Vector solitons of a Bose-Einstein condensate in a dynamical trap

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Evolution equations are presented and discussed for an atomic Bose-Einstein condensate in a dynamical trap formed by wide oscillating barriers. Their analytical solution reveals the existence of families of bright spatial vector solitons whose components can be in phase and antiphase.

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I. INTRODUCTION

Solitons of different physical nature fall into two broad classes: conservative and dissipative, the latter being the result of balance between the input and output of energy in the region of localization [1]. The features of the solitons of these two classes are radically different: The conservative solitons form families with a continuous spectrum of the main parameters, whereas for dissipative solitons this spectrum is discrete. In nonlinear optics, the dissipative solitons are known mainly as cavity solitons [2–5]. Using atomic Bose-Einstein condensates (BECs) [6], conservative bright [7–9] and dark [10–12] solitons of matter waves with nonlinear longitudinal localization and dissipative (cavity) solitons of light-matter waves [13] with nonlinear transverse localization were demonstrated experimentally.

In this paper, following the approach reported in [14,15], we consider nonlinearly localized structures (solitons) of matter waves with a different mechanism of energy input and output: due to collisions of atoms with oscillating cavity mirrors. In other words, the scheme is an atomic BEC in a dynamical trap with oscillating barriers. The barriers can be organized as optically induced movable mirrors [16]. Such a trap provides simultaneous localization and excitation of matter waves. Our simulations [17] show that in a one-dimensional geometry with linear transverse localization due to the trap potential, the barriers do not destroy the nonlinear longitudinal localization; however, oscillations of barriers can induce not only periodic or quasiperiodic motion, but also chaotic longitudinal motion of solitons similar to that of classical particles in the Fermi-Ulam model [18,19]. In the simplest case, these solitons are scalar (one component) and are Schrödinger-type solitons [20,21]. Below, a different case of transverse nonlinear localization under resonance conditions is considered [14,15]. Then the solitons are not of the Schrödinger type because they are described by two coupled nonlinear partial differential equations for amplitudes of two resonance levels; therefore, we have the case of vector solitons. The scheme resembles that of Faraday [22] and Unbanhowar *et al.* [23], where solitons known as oscillons were observed.

As indicated above, the terms "dissipative" and "cavity" solitons are widely used for spatial optical solitons where they were first demonstrated in simulations for wide-aperture-driven nonlinear interferometers [24] and lasers with saturable absorption [25]. However, a specific feature of BEC solitons in a dynamical trap is that, depending on the phase of barriers' oscillations, atoms can gain or lose their kinetic energy and the energy is nearly conserved when averaged over the oscillation period. Therefore, it is possible to suggest that these solitons take an intermediate position between conservative and dissipative ones. An additional argument in favor of this statement is that they can be described by conservative equations in the lowest-order approximation, as will be shown below (see also the discussion in Sec. IV).

In Sec. II, generalizing the results of Ref. [14] to the case of transversely distributed dynamical trap, governing equations are presented and the general features of these equations are discussed. In Sec. III we analyze solutions to these equations, mainly for transversely one-dimensional bright vector solitons. Finally, in Sec. IV the place of these localized structures among the classes of conservative and dissipative solitons is discussed.

II. GOVERNING EQUATIONS

Consider the atomic BEC in the dynamic trap, using the approach and results of Refs. [14,15], with the longitudinal coordinate notation changed from x to z [see Fig. 1(a)]. Then the BEC macroscopic wave function $\psi(\mathbf{r}_\perp,z,t)$ obeys the Gross-Pitaevskii equation (GPE) [6]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_p}\left(\frac{\partial^2\psi}{\partial z^2} + \Delta_\perp\psi\right) + U_0|\psi|^2\psi. \quad (1)$$

Here $\hbar$ is the reduced Planck constant, t is time, $\mathbf{r}_\perp = (x,y)$ is the vector of transverse coordinates x and y, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian, and m_p is the particle mass. The parameter of nonlinearity U_0 reflects weak interatomic interactions; U_0 can be positive or negative, depending on the external magnetic field. The GPE describes the dynamics of weakly nonideal diluted atomic gases at zero temperature.

As in [14,15], the boundary conditions for Eq. (1) correspond to the trap in the form of the infinite potential well with oscillating barriers

$$\psi(z = L_{\mathrm{left}}(t),t) = 0, \quad \psi(z = L_{\mathrm{right}}(t),t) = 0. \quad (2)$$

The modulation depth μ of these oscillations is small, $\mu \ll 1$. The transverse size of the trap is assumed to be large as compared with the transverse dimensions of the BEC structures considered below. In the case of a finite depth of the potential well, the lifetime of the BEC is also finite due to

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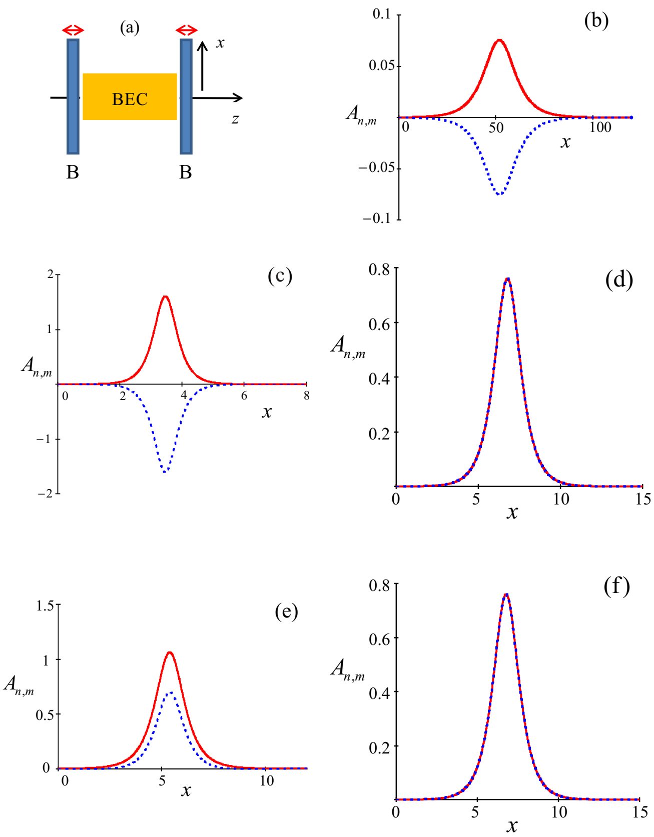

FIG. 1. (Color online) (a) Scheme of a dynamical trap with the BEC between oscillating barriers B. (b)–(f) Profiles of amplitudes $A_n(x)$ [solid (red) curves] and $A_m(x)$ [dotted (blue) curves]: (b) and (c) antiphase soliton pairs $A_n(x) = -A_m(x)$ and (d) and (f) in-phase pairs $A_n(x) = A_m(x)$, with (b) $\delta\omega = 0$ and $\delta\omega_0 = 1.02$, (c) $\delta\omega = 0$ and $\delta\omega_0 = 10$, (d) $\delta\omega = 0$ and $\delta\omega_0 = 1.02$, (e) $\delta\omega = 0.5$ and $\delta\omega_0 = 2$, and (f) $\delta\omega = 0$ and $\delta\omega_0 = 0.98$; $s = -1$ and $\nu = -1$.

its escape from the trap, so the consideration is valid for a time period less than the lifetime.

In the zeroth-order approximation with motionless barriers $\mu = 0$, $L_{\text{left}} = 0$, and $L_{\text{right}} = L_0$ and noninteracting atoms $U_0 = 0$, the solution to the problem gives a set of discrete states with energies $E_n^{(0)} = \frac{\pi^2\hbar^2}{2m_pL_0^2}n^2, n = 1,2,3,\ldots$, and wave functions

$$\psi_n^{(0)} \sim e^{-i(E_n^{(0)}t/\hbar)} \sin\left(\frac{\pi n}{L_0}z\right).$$

The energy levels' distribution is highly nonequidistant. Therefore, if modulation of the barriers' position is harmonic, with the frequency Ω close to the frequency of the transition between levels n and m,

$$\hbar\Omega = E_m^{(0)} - E_n^{(0)} + \hbar\delta\Omega, \quad |\delta\Omega|/\Omega \ll 1, \tag{3}$$

it is possible to achieve the resonance interaction of two levels with energies $E_n^{(0)}$ and $E_m^{(0)}$. Higher-order resonances are possible only in higher orders of the perturbation theory; thus they can be realized at very large time intervals only. Then the BEC wave function can be approximated by a superposition of two states with quantum numbers n and m with slowly varying amplitudes $a_{n,m}$:

$$\psi(\mathbf{r}_\perp,z,t) \approx a_n(\mathbf{r}_\perp,t)\psi_n^{(0)} + a_m(\mathbf{r}_\perp,t)\psi_m^{(0)}. \tag{4}$$

The high precision of the two-level approximation under resonance conditions is confirmed not only by estimations, but also by comparison with the direct solution to the GPE [14]. Then, after the replacement $i\hbar\frac{d}{dt} \to i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_p}\Delta_\perp$ in Eqs. (19) of Ref. [14], we get the governing equations for the case of a transversely distributed trap:

$$\begin{aligned}&\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_p}\Delta_\perp\right)a_n + s\mu nmE_1^{(0)}a_m\\ &\quad - U_0\left(\frac{3}{4}|a_n|^2 + |a_m|^2\right)a_n = 0,\\ &\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_p}\Delta_\perp\right)a_m + s\mu nmE_1^{(0)}a_n\\ &\quad + \left[\hbar\delta\Omega - U_0\left(\frac{3}{4}|a_m|^2 + |a_n|^2\right)\right]a_m = 0.\end{aligned} \tag{5}$$

For definiteness, we assume $L_{\text{left}} = 0$ and $L_{\text{right}} = L_0[1 + \mu\cos(\Omega t)]$; here $s = (-1)^{m-n}$. Below we use dimensionless units

$$\tilde{t} = t\frac{\mu nmE_1^{(0)}}{\hbar}, \quad (\tilde{x},\tilde{y}) = \sqrt{\frac{2m_p}{\hbar}}(x,y), \quad \delta\omega = \frac{\hbar\delta\Omega}{\mu nmE_1^{(0)}},$$

$$\tilde{a}_{n,m} = a_{n,m}\sqrt{\frac{V_0}{2N_3}}, \quad \nu = \frac{U_0N_3}{2V_0\mu nmE_1^{(0)}} = \text{sgn}U_0, \tag{6}$$

where $E_1^{(0)}$ is the energy of the first eigenstate, $V_0 = S_0L_0$ is the volume of trap with transverse section S_0, and N_3 is the total number of particles. Then the governing equations take the form of linearly (coherently) and nonlinearly (incoherently) coupled nonlinear Schrödinger equations

$$\begin{aligned}i\frac{\partial\tilde{a}_n}{\partial\tilde{t}} + \tilde{\Delta}_\perp\tilde{a}_n + s\tilde{a}_m - \nu(3|\tilde{a}_n|^2 + 4|\tilde{a}_m|^2)\tilde{a}_n = 0,\\ i\frac{\partial\tilde{a}_m}{\partial\tilde{t}} + \tilde{\Delta}_\perp\tilde{a}_m + s\tilde{a}_n + [\delta\omega - \nu(3|\tilde{a}_m|^2 + 4|\tilde{a}_n|^2)]\tilde{a}_m = 0.\end{aligned} \tag{7}$$

In the following we will omit the tilde. It follows from Eqs. (7) that the total number of particles is conserved for localized structures:

$$\int d\mathbf{r}_\perp(|a_n|^2 + |a_m|^2) = \text{const}. \tag{8}$$

Equations (7) have Galilean symmetry. This means that if functions $A_{n,m}(x,y,t)$ give a solution to Eqs. (7), then there is a family of solutions with an arbitrary transverse velocity V:

$$a_{n,m} = \exp\left(i\frac{V}{2}x - i\frac{V^2}{4}t\right)A_{n,m}(x - Vt,y,t). \tag{9}$$

Evidently, Eqs. (7) are invariant to a phase shift of both amplitudes $a_{n,m} \to a_{n,m}e^{i\delta\Phi}$ and $\delta\Phi = \text{const}$ and to shifts of coordinates $(x,y) \to (x + \delta x, y + \delta y)$. In the case of an exact resonance $\delta\omega = 0$, Eqs. (7) are also invariant to the replacement $n \leftrightarrow m$.

III. SOLUTIONS TO THE GOVERNING EQUATIONS

For the exact resonance $\delta\omega = 0$, there are solutions of Eqs. (7) with equal populations of the two resonance levels $a_m = \pm a_n \equiv a$. Then for a we have the equation

$$i\frac{\partial a}{\partial t} + \Delta_\perp a \pm sa - 7\nu|a|^2 a = 0. \quad (10)$$

After replacement $a = be^{\pm ist}$ it takes the form of the standard (transversely two-dimensional) nonlinear Schrödinger equation

$$i\frac{\partial b}{\partial t} + \Delta_\perp b - 7\nu|b|^2 b = 0. \quad (11)$$

In the opposite case, when mainly one level is populated $|a_m|^2 \ll |a_n|^2$, one gets also the nonlinear Schrödinger equation

$$i\frac{\partial a_n}{\partial t} + \Delta_\perp a_n - 3\nu|a_n|^2 a_n = 0 \quad (12)$$

in the lowest order and an inhomogeneous linear equation for a_m in the first order of perturbation theory. Correspondingly, these equations describe such phenomena as modulation instability, one-dimensional conservative bright ($\nu = -1$) and dark ($\nu = 1$) solitons, breathers, and cnoidal waves [1,20,21].

Transversely two-dimensional equations (7) are effectively reduced to one-dimensional equations if the system is linearly confined by the trap in the y direction. This confinement provides a way to avoid modulation instability and get a single-mode regime in this direction. A wide class of transversely one-dimensional solutions can be found in the form

$$a_{n,m} = A_{n,m}(x)e^{i\delta\omega_0 t}, \quad (13)$$

with real amplitudes $A_{n,m}(x)$ (see also [26]). Then it follows from Eqs. (7) that

$$\frac{d^2A_n}{dx^2} = -\frac{\partial U}{\partial A_n}, \quad \frac{d^2A_m}{dx^2} = -\frac{\partial U}{\partial A_m}, \quad (14)$$

where

$$U(A_n,A_m) = U_2(A_n,A_m) - \tfrac{3}{4}\nu\left(A_n^4 + A_m^4\right) - 2\nu A_n^2A_m^2, \quad (15)$$

$$U_2(A_n,A_m) = sA_nA_m - \frac{\delta\omega_0}{2}A_n^2 - \frac{\delta\omega_0 - \delta\omega}{2}A_m^2. \quad (16)$$

Equations (14) can be interpreted mechanically as equations of the two-dimensional motion of a particle with the unit mass under the action of force with the potential $U(A_n,A_m)$, with the transverse coordinate x for the BEC playing the role of time t for the mechanical particle. The corresponding Hamiltonian is

$$H = \frac{1}{2}\left(\frac{dA_n}{dx}\right)^2 + \frac{1}{2}\left(\frac{dA_m}{dx}\right)^2 + U(A_n,A_m). \quad (17)$$

Because the Hamiltonian does not depend on x explicitly, it is conserved: $H = \text{const}$, or $dH/dx = 0$ (the law of mechanical energy conservation).

For localized structures with a finite number of particles the amplitudes $A_{n,m}$ should vanish at infinity $A_{n,m}(x) \to 0$ for $x \to \pm\infty$. Therefore, $H = 0$ for these structures and in the vicinity of the point $A_n = A_m = 0$ the surface $U(A_n,A_m)$ is given by the quadratic form $U_2(A_n,A_m)$ [Eq. (16)]. It can be reduced to the canonic form $U_2 = \lambda_1 B_1^2 + \lambda_2 B_2^2$ by a unitary transformation $(A_n,A_m) \to (B_1,B_2)$ with real characteristic numbers

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{\delta\omega}{2} - \delta\omega_0 \pm \sqrt{1 + \left(\frac{\delta\omega}{2}\right)^2}\right]. \quad (18)$$

If both characteristic numbers are negative $\lambda_{1,2} < 0$, then the point $A_n = A_m = 0$ corresponds to a local maximum of function $U(A_n,A_m)$ and the field in the vicinity of this point has the form

$$A_n = c_1e^{p_1x} + c_2e^{p_2x},$$
$$A_m = -s\left[c_1\left(p_1^2 - \delta\omega_0\right)e^{p_1x} + c_2\left(p_2^2 - \delta\omega_0\right)e^{p_2x}\right], \quad (19)$$

where $p_{1,2}^2 = -2\lambda_{1,2} > 0$. Evidently, for $p_{1,2} > 0$ these amplitudes vanish at $x \to -\infty$ for any values of $c_{1,2}$. Below we fix the parameter of nonlinearity $\nu = -1$ and then the potential $U \to +\infty$ for large amplitudes $A_{n,m} \to \infty$. Under these conditions, one can find a family of solitons with a continuously varying parameter $\delta\omega_0$ (we ignore here the possibility of variation of a phase shift $\delta\Phi$, a shift of transverse coordinate δx, and a transverse velocity V). This means that for a fixed value of $\delta\omega_0$ it is possible to find the value c_2/c_1 that ensures the field vanishing for $x \to +\infty$. This case is realized under the conditions $\delta\omega_0 > 0$ and $\delta\omega < \delta\omega_0 - \frac{1}{\delta\omega_0}$.

Some examples are given in Fig. 1. For the exact resonance $\delta\omega = 0$, in accordance with Eq. (11), in-phase and antiphase soliton pairs with equal moduli of distributions $|A_n(x)| = |A_m(x)|$ can be found for the same value of $\delta\omega_0$, taking $c_1 = 0$ or $c_2 = 0$ [see Figs. 1(b)–1(d)]. For nonzero detuning $\delta\omega \neq 0$, the components of the soliton pairs have different amplitudes [see Fig. 1(e)].

In another variant $\lambda_1\lambda_2 < 0$, the point $A_n = A_m = 0$ corresponds to a saddle. Then only one constant, either c_1 or c_2 (corresponding to the negative characteristic value) in Eqs. (19), is nonzero. Nevertheless, even in this case there is a family of soliton pairs, but only of one type [see Fig. 1(f)]. For cases when $\lambda_{1,2} > 0$ with the point $A_n = A_m = 0$ corresponding to a local minimum, no bright structures are possible.

IV. DISCUSSION

The localized structures discussed herein present a type of vector spatial solitons with in-phase or antiphase components, transversely motionless or moving, with continuously varying carrier frequency. This is a signature of conservative solitons. On the other hand, they can be classified as cavity solitons due to the scheme of a trap with two barriers. Additionally, a BEC in the dynamical trap belongs to the class of open systems because collisions of particles (atoms) with moving barriers can result in an increase or a decrease of their kinetic energy. The possibility of describing this type of soliton by conservative equations is probably connected to their averaged description in terms of envelope solitons. In fact, the particle kinetic energy does not change practically due to collisions with barriers when averaged over an oscillation period. In a sense, these cavity oscillons form an intermediate class between conservative and dissipative solitons.

Due to the mathematical equivalence, the scheme considered could also be interpreted as an optical waveguide with a medium possessing the Kerr nonlinearity of refractive index; the oscillating barriers can be realized in this case by modulation of the transverse profile of electric voltage in an electro-optical medium. The governing equations presented have a general form, thus allowing the study of the interaction of vector solitons and features of two-dimensional vector structures. These interesting venues can be the subject of separate research.

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