c^{12}+d^{12})-408b^3(c^{15}+d^{15})]-1012a^3[5b^{18}(c^3-d^3)-204b^{15}(c^6-d^6)+1105b^{12}(c^9-d^9)-1105b^9(c^{12}-d^{12})+c^3d^3(5c^{15}-204c^{12}d^3+1105c^9d^6-1105c^6d^9+204c^3d^{12}-5d^{15})+204b^6(c^{15}-d^{15})-5b^3(c^{18}-d^{18})], \quad \text{(B3)}$$

$$J_{30}=584a^{30}+584b^{30}+584c^{30}-435c^{24}d^6-63365c^{18}d^{12}-63365c^{12}d^{18}-435c^6d^{24}+584d^{30}+440220a^{21}(b^3-c^3)(b^3-d^3)(c^3-d^3)-440220b^{21}c^3d^3(c^3+d^3)-435b^{24}(c^6+10c^3d^3+d^6)-25852920b^{15}c^3d^3(c^9+11c^6d^3+11c^3d^6+d^9)-63365b^{18}(c^{12}+110c^9d^3+462c^6d^6+110c^3d^9+d^{12})-1394030b^9c^3d^3(5c^{15}+204c^{12}d^3+1105c^9d^6+1105c^6d^9+204c^3d^{12}+5d^{15})-63365b^{12}(c^{18}+408c^{15}d^3+9282c^{12}d^6+24310c^9d^9+9282c^6d^{12}+408c^3d^{15}+d^{18})-290b^3c^3d^3(15c^{21}+1518c^{18}d^3+24035c^{15}d^6+89148c^{12}d^9+89148c^9d^{12}+24035c^6d^{15}+1518c^3d^{18}+15d^{21})-435b^6(c^{24}+1012c^{21}d^3+67298c^{18}d^6+653752c^{15}d^9+1352078c^{12}d^{12}+653752c^9d^{15}+67298c^6d^{18}+1012c^3d^{21}+d^{24})-435a^{24}[b^6+c^6-10c^3d^3+d^6-10b^3(c^3+d^3)]-63365a^{18}[b^{12}+c^{12}-110c^9d^3+462c^6d^6-110c^3d^9+d^{12}-110b^9(c^3+d^3)+462b^6(c^6+d^6)-110b^3(c^9+d^9)]+25852920a^{15}[b^{12}(c^3-d^3)-11b^9(c^6-d^6)+11b^6(c^9-d^9)+c^3d^3(c^9-11c^6d^3+11c^3d^6-d^9)+b^3(-c^{12}+d^{12})]-63365a^{12}[b^{18}+c^{18}-408c^{15}d^3+9282c^{12}d^6-24310c^9d^9+9282c^6d^{12}-408c^3d^{15}+d^{18}-408b^{15}(c^3+d^3)+9282b^{12}(c^6+d^6)-24310b^9(c^9+d^9)+9282b^6(c^{12}+d^{12})-408b^3(c^{15}+d^{15})]+1394030a^9[5b^{18}(c^3-d^3)-204b^{15}(c^6-d^6)+1105b^{12}(c^9-d^9)-1105b^9(c^{12}-d^{12})+c^3d^3(5c^{15}-204c^{12}d^3+1105c^9d^6-1105c^6d^9+204c^3d^{12}-5d^{15})+204b^6(c^{15}-d^{15})-5b^3(c^{18}-d^{18})]-435a^6[b^{24}+c^{24}-1012c^{21}d^3+67298c^{18}d^6-653752c^{15}d^9+1352078c^{12}d^{12}-653752c^9d^{15}+67298c^6d^{18}-1012c^3d^{21}+d^{24}-1012b^{21}(c^3+d^3)+67298b^{18}(c^6+d^6)-653752b^{15}(c^9+d^9)+1352078b^{12}(c^{12}+d^{12})-653752b^9(c^{15}+d^{15})+67298b^6(c^{18}+d^{18})-1012b^3(c^{21}+d^{21})]$$

$$+ 290a^3[15b^{24}(c^3 - d^3) - 1518b^{21}(c^6 - d^6) + 24035b^{18}(c^9 - d^9)$$
$$- 89148b^{15}(c^{12} - d^{12}) + 89148b^{12}(c^{15} - d^{15}) - 24035b^9(c^{18} - d^{18})$$
$$+ c^3d^3(15c^{21} - 1518c^{18}d^3 + 24035c^{15}d^6 - 89148c^{12}d^9 + 89148c^9d^{12}$$
$$- 24035c^6d^{15} + 1518c^3d^{18} - 15d^{21}) + 1518b^6(c^{21} - d^{21}) - 15b^3(c^{24} - d^{24})]. \tag{B4}$$

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