Reexamination of conservation of momentum and energy in partially coherent electromagnetic waves

Jinhyung Lee and S. M. Kim*

Department of Physics, Yonsei University, Wonju, Kangwon-Do 220-710, Korea

(Received 29 January 2014; published 17 March 2014)

We reviewed the conservation of momentum and energy in partially coherent electromagnetic wave fields in a unified perspective. We find that there is interference between the radiation from polarization and magnetization sources in the momentum flow unlike the result proposed in the previous study. The interference effect can be probed in an example by observing the angular distribution.

DOI: 10.1103/PhysRevA.89.035802 PACS number(s): 42.50.Wk, 37.10.−x

There has been significant progress in basic conservation relations in partially coherent wave fields recently [1–4] after first related studies [5–7]. The conservation laws of energy, momentum, and angular momentum of the wave fields were established and a few related examples were discussed in those studies.

The Maxwell stress tensor [8] has been useful in expressing energy, momentum, and angular momentum and also in dealing with spectral densities of the electromagnetic waves [9] in the studies. A careful review of the tensor corresponding to spectral densities of the electromagnetic waves in the far zone reveals clearer views for the energy, the momentum, and the angular momentum of the waves.

There were a couple of discussions [2,3] on the waves when there are both electric and magnetic sources. It is interesting to see the interference effect in the radiated fields from the sources. In this Brief Report, we review the interference effect on the momentum of the partially coherent electromagnetic waves previously studied in Ref. [2]. We will follow the conventions used in Refs. [2,3] throughout this Brief Report.

For the study of radiated electromagnetic fields, it is often convenient to use the Hertz vectors [10] from sources consisting of polarization and magnetization. For simplicity, we will consider (quasi-)homogeneous sources. The vectors are represented by the spatial integration of the products of source densities and three-dimensional Green's functions for the corresponding Helmholtz equations,

$$\boldsymbol{\pi}^e(\mathbf{r},\omega) = \int_V \mathbf{P}(\mathbf{r}',\omega)G(\mathbf{r}-\mathbf{r}')d^3r', \tag{1}$$

$$\boldsymbol{\pi}^m(\mathbf{r},\omega) = \int_V \mathbf{M}(\mathbf{r}',\omega)G(\mathbf{r}-\mathbf{r}')d^3r', \tag{2}$$

where e and m stand for polarization and magnetization, respectively, and $G(\mathbf{R})$ is the free-space Green's function for the Helmholtz equation,

$$G(\mathbf{R}) = \frac{e^{ikR}}{R}. \tag{3}$$

For further mathematical simplification, we introduce two differential operators as in [2],

$$\mathcal{M}_{ij} \equiv \partial_i\partial_j - \partial^2\delta_{ij}, \tag{4}$$

$$\mathcal{N}_{ij} \equiv ik\epsilon_{ilj}\partial_l, \tag{5}$$

where ∂_i is the partial derivative with respect to the ith Cartesian coordinate, $\partial/\partial x_i$. Here and throughout the remainder of this Brief Report, we will use the Einstein summation convention. By use of the differential operators, we may write the electromagnetic fields in the following forms:

$$E_i = \mathcal{M}_{il}\pi_l^e + \mathcal{N}_{il}\pi_l^m, \tag{6}$$

$$B_i = \mathcal{M}_{il}\pi_l^m - \mathcal{N}_{il}\pi_l^e. \tag{7}$$

In the far zone (the source size is negligible, $r \gg r'$), we may write the position vector as $\mathbf{r} = r\mathbf{u}$, where $\mathbf{u}$ is the unit vector in the direction of observation. Thus the Green function, $G(\mathbf{R})$, can be expressed as

$$G(\mathbf{R}) \approx \frac{e^{ikr}}{r}e^{-ik\mathbf{u}\cdot\mathbf{r}'}. \tag{8}$$

The resulting Hertz vectors can be written as the products of the outgoing waves and the Fourier transforms of the source densities,

$$\pi_i^e(\mathbf{r},\omega) \approx (2\pi)^3\frac{e^{ikr}}{r}\tilde{P}_i(k\mathbf{u},\omega), \tag{9}$$

$$\pi_i^m(\mathbf{r},\omega) \approx (2\pi)^3\frac{e^{ikr}}{r}\tilde{M}_i(k\mathbf{u},\omega), \tag{10}$$

where $\tilde{\mathbf{P}}(k\mathbf{u},\omega)$ and $\tilde{\mathbf{M}}(k\mathbf{u},\omega)$ are the three-dimensional Fourier transforms of the source polarization and the source magnetization, respectively,

$$\tilde{\mathbf{P}}(k\mathbf{u},\omega) \equiv \frac{1}{(2\pi)^3}\int_D \mathbf{P}(\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot\mathbf{r}}d^3r, \tag{11}$$

$$\tilde{\mathbf{M}}(k\mathbf{u},\omega) \equiv \frac{1}{(2\pi)^3}\int_D \mathbf{M}(\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot\mathbf{r}}d^3r. \tag{12}$$

After the applications of the differential operators $\mathcal{M}_{ij}$ and $\mathcal{N}_{ij}$ on the Hertz vectors and their conjugates, we have the

*minntenn@yonsei.ac.kr

following relations:

$$\mathcal{M}_{ij}\pi_j(\mathbf{r},\omega) = k^2(\delta_{ij} - u_i u_j)\pi_j(\mathbf{r},\omega), \tag{13}$$

$$\mathcal{M}^*_{ij}\pi^*_j(\mathbf{r},\omega) = k^2(\delta_{ij} - u_i u_j)\pi^*_j(\mathbf{r},\omega), \tag{14}$$

$$\mathcal{N}_{ij}\pi_j(\mathbf{r},\omega) = -k^2\epsilon_{ilj}u_l\pi_j(\mathbf{r},\omega), \tag{15}$$

$$\mathcal{N}^*_{ij}\pi^*_j(\mathbf{r},\omega) = -k^2\epsilon_{ilj}u_l\pi^*_j(\mathbf{r},\omega), \tag{16}$$

where the last equation is corrected from Eq. (33) of the previously published result [2] as was corrected in [11]. We will see that this correction gives rise to the interference effect between the radiated fields from the polarization and magnetization sources.

From the above relations, we can see that the differential operators are always perpendicular to the vector $\mathbf{u}$ in the far zone from the sources,

$$u_i\mathcal{M}^{(*)}_{ij} \to k^2(u_i\delta_{ij} - u^2 u_j) \to 0, \tag{17}$$

$$u_i\mathcal{N}^{(*)}_{ij} \to -k^2\epsilon_{ilj}u_i u_l \to 0. \tag{18}$$

In other words, the electromagnetic fields in the far zone are always transverse to the propagating wave vector, $\mathbf{k}$.

We now consider the partially coherent electromagnetic fields. In the space-frequency representation, the cross-spectral density tensors of the electric and the magnetic fields are given by

$$W^{EE}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E^*_i(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle_\omega, \tag{19}$$

$$W^{BB}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle B^*_i(\mathbf{r}_1,\omega)B_j(\mathbf{r}_2,\omega)\rangle_\omega, \tag{20}$$

where the bracket $\langle\cdot\rangle_\omega$ represents averaging over the ensemble of space-frequency realization and we will simply write it as $\langle\cdot\rangle$ throughout this Brief Report. By using the cross-spectral density tensors of the Hertz vectors, the cross-spectral density tensors of the electric and the magnetic fields can be written as

$$\begin{aligned} W^{EE}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = {} & \mathcal{M}^{(1)*}_{il}\mathcal{M}^{(2)}_{jm}W^{ee}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & + \mathcal{M}^{(1)*}_{il}\mathcal{N}^{(2)}_{jm}W^{em}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & + \mathcal{N}^{(1)*}_{il}\mathcal{M}^{(2)}_{jm}W^{me}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & + \mathcal{N}^{(1)*}_{il}\mathcal{N}^{(2)}_{jm}W^{mm}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega), \end{aligned} \tag{21}$$

$$\begin{aligned} W^{BB}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = {} & \mathcal{M}^{(1)*}_{il}\mathcal{M}^{(2)}_{jm}W^{mm}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & - \mathcal{M}^{(1)*}_{il}\mathcal{N}^{(2)}_{jm}W^{me}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & - \mathcal{N}^{(1)*}_{il}\mathcal{M}^{(2)}_{jm}W^{em}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ & + \mathcal{N}^{(1)*}_{il}\mathcal{N}^{(2)}_{jm}W^{ee}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega), \end{aligned} \tag{22}$$

where the superscript indices of $\mathcal{M}^{(q)}_{ij}$ and $\mathcal{N}^{(q)}_{ij}$ indicate the operators acting only on the corresponding variables, $\mathbf{r}_q$, of the cross-spectral densities W. Also, W^{ab}_{lm} is defined by

$$W^{ab}_{lm}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \langle\pi^{a*}_l(\mathbf{r}_1,\omega)\pi^b_m(\mathbf{r}_2,\omega)\rangle \tag{23}$$

for all possible combinations of $a = e,m$ and $b = e,m$.

For monochromatic fields, after taking $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, the Maxwell stress tensor can be written as

$$\begin{aligned} T_{ij}(\mathbf{r},\omega) = \frac{1}{4\pi}\Big\{ & E^*_i(\mathbf{r},\omega)E_j(\mathbf{r},\omega) + B^*_i(\mathbf{r},\omega)B_j(\mathbf{r},\omega) \\ & -\frac{1}{2}\delta_{ij}[E^*_l(\mathbf{r},\omega)E_l(\mathbf{r},\omega) + B^*_l(\mathbf{r},\omega)B_l(\mathbf{r},\omega)]\Big\}. \end{aligned} \tag{24}$$

Trace of the stress tensor yields the negative total energy density $U(\mathbf{r},\omega)$,

$$\begin{aligned} T_{ii}(\mathbf{r},\omega) &= -\frac{1}{8\pi}[E^*_i(\mathbf{r},\omega)E_i(\mathbf{r},\omega) + B^*_i(\mathbf{r},\omega)B_i(\mathbf{r},\omega)] \\ &= -U(\mathbf{r},\omega). \end{aligned} \tag{25}$$

Therefore, for partially coherent fields, the ensemble average of the stress tensor becomes

$$\begin{aligned} \langle T_{ij}\rangle &= \frac{1}{4\pi}\Big\{W^{EE}_{ij}(\mathbf{r},\omega) + W^{BB}_{ij}(\mathbf{r},\omega) \\ &\quad -\frac{1}{2}\delta_{ij}\big[W^{EE}_{kk}(\mathbf{r},\omega) + W^{BB}_{kk}(\mathbf{r},\omega)\big]\Big\} \\ &= \frac{1}{4\pi}\Big[W^{EE}_{ij}(\mathbf{r},\omega) + W^{BB}_{ij}(\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}\langle U(\mathbf{r},\omega)\rangle\Big], \end{aligned} \tag{26}$$

where $\langle U(\mathbf{r},\omega)\rangle$ is the ensemble average of the energy density. Now we apply the far-field approximation to $\langle U(\mathbf{r},\omega)\rangle$. In the far zone from the sources, after the differential operations on the Hertz vectors, we have

$$\begin{aligned} \mathcal{M}^{(1)*}_{kl}\mathcal{M}^{(2)}_{km} &\to k^4(\delta_{kl} - u_k u_l)(\delta_{km} - u_k u_m) \\ &= k^4(\delta_{lm} - u_l u_m), \end{aligned} \tag{27}$$

$$\begin{aligned} \mathcal{M}^{(1)*}_{kl}\mathcal{N}^{(2)}_{km} &\to -k^4(\delta_{kl} - u_k u_l)\epsilon_{knm}u_n \\ &= -k^4\epsilon_{lnm}u_n, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathcal{N}^{(1)*}_{kl}\mathcal{M}^{(2)}_{km} &\to -k^4\epsilon_{knl}u_n(\delta_{km} - u_k u_m) \\ &= -k^4\epsilon_{mnl}u_n, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{N}^{(1)*}_{kl}\mathcal{N}^{(2)}_{km} &\to k^4\epsilon_{knl}u_n\epsilon_{kpm}u_p \\ &= k^4(\delta_{lm} - u_l u_m), \end{aligned} \tag{30}$$

where we set $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ during the calculations. By using the evaluations of the operators, $\langle U(\mathbf{r},\omega)\rangle$ can be simplified as

$$\begin{aligned} \langle U(\mathbf{r},\omega)\rangle = \frac{k^4}{4\pi}\big\{ & (\delta_{ij} - u_i u_j)\big[W^{ee}_{ij}(\mathbf{r},\omega) + W^{mm}_{ij}(\mathbf{r},\omega)\big] \\ & + \epsilon_{jni}u_n W^{em}_{ij}(\mathbf{r},\omega) + \epsilon_{inj}u_n W^{me}_{ij}(\mathbf{r},\omega)\big\}. \end{aligned} \tag{31}$$

As discussed in Ref. [1], momentum flow may be written as a normal component of a unit vector, $\mathbf{u}$, on a sphere whose radius is much larger than the size of a localized source at the center. The momentum flow becomes

$$u_i\langle T_{ij}(\mathbf{r},\omega)\rangle = \frac{1}{4\pi}u_i\left[W_{ij}^{EE}(\mathbf{r},\omega)+W_{ij}^{BB}(\mathbf{r},\omega)\right] - \frac{1}{8\pi}u_j\langle U(\mathbf{r},\omega)\rangle. \quad (32)$$

Since the electromagnetic field produced by the polarization and the magnetization is always transverse on the sphere in the far zone, the normal components of the cross-spectral density tensors of the fields always vanish in the far zone. As a result of this, the momentum flow can be reduced to

$$u_i\langle T_{ij}(\mathbf{r},\omega)\rangle = -u_j\langle U(\mathbf{r},\omega)\rangle, \quad (33)$$

which means that the momentum flow is simply the energy density transfer to the normal direction in the far zone. It is straightforward to show that the momentum flow in the far zone has the following form by using the spectral energy density:

$$\begin{aligned}u_i\langle T_{ij}(\mathbf{r},\omega)\rangle = -\frac{(2\pi)^6k^4}{4\pi r^2}\big\{&u_j(\delta_{lm}-u_lu_m)\\ &\times\left[\tilde{W}_{lm}^{PP}(-k\mathbf{u},k\mathbf{u})+\tilde{W}_{lm}^{MM}(-k\mathbf{u},k\mathbf{u})\right]\\ &+u_j\epsilon_{lmn}u_m\big[\tilde{W}_{nl}^{PM}(-k\mathbf{u},k\mathbf{u})\\ &+\tilde{W}_{ln}^{MP}(-k\mathbf{u},k\mathbf{u})\big]\big\},\end{aligned} \quad (34)$$

where $\tilde{W}_{ij}^{XY}$ is the sixfold Fourier transform of a cross-spectral density of a combination of the polarization and the magnetization,

$$\tilde{W}_{ij}^{XY}(\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{(2\pi)^6}\int d^3r_1d^3r_2e^{i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)} \times\langle X^*(\mathbf{r}_1,\omega)Y(\mathbf{r}_2,\omega)\rangle, \quad (35)$$

where X and Y can be P or M. This result is exactly the same as that of the result in Ref. [3]. The first term in this formula is exactly the same as the result in Eq. (26) in Ref. [1], which represents the momentum flow only from the polarization source, while the second term is only from the magnetization source. The last two terms are produced from the interference of the polarization and the magnetization sources. In general, the last term does not vanish unlike the result in Ref. [2]. Therefore, this result corrects the previously published result in Ref. [2], while the result in Ref. [3] still remains valid.

To elucidate the result, we consider a simple example. We assume that a polarization $\mathbf{P} = P\hat{z}\delta(\mathbf{r}-\mathbf{r}_P)$ and a magnetization $\mathbf{M} = M\hat{x}\delta(\mathbf{r}-\mathbf{r}_M)$ are located at $\mathbf{r}_P = (0,0,a/2)$ and $\mathbf{r}_M = (a/2,0,0)$, respectively. There is no special reason for this choice as an example. The purpose is only to show the interference between the polarization and magnetization.

For this example, it is useful to use the Fourier transforms of the cross spectral densities,

$$\tilde{W}_{ij}^{PP}(-\mathbf{k},\mathbf{k}) = P_i^*P_j = P^2\delta_{i3}\delta_{j3}, \quad (36)$$

$$\tilde{W}_{ij}^{MM}(-\mathbf{k},\mathbf{k}) = M_i^*M_j = M^2\delta_{i1}\delta_{j1}, \quad (37)$$

$$\begin{aligned}\tilde{W}_{ij}^{PM}(-\mathbf{k},\mathbf{k}) &= \mu P_i^*M_je^{-ik\mathbf{u}\cdot(\mathbf{r}_P-\mathbf{r}_M)}\\ &= PMe^{ika(\sin\theta\cos\phi-\cos\phi)/2}\delta_{i3}\delta_{j1},\end{aligned} \quad (38)$$

$$\tilde{W}_{ji}^{MP}(-\mathbf{k},\mathbf{k}) = \tilde{W}_{ij}^{PM*}(-\mathbf{k},\mathbf{k}), \quad (39)$$

for real P, M, and μ. The momentum flow in Eq. (34) can be expressed as a product of the angular part and the nonangular part;

$$u_i\langle T_{ij}(\mathbf{r},\omega)\rangle = -u_j\frac{(2\pi)^6k^4}{4\pi r^2}H(\mathbf{u}), \quad (40)$$

where the angular function $H(\mathbf{u})$ can be written as

$$\begin{aligned}H(\mathbf{u}) = &P^2\sin^2\theta+M^2(1-\sin^2\theta\cos^2\phi)\\ &+2\mu PM\cos[ka(\cos\theta-\sin\theta\cos\phi)/2]\\ &\times\sin\theta\sin\phi.\end{aligned} \quad (41)$$

The angular distribution of the momentum flow depends not only on the strengths of polarization and magnetization, P and M, but also on the degree of coherence between them, μ.

The magnitude of the radiation from magnetization is relatively small compared to that from polarization in most physical radiation sources. We take $M = 0.1P$ to reflect this. Then the contribution from the second term in Eq. (41) is much smaller than that from the first term. However, the third term gives non-negligible interference effects since the magnitude is about 20% of the radiation from polarization. Specifically, the interference effect produces an azimuthal dependency to the angular distribution in this example. The resulting angular distribution of the momentum flow at $\theta = \pi/2$ is seen in Fig. 1 for various values of μ with $ka = 2\pi$. The interference effect can be most notable for the non-negligible fluctuation in azimuthal direction as shown in the figure.

The momentum flow from the polarization, which is the first term in Eq. (41), has no azimuthal variation. When we take a small magnetization ($M = 0.1P$), the magnetization effect [the second terms in Eq. (41)] makes almost no azimuthal variation to the momentum flow by the polarization. As seen in Fig. 1, the momentum flow of completely incoherent

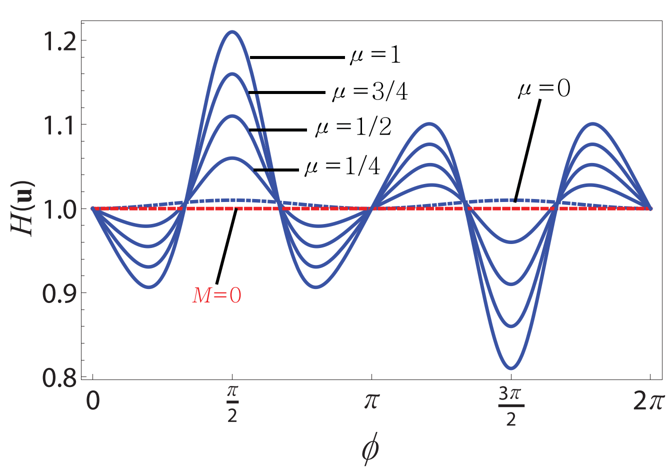

FIG. 1. (Color online) The angular distribution of momentum flow for the source with both polarization and magnetization as a function of ϕ for $P = 1$, $ka = 2\pi$, and $\theta = \pi/2$.

polarization and magnetization sources ($\mu = 0$) has almost no azimuthal variation. Therefore, the presence of the azimuthal variation of the momentum flow indicates the interference between both sources [the third term in Eq. (41)]. We can also use this example to determine the degree of coherence between the polarization and the magnetization.

We investigated the momentum flow and the energy density of the partially coherent electromagnetic fields produced by (quasi-)monochromatic electric and magnetic sources. The electromagnetic fields from a localized source with both polarization and magnetization in the far zone are always transverse to the outgoing normal vector. The momentum flow is the energy density transfer to the outgoing normal directions. We found the interference effect in the radiations from the electric and the magnetic sources in disagreement with what was found in the previous study [2]. This effect can be probed in a special example most notably by observing the presence of an azimuthal dependence.

We thank to Dr. Gbur for useful discussions. This research was supported in part by Basic Science Research Program through the NRF of Korea funded by MEST (Grant No. 2011-0014308) and in part by Yonsei University Research Fund of 2013.

[1] S. M. Kim and G. Gbur, Phys. Rev. A **79**, 033844 (2009).
[2] G. Gbur and S. M. Kim, Phys. Rev. A **82**, 043807 (2010).
[3] O. Bak and S. M. Kim, Phys. Rev. A **82**, 063819 (2010).
[4] S. M. Kim and G. Gbur, Phys. Rev. A **86**, 043814 (2012).
[5] P. Roman and E. Wolf, Nuovo Cimento **17**, 477 (1960).
[6] G. S. Agarwal and E. Wolf, Phys. Rev. A **54**, 4424 (1996).
[7] G. Gbur, D. F. V. James, and E. Wolf, Phys. Rev. E **59**, 4594 (1999).
[8] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Sec. 6.8.
[9] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Chap. 4.
[10] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, 2006), Sec. 2.2.
[11] G. Gbur and S. M. Kim, Phys. Rev. A **89**, 039903(E) (2014).