Optimal extraction of information from two spins

Lina Chang
National Laboratory for Science and Technology on Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, People's Republic of China

Nan Li, Shunlong Luo,* and Hongting Song
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

(Received 20 January 2014; published 18 April 2014)

We revisit the issue of extracting information from parallel and antiparallel spins, as initiated by Gisin and Popescu [Phys. Rev. Lett. **83**, 432 (1999)], from the viewpoint of parameter estimation. By comparing two fundamental figures of merit based on fidelity and quantum Fisher information for assessing information quality, we demonstrate that these criteria yield different strategies for optimally extracting spin information. A surprising observation is that while for a single spin, as well as for parallel spins, quantum Fisher information cannot be fully extracted by any uniform measurement, this is not the case for antiparallel spins. A simple uniform measurement fully extracting quantum Fisher information in antiparallel spins is identified. This reveals a significant feature of antiparallel spins from the perspective of Fisher information and provides an alternative illustration of the idea that antiparallel spins carry more information than parallel spins.

DOI: 10.1103/PhysRevA.89.042110 PACS number(s): 03.65.Ta, 03.67.−a

I. INTRODUCTION

Information has to be encoded in physical systems and can be extracted via measurements. With the emergence of quantum information theory, many quantum characteristics for information processing differing from the classical scenario have been revealed [1–5]. A remarkable result in this line is that antiparallel spins may carry more information than parallel spins when the information quality is assessed by fidelity [6–12]. In this work, we will address this information extraction issue from an alternative viewpoint by use of quantum Fisher information.

Consider a spin-1/2 particle, whose state

$$\rho = \frac{1}{2}(\mathbf{1} + \vec{n}\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+n_3 & n_1 - in_2 \\ n_1 + in_2 & 1-n_3 \end{pmatrix}$$

in the Bloch sphere representation takes the vector form $\vec{n} = (n_1,n_2,n_3) \in R^3$. Here σ_j are the Pauli spin operators. It is well known that $|\vec{n}|^2 := \sum_j n_j^2 < 1$ corresponds to mixed states, while $|\vec{n}|^2 = 1$ corresponds to pure states, and in this latter case, we may parametrize the direction vector $\vec{n}$ as $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \theta \in [0,\pi), \phi \in [0,2\pi)$. The corresponding pure state is $\rho = |\vec{n}\rangle\langle\vec{n}|$. The flipped direction of $\vec{n}$ is $-\vec{n}$, and the corresponding flipped (orthogonal) spin state is $|-\vec{n}\rangle$. Spins are natural quantum objects for encoding direction information, and for a single spin, it makes no essential difference for using $|\vec{n}\rangle$ or $|-\vec{n}\rangle$ for many information tasks.

Now consider a two-spin system. It is naive to regard, at least from a mathematical point of view, that the parallel spins $|\vec{n},\vec{n}\rangle$ and the antiparallel spins $|\vec{n},-\vec{n}\rangle$ share similar properties in quantum information tasks. However, since there is no universal spin-flip operation [7,13,14], the parallel spins and antiparallel spins differ substantially from the physical point of view. Then the natural problem arises: Which is better for some particular information task? This line of study was initiated by Gisin and Popescu [7]. Following their seminal work, we will investigate this issue in terms of quantum Fisher information and compare it with the Gisin-Popescu approach based on fidelity. We highlight several interesting features: First, we show that the optimal strategy (in terms of maximum fidelity) for extracting information in parallel spins is not optimal when the quality is evaluated by use of quantum Fisher information. Second, we provide a simple example to illustrate the superiority of antiparallel spins over parallel spins in carrying direction information when the quality is quantified by use of quantum Fisher information, and an optimal measurement is identified. This corroborates the results of Gisin and Massar [7]. Third, we show that although quantum Fisher information encoded in a general single spin cannot be fully extracted by any uniform measurement (in the sense that the measurement does not depend on the state parameters), quantum Fisher information in antiparallel spins can be fully extracted by a uniform measurement.

The work is organized as follows. In Sec. II, we review and elucidate the remarkable results of Gisin and Popescu [7], which show the superiority of antiparallel spins over parallel spins in carrying information within the fidelity framework. In Sec. III, we show that the optimal strategy (in terms of fidelity) for extracting information from parallel spins is not optimal in extracting Fisher information and illustrate the advantage of antiparallel spins in encoding direction information when the quality is quantified by quantum Fisher information. Finally, we conclude with a discussion in Sec. IV.

II. FIDELITY

In this section, to motivate our approach and to supply some missing phase details in Ref. [7], we review and recapitulate the seminal results in Refs. [7,8] for the convenience of comparison with the approach based on quantum Fisher information in Sec. III. For information extraction strategies, we restrict ourselves to von Neumann measurements.

*luosl@amt.ac.cn

Suppose that Alice wants to communicate to Bob the parameter information about the space direction $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ which is encoded in the pure spin state $\rho = |\vec{n}\rangle\langle\vec{n}| = \frac{1}{2}(\mathbf{1} + \vec{n}\vec{\sigma})$. If only one spin is available, then by the symmetric characteristics of the problem and under the condition of no prior knowledge, the optimal strategy for Bob is to measure the spin along any direction with a von Neumann measurement $\{|\vec{m}_k\rangle : k = 0,1\}$. If the result corresponds to k occurs, whose probability is $p_k = |\langle\vec{n}|\vec{m}_k\rangle|^2$, Bob then guesses the direction to be $\vec{m}_k$. The fidelity for such a method is $|\langle\vec{n}|\vec{m}_k\rangle|^2 = \frac{1+\vec{n}\cdot\vec{m}_k}{2}$, and, consequently, the optimal mean fidelity is [7]

$$\bar{F} = \int \sum_{k=0}^{1} p_k \frac{1 + \vec{n}\cdot\vec{m}_k}{2} d\Omega = \frac{2}{3},$$

where $d\Omega = \frac{\sin\theta}{4\pi} d\theta d\phi$ is the uniform probability measure on the unit sphere parameterized by $\theta \in [0,\pi), \phi \in [0,2\pi)$. This result is quite simple and straightforward.

Some interesting and unusual phenomena manifest when two spins, rather than a single spin, are used to encode the information. Now Alice has two natural and simple ways to encode the information, i.e., by sending two parallel spins $|\vec{n},\vec{n}\rangle$, both polarized along $\vec{n}$, or by sending two antiparallel spins $|\vec{n},-\vec{n}\rangle$, with the first spin polarized along $\vec{n}$ but the second polarized in the opposite direction. To simplify notation, let us put

$$\rho_\parallel = |\vec{n},\vec{n}\rangle\langle\vec{n},\vec{n}|, \quad \rho_\perp = |\vec{n},-\vec{n}\rangle\langle\vec{n},-\vec{n}|,$$

which are the corresponding density matrices.

In the framework of maximizing fidelity between the original states and the guessed states after measurements, it was shown by Massar and Popescu that the optimal strategy for extracting information from $\rho_\parallel = |\vec{n},\vec{n}\rangle\langle\vec{n},\vec{n}|$ is via the following von Neumann measurement $\Phi = \{|\Phi_j\rangle : j = 0,1,2,3\}$ with [6,7]

$$|\Phi_j\rangle = \frac{\sqrt{3}}{2}|\vec{n}_j,\vec{n}_j\rangle + \frac{1}{2}|\Psi^-\rangle, \tag{1}$$

where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ denotes the singlet state, and

$$|\vec{n}_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\vec{n}_1\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix},$$
$$|\vec{n}_2\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}e^{i2\pi/3}\end{pmatrix}, \quad |\vec{n}_3\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}-1\\\sqrt{2}e^{i\pi/3}\end{pmatrix}.$$

The corresponding Bloch vectors $\vec{n}_0 = (0,0,1)$, $\vec{n}_1 = \frac{1}{3}(\sqrt{8},0,-1)$, $\vec{n}_2 = -\frac{1}{3}(\sqrt{2},-\sqrt{6},1)$, $\vec{n}_3 = -\frac{1}{3}(\sqrt{2},\sqrt{6},1)$ actually point to the four vertices of a tetrahedron. Note that the phases of $|\vec{n}_j\rangle$ are so chosen such that $|\Phi_j\rangle$ defined by Eq. (1) are mutually orthogonal.

Bob's task is to identify the direction information $\vec{n}$ as well as possible. When he performs the measurement Φ, and if the measurement result corresponding to j occurs, which happens with probability $u_j = |\langle\Phi_j|\vec{n},\vec{n}\rangle|^2$, then he guesses the direction $\vec{n}_j$. The postmeasurement average state is $\Phi(\rho_\parallel) = \sum_{j=0}^{3} u_j|\Phi_j\rangle\langle\Phi_j|$. The mean fidelity after performing the measurement Φ on the parallel spin $\rho_\parallel = |\vec{n},\vec{n}\rangle\langle\vec{n},\vec{n}|$ is [7]

$$\bar{F}(\Phi(\rho_\parallel)) = \int \sum_{j=0}^{3} u_j \frac{1 + \vec{n}\cdot\vec{n}_j}{2} d\Omega = \frac{3}{4},$$

which has been proved to be the optimal fidelity [8].

Next, consider encoding the direction information $\vec{n}$ via the antiparallel spins $\rho_\perp = |\vec{n},-\vec{n}\rangle\langle\vec{n},-\vec{n}|$, and Bob's task is still to extract as much of the direction information $\vec{n}$ as possible from $\rho_\perp = |\vec{n},-\vec{n}\rangle\langle\vec{n},-\vec{n}|$. How much fidelity can he achieve? Can he surpass the above optimal fidelity 3/4 for the parallel spins? Gisin and Popescu constructed a strategy which indeed outperforms the optimal measurement for parallel spins [7]. Before reviewing their results, let us first proceed in a direct and simple way. Analogously to Eq. (1), it seems natural to consider an alternative von Neumann measurement $\Pi = \{|\Pi_j\rangle : j = 0,1,2,3\}$ with

$$|\Pi_j\rangle = \frac{\sqrt{3}}{2}|\Omega_j\rangle + \frac{1}{2}|\Psi^-\rangle, \tag{2}$$

where $|\Omega_j\rangle = (|\vec{n}_j,-\vec{n}_j\rangle + |-\vec{n}_j,\vec{n}_j\rangle)/\sqrt{2}$, and $|\vec{n}_j\rangle$ are as the above, while

$$|-\vec{n}_0\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad |-\vec{n}_1\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix},$$
$$|-\vec{n}_2\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}\sqrt{2}e^{-2i\pi/3}\\-1\end{pmatrix},$$
$$|-\vec{n}_3\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix}\sqrt{2}e^{-i\pi/3}\\1\end{pmatrix},$$

which guarantee that $|\Pi_j\rangle$ defined by Eq. (2) are mutually orthogonal.

The above measurement Π actually coincides with the measurement $\Theta = \{\theta_j\}$ defined by Eq. (4) in Ref. [7], which is shown to be optimal for extracting information from antiparallel spins within the fidelity framework [8]. To establish that $\Pi = \Theta$, first noting that $\frac{1}{\sqrt{2}}(|\vec{n}_j,-\vec{n}_j\rangle - |-\vec{n}_j,\vec{n}_j\rangle) = |\Psi^-\rangle$ for any j (which can be directly checked), it follows that

$$\begin{aligned}|\Pi_j\rangle &= \frac{\sqrt{3}}{2}|\Omega_j\rangle + \frac{1}{2}|\Psi^-\rangle\\ &= \frac{\sqrt{3}}{2\sqrt{2}}(|\vec{n}_j,-\vec{n}_j\rangle + |-\vec{n}_j,\vec{n}_j\rangle)\\ &\quad + \frac{1}{2\sqrt{2}}(|\vec{n}_j,-\vec{n}_j\rangle - |-\vec{n}_j,\vec{n}_j\rangle)\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}|\vec{n}_j,-\vec{n}_j\rangle + \frac{\sqrt{3}-1}{2\sqrt{2}}|-\vec{n}_j,\vec{n}_j\rangle.\end{aligned}$$

On the other hand, the measurement in Eq. (4) of Ref. [7] is given by

$$\theta_j = \alpha|\vec{n}_j,-\vec{n}_j\rangle - \beta\sum_{k\neq j}|\vec{n}_k,-\vec{n}_k\rangle,$$

where $\alpha = \frac{13}{6\sqrt{6}-2\sqrt{2}}, \beta = \frac{5-2\sqrt{3}}{6\sqrt{6}-2\sqrt{2}}$. By direct calculation, we have

$$\frac{1}{2}\sum_{k=0}^{3}|\vec{n}_k,-\vec{n}_k\rangle = |\vec{n}_j,-\vec{n}_j\rangle - |-\vec{n}_j,\vec{n}_j\rangle$$

for $j = 0,1,2,3$. It then follows, after some direct algebraic manipulations, that

$$\begin{aligned}\theta_j &= \alpha|\vec{n}_j, -\vec{n}_j\rangle - \beta\sum_{k\neq j}|\vec{n}_k, -\vec{n}_k\rangle\\ &= (\alpha-\beta)|\vec{n}_j, -\vec{n}_j\rangle + 2\beta|-\vec{n}_j,\vec{n}_j\rangle\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}|\vec{n}_j, -\vec{n}_j\rangle + \frac{\sqrt{3}-1}{2\sqrt{2}}|-\vec{n}_j,\vec{n}_j\rangle,\end{aligned}$$

which coincides with $|\Pi_j\rangle$.

The postmeasurement average state after the measurement Π on $\rho_\perp$ is $\Pi(\rho_\perp) = \sum_{j=0}^{3} t_j|\Pi_j\rangle\langle\Pi_j|$ with $t_j = |\langle\Pi_j|\vec{n}, -\vec{n}\rangle|^2$. The mean fidelity can be evaluated as [7,8]

$$\bar{F}(\Pi(\rho_\perp)) = \int\sum_{j=0}^{3} t_j\frac{1+\vec{n}\cdot\vec{n}_j}{2}d\Omega = \frac{3+\sqrt{3}}{6} \simeq 0.7886,$$

which is larger than the optimal mean fidelity $\bar{F}(\Phi(\rho_\parallel)) = 0.75$ for parallel spins. This means that encoding the direction information in antiparallel spins will yield larger extractable information as long as the quality is quantified by fidelity.

III. QUANTUM FISHER INFORMATION

Now we turn to quantum Fisher information and compare different ways to encode the direction information by parallel and antiparallel spins.

Fisher information arises from statistics [15], quantifies the estimation precision of parameters, as shown by the Cramér-Rao bounds, and determines the asymptotically optimal rate at which neighboring states can be distinguished based on measurements. Its quantum extensions are playing an increasingly important and ubiquitous role in quantum detection and estimation theory [16–30] and, in particular in quantum metrology [31,32].

Recall that quantum Fisher information of parameterized quantum states $\rho = \rho_\gamma$ (we suppress the subscript γ for latter convenience) is defined as [17–19]

$$Q_\gamma(\rho) = \mathrm{tr}\rho L_\gamma^2,$$

with the symmetric logarithmic derivative L_γ determined by $\frac{\partial}{\partial\gamma}\rho = \frac{1}{2}(L_\gamma\rho + \rho L_\gamma)$, and can be evaluated as [20,27]

$$Q_\gamma(\rho) = \sum_{jk}\frac{2}{\lambda_j+\lambda_k}\left|\langle j|\frac{\partial}{\partial\gamma}\rho|k\rangle\right|^2 \tag{3}$$

by means of the spectral decomposition $\rho = \sum_j \lambda_j|j\rangle\langle j|$. Quantum Fisher information sets a fundamental bound to the estimation precision via the celebrated Cramér-Rao inequality. In this work, we will employ quantum Fisher information as a basic concept in information extraction.

If $M = \{|m_j\rangle\}$ is a von Neumann measurement independent of the parameter, then quantum Fisher information of the postmeasurement average state $M(\rho) = \sum_j p_j(\gamma)|m_j\rangle\langle m_j|$ turns out to be the measurement-induced Fisher information [28],

$$Q_\gamma(M(\rho)) = \sum_j p_j(\gamma)\left(\frac{\partial}{\partial\gamma}\ln p_j(\gamma)\right)^2,$$

which is the classical Fisher information of the measurement-induced probability distribution $p_j(\gamma) = \langle m_j|\rho|m_j\rangle$.

For the single spin state $|\vec{n}\rangle\langle\vec{n}|$ (recall that $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$), it can be easily evaluated that the quantum Fisher information of this original state with respect to the parameters θ and ϕ is, respectively,

$$Q_\theta(|\vec{n}\rangle\langle\vec{n}|) = 1, \quad Q_\phi(|\vec{n}\rangle\langle\vec{n}|) = \sin^2\theta.$$

If the von Neumann measurement $M = \{|\vec{m}\rangle, |-\vec{m}\rangle\}$ with $\vec{m} = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0)$ is performed on $|\vec{n}\rangle$, then the quantum Fisher information of the postmeasurement state can be evaluated as

$$\begin{aligned}Q_\theta(M(|\vec{n}\rangle\langle\vec{n}|)) &= 1 - A\sin^2\theta_0\sin^2(\phi-\phi_0),\\ Q_\phi(M(|\vec{n}\rangle\langle\vec{n}|)) &= A\sin^2\theta_0\sin^2(\phi-\phi_0)\sin^2\theta,\end{aligned}$$

where $1/A = 1 - [\cos\theta\cos\theta_0 + \cos(\phi-\phi_0)\sin\theta\sin\theta_0]^2$. From the above expressions, we readily see that if $\theta_0 = 0$, then $Q_\theta(M(|\vec{n}\rangle\langle\vec{n}|)) = 1$ for any ϕ and θ, i.e., such a measurement extracts the full quantum Fisher information about the parameter θ. However, $Q_\phi(M(|\vec{n}\rangle\langle\vec{n}|)) = 0$ for such a measurement, and we cannot gain any information about the phase parameter ϕ. Alternatively, if $\theta_0 = \frac{\pi}{2}$ and $\phi_0 = \phi - \frac{\pi}{2}$, then $Q_\phi(M(|\vec{n}\rangle\langle\vec{n}|)) = \sin^2\theta$ but $Q_\theta(M(|\vec{n}\rangle\langle\vec{n}|)) = 0$. These relations are apparently manifestations of the uncertainty principle. In general, we have the following Fisher information conservation relation:

$$Q_\theta(M(|\vec{n}\rangle\langle\vec{n}|)) + \frac{1}{\sin^2\theta}Q_\phi(M(|\vec{n}\rangle\langle\vec{n}|)) = 1, \tag{4}$$

which in turn implies a simple uncertainty relation,

$$Q_\theta(M(|\vec{n}\rangle\langle\vec{n}|)) + Q_\phi(M(|\vec{n}\rangle\langle\vec{n}|)) \leqslant 1. \tag{5}$$

Equation (4) is special to the current parametrization of spin states and is reminiscent of the Fisher information conservation relation for the Husimi distribution [33]. Inequality (5) is also special to the present setup and differs markedly from the general Cramér-Rao inequality.

For the flipped spin $|-\vec{n}\rangle\langle-\vec{n}|$, we have similar results,

$$Q_\theta(|-\vec{n}\rangle\langle-\vec{n}|) = 1, \quad Q_\phi(|-\vec{n}\rangle\langle-\vec{n}|) = \sin^2\theta,$$

for quantum Fisher information before measurements, and

$$\begin{aligned}Q_\theta(M(|-\vec{n}\rangle\langle-\vec{n}|)) &= 1 - A\sin^2\theta_0\sin^2(\phi-\phi_0),\\ Q_\phi(M(|-\vec{n}\rangle\langle-\vec{n}|)) &= A\sin^2\theta_0\sin^2(\phi-\phi_0)\sin^2\theta,\end{aligned}$$

after the measurement M.

After presenting some aspects for one spin, we move to two spins. In the following two subsections, we will study quantum Fisher information for the postmeasurement average states after performing the measurements Φ and Π on the parallel and antiparallel spins (rather than a single spin) and will reveal new features with respect to information extraction in terms of quantum Fisher information.

A. Parallel spins

For parallel spins $\rho_\parallel = |\vec{n},\vec{n}\rangle\langle\vec{n},\vec{n}|$, the original quantum Fisher information can be readily evaluated, via Eq. (3), as

$$Q_\theta(\rho_\parallel) = 2, \quad Q_\phi(\rho_\parallel) = 2\sin^2\theta. \tag{6}$$

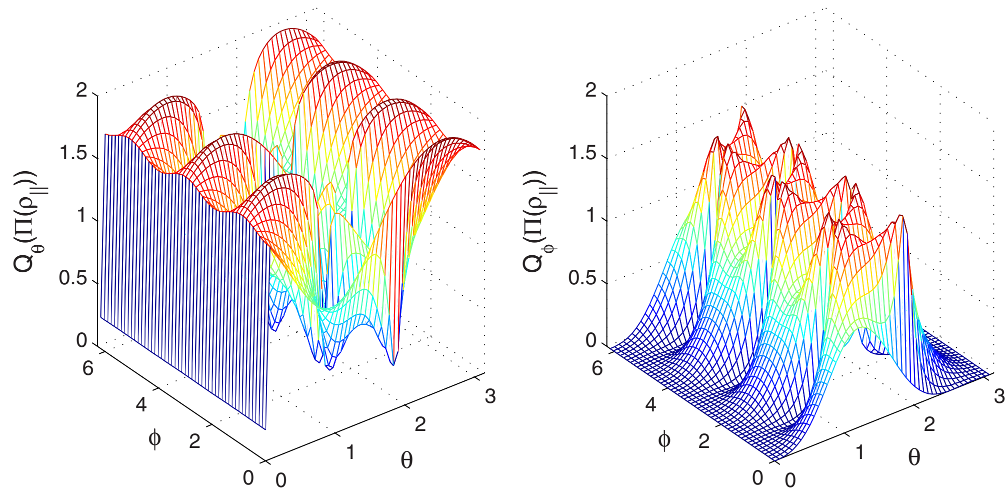

FIG. 1. (Color online) Quantum Fisher information $Q_\theta(\Pi(\rho_{\|}))$ and $Q_\phi(\Pi(\rho_{\|}))$ as functions of $\theta \in [0,\pi),\phi \in [0,2\pi)$. This should be compared with the quantum Fisher information $Q_\theta(\Phi(\rho_{\|})) = 1$ and $Q_\phi(\Phi(\rho_{\|})) = \sin^2\theta$. Their average values (with respect to the uniform probability measure on the Bloch sphere) have the following relations: $\bar{Q}_\theta(\Phi(\rho_{\|})) = 1 < \bar{Q}_\theta(\Pi(\rho_{\|})) \simeq 1.1357$, $\bar{Q}_\phi(\Phi(\rho_{\|})) = \frac{2}{3} = \bar{Q}_\phi(\Pi(\rho_{\|}))$. The measurement Π performs better than the measurement Φ in extracting Fisher information from parallel spins.

Now consider how much information can be extracted from parallel spins. After the fidelity-optimal measurement $\Phi = \{|\Phi_j\rangle\}$, the postmeasurement average state turns out to be $\Phi(\rho_{\|}) = \sum_{j=0}^{3} u_j|\Phi_j\rangle\langle\Phi_j|, u_j = |\langle\Phi_j|\vec{n},\vec{n}\rangle|^2$, with the quantum Fisher information (see the Appendix for detailed calculations) as follows:

$$Q_\theta(\Phi(\rho_{\|})) = 1, \quad Q_\phi(\Phi(\rho_{\|})) = \sin^2\theta.$$

Both are half of the amounts for the original parallel spins in view of Eq. (6). Thus the measurement Φ destroys half of the original quantum Fisher information.

However, after performing the measurement $\Pi = \{|\Pi_j\rangle\}$ defined by Eq. (2) on the parallel spins, the postmeasurement average state is $\Pi(\rho_{\|}) = \sum_{j=0}^{3} v_j|\Pi_j\rangle\langle\Pi_j|$, $v_j = |\langle\Pi_j|\vec{n},\vec{n}\rangle|^2$, whose quantum Fisher information $Q_\theta(\Pi(\rho_{\|}))$ and $Q_\phi(\Pi(\rho_{\|}))$ can be evaluated straightforwardly by use of Eq. (3). Since the expressions are rather complicated and lengthy, we content ourselves by depicting their graphs in Fig. 1, which illustrate clearly their characteristics. This should be compared with $Q_\theta(\Phi(\rho_{\|})) = 1$ and $Q_\phi(\Phi(\rho_{\|})) = \sin^2\theta$.

The maximum and minimum values for $Q_\theta(\Pi(\rho_{\|}))$ are 2 and 0.25, respectively, when θ and ϕ vary: $Q_\theta(\Pi(\rho_{\|})) = 2$ for $\theta = \arcsin(\sqrt{3}/3) = 0.6155$, $\phi = \pi$; $Q_\theta(\Pi(\rho_{\|})) = 0.25$ for $\theta = \arcsin(2\sqrt{2}/3) = 1.2310$, $\phi = \pi$. The maximum and minimum values for $Q_\phi(\Pi(\rho_{\|}))$ are 1.5556 and 0, respectively: $Q_\phi(\Pi(\rho_{\|})) = 1.5556$ for $\theta = \arcsin(2\sqrt{2}/3) = 1.2310$, $\phi = \pi$; $Q_\phi(\Pi(\rho_{\|})) = 0$ for $\theta = 0$ and any ϕ. We see that $Q_\theta(\Pi(\rho_{\|})) > Q_\theta(\Phi(\rho_{\|}))$, at least for some θ. Moreover, when the direction $|\vec{n}\rangle$ is randomly and uniformly distributed over the Bloch sphere, we calculate the mean quantum Fisher information as follows:

$$\bar{Q}_\theta(\Phi(\rho_{\|})) = \int Q_\theta(\Phi(\rho_{\|}))d\Omega = 1,$$

$$\bar{Q}_\theta(\Pi(\rho_{\|})) = \int Q_\theta(\Pi(\rho_{\|}))d\Omega \simeq 1.1357,$$

$$\bar{Q}_\phi(\Phi(\rho_{\|})) = \int Q_\phi(\Phi(\rho_{\|}))d\Omega = \frac{2}{3},$$

$$\bar{Q}_\phi(\Pi(\rho_{\|})) = \int Q_\phi(\Pi(\rho_{\|}))d\Omega = \frac{2}{3}.$$

Accordingly, on average the measurement Π performs better than Φ for estimating the parameter θ, although they are equally good for estimating the phase parameter ϕ. This stands in sharp contrast to the Gisin-Popescu result, which states that the measurement Φ is optimal (and thus is better than the measurement Π) in extracting fidelity information.

B. Antiparallel spins

For the antiparallel spins $\rho_\perp = |\vec{n}, -\vec{n}\rangle\langle\vec{n}, -\vec{n}|$, quantum Fisher information of the original states can be evaluated, via Eq. (3), as

$$Q_\theta(\rho_\perp) = 2, \quad Q_\phi(\rho_\perp) = 2\sin^2\theta. \tag{7}$$

To investigate how much information can be extracted form the antiparallel spins, consider the measurement Φ and Π on the antiparallel spins. The postmeasurement average states are $\Phi(\rho_\perp) = \sum_{j=0}^{3} s_j|\Phi_j\rangle\langle\Phi_j|$ with $s_j = |\langle\Phi_j|\vec{n}, -\vec{n}\rangle|^2$ and $\Pi(\rho_\perp) = \sum_{j=0}^{3} t_j|\Pi_j\rangle\langle\Pi_j|$ with $t_j = |\langle\Pi_j|\vec{n}, -\vec{n}\rangle|^2$, respectively.

Quantum Fisher information $Q_\theta(\Phi(\rho_\perp))$ and $Q_\phi(\Phi(\rho_\perp))$ can be evaluated straightforwardly by use of Eq. (3). We omit the lengthy expressions and depict their graphs in Fig. 2, which display their basic features. The maximum and minimum values for $Q_\theta(\Phi(\rho_\perp))$ are 2 and 0, respectively, when θ and ϕ vary: $Q_\theta(\Phi(\rho_\perp)) = 2$ for $\theta = 2.1860$, $\phi = \pi$; $Q_\theta(\Phi(\rho_\perp)) = 0$ for $\theta = \arcsin(\sqrt{6}/3) = 0.9553$, $\phi = 0$. The maximum and minimum values for $Q_\phi(\Phi(\rho_\perp))$ are 1.5793 and 0, respectively: $Q_\phi(\Phi(\rho_\perp)) = 1.5793$ for $\theta = 1.5269$, $\phi = 0.7861$ or $2\pi - 0.7861 = 5.4971$; $Q_\phi(\Phi(\rho_\perp)) = 0$ for $\theta = 0$ and any ϕ.

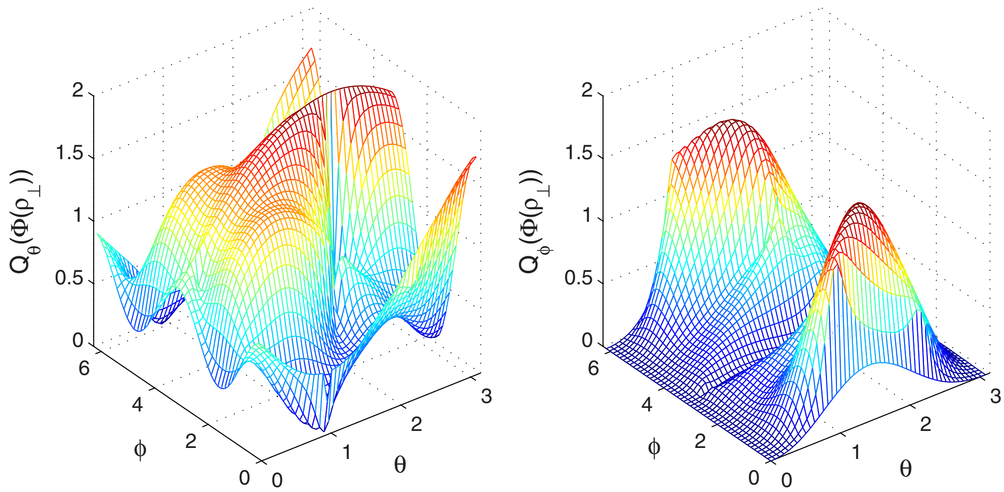

FIG. 2. (Color online) Quantum Fisher information $Q_\theta(\Phi(\rho_\perp))$ and $Q_\phi(\Phi(\rho_\perp))$ as functions of $\theta \in [0,\pi),\phi \in [0,2\pi)$. This should be compared with the quantum Fisher information $Q_\theta(\Pi(\rho_\perp)) = 2$ and $Q_\phi(\Pi(\rho_\perp)) = 2\sin^2\theta$. Their average values (with respect to the uniform probability measure on the Bloch sphere) have the following relations: $\bar{Q}_\theta(\Phi(\rho_\perp)) \simeq 0.9025 < \bar{Q}_\theta(\Pi(\rho_\perp)) = 2, \bar{Q}_\phi(\Phi(\rho_\perp)) \simeq 0.6026 < \bar{Q}_\phi(\Pi(\rho_\perp)) = \frac{4}{3}$. The measurement Π performs better than the measurement Φ in extracting Fisher information from antiparallel spins.

Quantum Fisher information of $\Pi(\rho_\perp)$ can be evaluated as (see the Appendix)

$$Q_\theta(\Pi(\rho_\perp)) = 2, \quad Q_\phi(\Pi(\rho_\perp)) = 2\sin^2\theta,$$

which, surprisingly, turns out to be equal to the original quantum Fisher information, which are maximally possible in view of the Braunstein-Caves relation [20]. Thus Π is an optimal measurement for extracting quantum Fisher information in antiparallel spins, and the upper bounds for both the parameters θ and ϕ are attained simultaneously.

In sharp contrast, there is no measurements (even enlarging the von Neumann measurements to POVMs) which can extract the full quantum Fisher information about the parameters θ and ϕ simultaneously in the parallel spins case, and thus the quantum Fisher information for the parallel spins is not attainable.

To establish this, suppose that, on the contrary, there is a POVM $E = \{E_j\}$ (i.e., E_j are nonnegative operators satisfying $\sum_j E_j = \mathbf{1}$) which extracts the full quantum Fisher information about both the parameters θ and ϕ, then by the results in Refs. [25,26], it holds that

$$E_j^{\frac{1}{2}} L_\theta \rho_{||}^{\frac{1}{2}} = a_{j,\theta,\phi} E_j^{\frac{1}{2}} \rho_{||}^{\frac{1}{2}}, \tag{8}$$

$$E_j^{\frac{1}{2}} L_\phi \rho_{||}^{\frac{1}{2}} = b_{j,\theta,\phi} E_j^{\frac{1}{2}} \rho_{||}^{\frac{1}{2}}. \tag{9}$$

Here $a_{j,\theta,\phi}$ and $b_{j,\theta,\phi}$ are real numbers depending on j,θ and ϕ, while L_θ and L_ϕ are the symmetric logarithm derivatives of $\rho_{||}$ with respect to the parameters θ and ϕ, respectively. From Eqs. (8) and (9), we have

$$\mathrm{tr}\rho_{||} L_\theta E_j L_\phi = a_{j,\theta,\phi} b_{j,\theta,\phi} \mathrm{tr} E_j \rho_{||}.$$

Noting $\sum_j E_j = \mathbf{1}$, we obtain

$$\mathrm{tr}\rho_{||} L_\theta L_\phi = \sum_j a_{j,\theta,\phi} b_{j,\theta,\phi} \mathrm{tr} E_j \rho_{||}.$$

Since $\rho_{||}$ and E_j are non-negative operators, it follows that $\mathrm{tr}E_j\rho_{||}$ is a real number. We conclude that a necessary condition for the existence of an optimal measurement simultaneously extracting the full quantum Fisher information of both the parameters θ and ϕ is that $\mathrm{tr}\rho_{||}L_\theta L_\phi$ must be real for any θ and ϕ. This term (rather than its real part, as is usually taken) may be defined as an off-diagonal element of the quantum Fisher information matrix and quantifies the interference associated with the two parameters from an estimation perspective.

Now by direct calculation, we have

$$\mathrm{tr}\rho_{||} L_\theta L_\phi = -i \sin\theta, \tag{10}$$

which is always imaginary except for $\theta = 0$. Consequently, a measurement (independent of the parameters) that can simultaneously extract the full quantum Fisher information of both the parameters θ and ϕ in the parallel spins case does not exist, that is, a measurement whereby the measurement-induced Fisher information coincides with the quantum Fisher information does not exist. The purely imaginary nature of the term in Eq. (10) may be interpreted as an interference obstacle to the full extraction of the quantum Fisher information. In particular, neither Φ nor Π can extract the full quantum Fisher information in the parallel spins.

In contrast, the antiparallel spins satisfy the necessary condition, since $\mathrm{tr}\rho_\perp L_\theta L_\phi = 0$ indeed is real (here L_θ and L_ϕ are the symmetric logarithmic derivatives of $\rho_\perp$ with respect to the parameters θ and ϕ, respectively). This is consistent with the fact that there exists an optimal measurement (i.e., the measurement Π) that extracts the full quantum Fisher information in the antiparallel spins.

To gain an overall understanding and to make a fair comparison, we evaluate the mean quantum Fisher information as

$$\bar{Q}_\theta(\Phi(\rho_\perp)) = \int Q_\theta(\Phi(\rho_\perp)) d\Omega \simeq 0.9025,$$

$$\bar{Q}_\theta(\Pi(\rho_\perp)) = \int Q_\theta(\Pi(\rho_\perp)) d\Omega = 2,$$

$$\bar{Q}_\phi(\Phi(\rho_\perp)) = \int Q_\phi(\Phi(\rho_\perp))d\Omega \simeq 0.6026,$$

$$\bar{Q}_\phi(\Pi(\rho_\perp)) = \int Q_\phi(\Pi(\rho_\perp))d\Omega = \frac{4}{3}.$$

Finally, we summarize our main results in a table and come to the following observations.

	$\Phi(\rho_\parallel)$	$\Pi(\rho_\parallel)$	$\Phi(\rho_\perp)$	$\Pi(\rho_\perp)$
$\bar{F}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0.7886
$\bar{Q}_\theta$	1	1.1357	0.9025	2
$\bar{Q}_\phi$	$\frac{2}{3}$	$\frac{2}{3}$	0.6026	$\frac{4}{3}$

(a) For parallel spins $\rho_\parallel$, although the measurement Φ extracts more fidelity information than the measurement Π, $\bar{F}(\Phi(\rho_\parallel)) = 3/4 > \bar{F}(\Pi(\rho_\parallel)) = 1/2$, the contrary is true for the criteria based on quantum Fisher information since the measurement Π extracts more quantum Fisher information about the parameter θ than the measurement Φ: $\bar{Q}_\theta(\Phi(\rho_\parallel)) = 1 < \bar{Q}_\theta(\Pi(\rho_\parallel)) \simeq 1.1357$. However, they extract the same amount of quantum Fisher information about the parameter ϕ: $\bar{Q}_\phi(\Phi(\rho_\parallel)) = \bar{Q}_\phi(\Pi(\rho_\parallel)) = 2/3$.

(b) For antiparallel spins $\rho_\perp$, the measurement Π extracts both more fidelity and more quantum Fisher information than Φ. This means that antiparallel spins carry more information than parallel spins for both the criteria based on fidelity and on quantum Fisher information.

(c) There is a simple uniform measurement Π which fully extracts quantum Fisher information encoded in antiparallel spins. This result is quite surprising since this is not the case for parallel spins. Moreover, for a single spin, there is no measurement which can fully extract quantum Fisher information encoded in the original single spin, and quantum Fisher information encoded in antiparallel spins is just twice as quantum Fisher information of a single spin. The underlying reason may lie in the special structure of antiparallel spins and the entanglement in the measurement elements (in fact, each $|\Pi_j\rangle$ is an entangled state) since the antiparallel spins are product states, and any product measurement on antiparallel spins reduces the overall quantum Fisher information in general. This instance provides a significant demonstration of the roles of antiparallel spins and entanglement in encoding and decoding information.

IV. DISCUSSION

We have addressed the issues of encoding and decoding direction information in parallel and antiparallel spins. It turns out that the optimal strategy for extracting information depends on the criteria. We have employed two natural and widely used figures of merits in quantifying the quality of information extraction: one is fidelity and the other is quantum Fisher information. We have shown that for parallel spins, the optimal measurement for fidelity is not optimal for quantum Fisher information. Moreover, the antiparallel spins have advantages in carrying direction information in terms of both the criteria based on fidelity and on quantum Fisher information. We have identified an optimal von Neumann measurement for extracting quantum Fisher information in antiparallel spins. Our results reveal new aspects of information extraction and complement the pioneering results of Gisin and Popescu [7]. We remark that in the present setup, if we employ the Fisher information matrix, the comparison analysis and results are similar, and we omit the details, which are straightforward calculations.

Some interesting problems remain to be investigated. For example, for general multiple spins, what are the optimal strategies for extracting quantum Fisher information? What is the situation if we allow prior knowledge and more general POVMs rather than von Neumann measurements? How can we design general procedures to determine optimal measurements which achieve a certain trade-off of quantum Fisher information for different parameters? How can we quantify the interplay between Fisher-information uncertainty relations and information extraction? How will these results guide future and practical measurement strategies? It will also be interesting to explore the relations between information extraction and quantum Fisher information broadcasting and cloning [29,30].

ACKNOWLEDGMENTS

This work was supported by the Science Fund for Creative Research Groups; the National Natural Science Foundation of China (Grants No. 11375259 and No. 61134008); and the National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences (Grant No. Y029152K51).

APPENDIX

Here we present some detailed calculations of the quantum Fisher information.

First, for computational convenience, we express the parallel and antiparallel spins in the standard base $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ as (up to global phases)

$$|\vec{n},\vec{n}\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ \frac{1}{2}e^{i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta \\ e^{2i\phi}\sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}e^{i\phi}\sin\theta \begin{pmatrix} \frac{1}{t}e^{-i\phi} \\ 1 \\ 1 \\ te^{i\phi} \end{pmatrix},$$

$$|\vec{n},-\vec{n}\rangle = \begin{pmatrix} -\frac{1}{2}\sin\theta \\ e^{i\phi}\cos^2\frac{\theta}{2} \\ -e^{i\phi}\sin^2\frac{\theta}{2} \\ \frac{1}{2}e^{2i\phi}\sin\theta \end{pmatrix} = \frac{1}{2}e^{i\phi}\sin\theta \begin{pmatrix} -e^{-i\phi} \\ \frac{1}{t} \\ -t \\ e^{i\phi} \end{pmatrix},$$

where $t = \mathrm{tg}\frac{\theta}{2}$.

Next, we write down explicitly the measurement vectors for both Φ and Π. The von Neumann measurement $\Phi = \{|\Phi_j\rangle\}$ with mutually orthogonal $|\Phi_j\rangle = \frac{\sqrt{3}}{2}|\vec{n}_j,\vec{n}_j\rangle + \frac{1}{2}|\Psi^-\rangle$

are explicitly given by (in the standard base)

$$|\Phi_0\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{6}\\1\\-1\\0\end{pmatrix}, \quad |\Phi_1\rangle = \frac{-1}{2\sqrt{3}}\begin{pmatrix}1\\ \frac{2-\sqrt{3}}{\sqrt{2}}\\ \frac{2+\sqrt{3}}{\sqrt{2}}\\ 2\end{pmatrix},$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}-1/2\\ \frac{e^{-i\pi/3}+\frac{\sqrt{3}}{2}}{\sqrt{2}}\\ \frac{e^{-i\pi/3}-\frac{\sqrt{3}}{2}}{\sqrt{2}}\\ e^{i\pi/3}\end{pmatrix}, |\Phi_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}-1/2\\ \frac{e^{i\pi/3}+\frac{\sqrt{3}}{2}}{\sqrt{2}}\\ \frac{e^{i\pi/3}-\frac{\sqrt{3}}{2}}{\sqrt{2}}\\ e^{-i\pi/3}\end{pmatrix}.$$

The von Neumann measurement $\Pi = \{|\Pi_j\rangle\}$ with mutually orthogonal $|\Pi_j\rangle$ are explicitly given by

$$|\Pi_0\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}0\\ \sqrt{3}+1\\ \sqrt{3}-1\\ 0\end{pmatrix}, |\Pi_1\rangle = \frac{-1}{\sqrt{3}}\begin{pmatrix}1\\ \frac{1-\sqrt{3}}{2\sqrt{2}}\\ \frac{1+\sqrt{3}}{2\sqrt{2}}\\ -1\end{pmatrix},$$

$$|\Pi_2\rangle = \frac{-1}{\sqrt{3}}\begin{pmatrix}e^{-i2\pi/3}\\ \frac{1-\sqrt{3}}{2\sqrt{2}}\\ \frac{1+\sqrt{3}}{2\sqrt{2}}\\ e^{-i\pi/3}\end{pmatrix}, |\Pi_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}e^{-i\pi/3}\\ \frac{-1+\sqrt{3}}{2\sqrt{2}}\\ \frac{-1-\sqrt{3}}{2\sqrt{2}}\\ e^{-i2\pi/3}\end{pmatrix}.$$

To evaluate the quantum Fisher information $Q_\theta(\Phi(\rho_\parallel))$ and $Q_\phi(\Phi(\rho_\parallel))$ of the parallel spins after the measurement Φ, note that the postmeasurement average state is $\Phi(\rho_\parallel) = \sum_{j=0}^3 u_j|\Phi_j\rangle\langle\Phi_j|$ with $u_j = |\langle\Phi_j|\vec{n},\vec{n}\rangle|^2$. Explicitly, $u_0 = \frac{3}{4}\cos^4\frac{\theta}{2}$ and

$$u_j = \frac{1}{12}\left\{1+\sin^2\frac{\theta}{2}+\sqrt{2}\sin\theta\cos\left[\phi-\frac{2(j-1)\pi}{3}\right]\right\}^2$$

for $j = 1,2,3$. From Eq. (3) and

$$\frac{\partial\Phi(\rho_\parallel)}{\partial\theta} = \sum_{j=0}^{3}\frac{\partial u_j}{\partial\theta}|\Phi_j\rangle\langle\Phi_j|$$

we obtain, by direct manipulation, that the quantum Fisher information of $\Phi(\rho_\parallel)$ with respect to the parameter θ is $Q_\theta(\Phi(\rho_\parallel)) = \sum_{j=0}^3 \frac{1}{u_j}(\frac{\partial u_j}{\partial\theta})^2 = 1$.

Similarly, the quantum Fisher information of $\Phi(\rho_\parallel)$ with respect to the parameter ϕ can be evaluated as $Q_\phi(\Phi(\rho_\parallel)) = \sum_{j=0}^3 \frac{1}{u_j}(\frac{\partial u_j}{\partial\phi})^2 = \sin^2\theta$.

To evaluate the quantum Fisher information $Q_\theta(\Pi(\rho_\parallel))$ and $Q_\phi(\Pi(\rho_\parallel))$ for the parallel spins after the measurement Π, note that the postmeasurement average state after performing the von Neumann measurement Π is $\Pi(\rho_\parallel) = \sum_{j=0}^3 v_j|\Pi_j\rangle\langle\Pi_j|$ with $v_j = |\langle\Pi_j|\vec{n},\vec{n}\rangle|^2$. Explicitly, $v_0 = \frac{3}{8}\sin^2\theta$ and

$$v_j = \frac{3}{8} - \frac{1}{3}\left\{\sin\theta\cos\left[\phi-\frac{(j-1)2\pi}{3}\right]-\frac{\cos\theta}{2\sqrt{2}}\right\}^2$$

for $j = 1,2,3$. It follows from Eq. (3) that quantum Fisher information $Q_\theta(\Pi(\rho_\parallel))$ and $Q_\phi(\Pi(\rho_\parallel))$ can be similarly evaluated. We omit the complicated expressions. The graphs are depicted in Fig. 1.

To evaluate the quantum Fisher information $Q_\theta(\Phi(\rho_\perp))$ and $Q_\phi(\Phi(\rho_\perp))$ for the antiparallel spins after the measurement Φ, note that the postmeasurement average state is $\Phi(\rho_\perp) = \sum_{j=0}^3 s_j|\Phi_j\rangle\langle\Phi_j|$ with $s_j = |\langle\Phi_j|\vec{n},-\vec{n}\rangle|^2$. Explicitly,

$$s_0 = \frac{1}{16}(2+3\sin^2\theta-2\sqrt{6}\sin\theta\cos\phi),$$

$$s_1 = \frac{1}{48}(\sin\theta\cos\phi+2\sqrt{2}\cos\theta-\sqrt{6})^2 + \frac{3}{16}\sin^2\theta\sin^2\phi,$$

$$s_2 = \frac{1}{48}\left[\sin\theta\cos\left(\phi+\frac{\pi}{3}\right)-2\sqrt{2}\cos\theta-\frac{\sqrt{6}}{2}\right]^2 + \frac{3}{16}\left[\sin\theta\sin\left(\phi+\frac{\pi}{3}\right)+\frac{\sqrt{2}}{2}\right]^2,$$

$$s_3 = \frac{1}{48}\left[\sin\theta\cos\left(\phi-\frac{\pi}{3}\right)-2\sqrt{2}\cos\theta-\frac{\sqrt{6}}{2}\right]^2 + \frac{3}{16}\left[\sin\theta\sin\left(\phi-\frac{\pi}{3}\right)-\frac{\sqrt{2}}{2}\right]^2.$$

It follows from Eq. (3) that the quantum Fisher information $Q_\theta(\Phi(\rho_\perp))$ and $Q_\phi(\Phi(\rho_\perp))$ can be directly evaluated. We omit the complicated expressions, and the graphs are depicted in Fig. 2.

To evaluate the quantum Fisher information $Q_\theta(\Pi(\rho_\perp))$ and $Q_\phi(\Pi(\rho_\perp))$ after performing measurement Π on the antiparallel spins, note that the postmeasurement average state is $\Pi(\rho_\perp) = \sum_{j=0}^3 t_j|\Pi_j\rangle\langle\Pi_j|$ with $t_j = |\langle\Pi_j|\vec{n},-\vec{n}\rangle|^2$. Explicitly, $t_0 = \frac{1}{8}(\sqrt{3}\cos\theta+1)^2$ and

$$t_j = \frac{1}{3}\left\{\sin\theta\cos\left[\phi-\frac{(j-1)2\pi}{3}\right]-\frac{\sqrt{2}}{4}\cos\theta+\frac{\sqrt{6}}{4}\right\}^2$$

for $j = 1,2,3$. It follows from Eq. (3) and direct calculation that

$$Q_\theta(\Pi(\rho_\perp)) = 2, \quad Q_\phi(\Pi(\rho_\perp)) = 2\sin^2\theta.$$

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