Quantum-information engines with many-body states attaining optimal extractable work with quantum control

J. M. Diaz de la Cruz[1,*] and M. A. Martin-Delgado[2]
[1]*Departmento de Fisica Aplicada, Universidad Politecnica, 28006 Madrid, Spain*
[2]*Departamento de Fisica Teorica I, Universidad Complutense, 28040 Madrid, Spain*
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We introduce quantum information engines that extract work from quantum states and a single thermal reservoir. They may operate under three general conditions—(1) unitarily steered evolution (US), driven by a restricted set of available Hamiltonians; (2) irreversible thermalization (IT), and (3) isothermal relaxation (IR)—and hence are called USITIR machines. They include novel engines without traditional feedback control mechanisms, as well as versions which also include them. Explicit constructions of USITIR engines are presented for one- and two-qubit states and their maximum extractable work is computed, which is optimal. Optimality is achieved when the notions of controllable thermalizability and density matrix controllability are fulfilled. Then many-body extensions of USITIR engines are also analyzed and conditions for optimal work extraction are identified. When they are not met, we measure their lack of optimality by means of newly defined *uncontrollable entropies,* which are explicitly computed for some selected examples. This includes cases of distinguishable and indistinguishable particles.

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I. INTRODUCTION

The goal of increasing the speed of a universal computer has played a central role in computer design, but this comes along with a big power bill that also scales up the faster the computer is. Thus, power saving has become a central issue in the new generations of supercluster computers. This has led to the creation of the Green500 list [1] of the world's most energy-efficient supercomputers, which complements the traditional TOP500 list [2] ranking the most powerful nondistributed computers. The possibility of doing classical universal computation in a fully reversible way [3] was motivated by the quest of a dissipationless computer [4–7]. Interestingly enough, the early origins of quantum computing are linked to the research on the energy requirements of standard computers since reversibility was considered as a resource to reduce energy dissipation [8]. The first hint at the idea of a quantum computer was to use the unitary evolution of quantum mechanics as a natural way to achieve the goal of reversibility in computation [9,10]. Yet, this missed the new possibilities offered by quantum superposition and entanglement properties of quantum states. However, soon thereafter the surprising new capabilities offered by quantum computers [11,12], with speedups over classical algorithms, took over the original energy considerations and have been the predominant topic in the theory of quantum computation. The experimental feasibility of realizing a real quantum computer [13] has been another essential breakthrough for the development of quantum computation as a new field of research.

Soon after the model of universal quantum computer became accepted as a new model for computation, it was evident that a realistic experimental realization of it could only be possible by addressing again the problems of dissipation, this time in the context of battling the decoherence disturbing effects [14,15] that inexorably affect the entangling properties of quantum states needed to carry out a powerful quantum computation. In this way, thermodynamical aspects of quantum computation are making a comeback. A quantum computer can be thought of as a special type of engine that trades free energy for mathematical operations at the expense of dissipating some heat. In this work, we focus on another type of machines known as quantum-information engines whose main goal is to extract the maximum amount of work from possibly entangled many-body states regardless of the computational work that they may carry out. Quantum engines play a fundamental role in investigating quantum effects in thermodynamical laws, in particular, the second law.

An essential question in thermodynamics is, What are the limits for the work we can extract from a given system, be it classical or quantum? The traditional machine used to address this question is the Szilard engine (SZE) [16]. The classical case of the SZE was solved by Bennett [17], realizing that the second law of thermodynamics can be recovered once we take into account the dissipation introduced in the erasure of the register needed to reset the thermodynamic cycle of the engine. The necessity of this energy cost dissipated as heat is called the Landauer principle [7]. In the quantum world, several studies have been carried out with quantum versions of the SZE [18–31]. Contrary to the classical case, the energy analysis in the quantum engine is more involved, since the role of the piston or barrier present in the SZE introduces energy costs in several stages of the engine cycle that are not present in the classical engine. Moreover, the quantum SZE operates with a feedback control mechanism [27,30] that is an essential ingredient to extract work from quantum information.

In this paper we use different variations of a relatively simple quantum-information heat engine (QIHE) based on magnetic qubits [18,32] as concept models to process information contained in multiqubit systems. We show that the premises for optimality of the extractable work refer to quantum control theory [33] and to a problem that we call controllable thermalizability (CT), put forward in Sec. IV. It is

*jose.diazdelacruz@upm.es

shown that the latter implies a specially demanding condition when many-body quantum states are used. Consideration of restricted sets of available Hamiltonians follows from this difficulty.

Under nonoptimal conditions, we also determine the relation between the obtainable work and a new variation of relative entropy that we call uncontrollable entropy. The QIHE that we propose avoids the use of a barrier as a basic component in the quantum engine. Information heat engines are devices that extract work from a thermal reservoir in exchange for an increment in entropy in some physical system at the core of the machine. Cyclic workflows imply procedures to reset the entropy of such system. The best-known way to do this is to purify the quantum state of the internal system by performing a Von Neumann measurement, at the cost of increasing the entropy of the qubits that hold the result. A less visited way is to swap the entropy of internal and fresh quantum systems that are cyclically fed to the machine. Considering the connection between information theoretic concepts and thermodynamic entropies, some papers [29,30] have explored the extraction of work from the entanglement of a two-qubit system. We analyze a many-body generalization and conclude that there is a nontrivial issue concerning the set of Hamiltonians that drive the evolution of the system. The new QIHEs are introduced for simple cases using one- and two-qubit systems. Feedback control versions of them are also constructed. In both cases, the extractable work analysis shows that it is maximal for these quantum engines. As this maximum work coincides with that compatible with the second law of thermodynamics, it is optimal. We generalize these QIHEs based on swap operations to systems comprising many-body states and perform the extractable work analysis yielding the optimal value. We call them USITIR machines, after the three kinds of evolution under which they may operate: unitary steering (US), irreversible thermalization (IT), and isothermal relaxation (IR). This generalization includes novel models as well as previously well-known models of quantum engines. We note that in order to achieve optimality for these USITIR engines with more than two qubits, general k-body control Hamiltonians are needed.

In a quantum many-body system, there is the issue of whether or not the particles are identical [27]. Thus, for the many-body USITIR engines we also derive the corresponding formulas of the optimal extractable work for identical particles [27,28].

This paper is organized as follows: in Sec. II, we present simple and scalable models of magnetic information heat engines, which motivate the issues of controllability and controllable thermalizability developed in Secs. III and IV, respectively. These properties prompt a more abstract and general type of information heat engine, which is introduced in Sec. V. We refer to them as USITIR machines, and the work that they can process is analyzed in Sec. VI. Section VII is devoted to conclusions. In Appendixes A–E we provide detailed calculations for the uncontrollable entropies and extractable work of some relevant USITIR engines.

An extensive study of the rich relations between relative entropy and thermodynamics has been published [34], and also studies from a resource perspective [35,36]. Recently a number of publications extend the validity of relations between work and entropy by introducing generalizations of the Shannon or Von Neumann entropy to the so-called *smooth* entropies. They can handle situations where the goal is maximizing not merely the expected value of the extracted work, but also other aspects with regard to its fluctuations and probabilities of failure [37–39]. However, our paper only considers machines that work in cycles and focuses only on the expected value of the extracted work.

II. QUANTUM ENGINES BASED ON MAGNETIC QUBITS

This section presents an alternative to previous models of QIHEs [27,40,41]. In the following paragraphs we describe a magnetic quantum-information heat engine (henceforth MQIHE) for both one-qubit and two-qubit inputs which is different from other magnetic alternatives [18,32]. Unlike most previously described SZEs, some of our models contain neither an explicit measurement nor a feedback control. It is simpler than the previously described information heat engines and shows clearly how it serves its purpose: to trade entropy for work, taking energy from a heat reservoir at a definite temperature using up the information provided by input qubits. We describe two kinds of devices: swap and feedback engines. In swap machines, the density matrix of the input system (one qubit in Sec. II A or two qubits in Sec. II C) and that of internal magnetic qubits are swapped. In feedback engines, the internal qubits are entangled with the input ones (one qubit in Sec. II B or two qubits in Sec. II D), which are subsequently measured, and the outcomes govern further action. The degree of purity of the input qubits determines the quality of the measurement and thus the extractable work. We also provide physical representations of the engines.

A. One-qubit magnetic quantum engine

In this subsection we describe a one-qubit MQIHE (1MQIHE), showing how, after each cycle, some energy is stored in an electrical battery provided that the system is fed with a non-fully-depolarized magnetic qubit; this qubit exits the cycle completely depolarized. The system depicted in Fig. 1 represents a 1MQIHE.

There are two important qubits in the system: the *ancilla* or *input* A and the *system* or *internal* qubit S. The ancilla A is an external qubit that enters the machine in a known, possibly mixed, state; it is shown that each ancilla in a pure state will enable us to obtain an energy $k_B(\ln 2)T$ from the thermal reservoir $R(T)$, which is always supposed to be in equilibrium at temperature T. The S qubit is a magnetic spin $\frac{1}{2}$ whose state may belong to the Hilbert space spanned by the kets $|\uparrow\rangle,|\downarrow\rangle$ or may even be a mixed state without further restriction. The magnetic qubit S lies in a magnetic field $\vec{B}(t) = B(t)\,\vec{u}_z$ generated by the electrical current $I_C(t)$ running through coil C. The thermal connection between the S qubit and the reservoir $R(T)$ is accomplished through a wall P, which may be adiabatic or diathermal, according to a cyclic timing that is described below.

In order to appreciate more clearly the trade-off between information and energy, we assume that the energies of the S and A qubits are completely degenerate in the initial and the final instants. This means that the energy of qubit A is the same

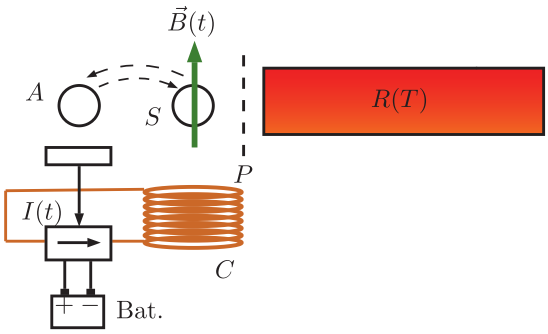

FIG. 1. (Color online) Main components of the 1MQIHE: the information of input qubit A is traded for that of a magnetic qubit spin-$\frac{1}{2}$ S that lies in the magnetic induction $\vec{B}(t)$ created by the current $I(t)$ running through coil C. The power needed to drive the current is taken from or to an electrical battery. The qubit S may be set in thermal equilibrium with or isolated from a heat bath $R(T)$ at a fixed temperature T.

irrespective of whether it is in the $|\uparrow\rangle$ or the $|\downarrow\rangle$ state, or any mixture thereof. This is also true for qubit S.

Next, a description of the three stages of the 1MQIHE cycle (Fig. 2) is given, along with the evolution of the states of the A and S qubits; the energy flow is traced in Appendix A to draw a final balance.

(i) At this stage, the ancilla A and the system qubit S stay thermally isolated from the reservoir $R(T)$ and undergo a reversible swap operation whereby $\forall \rho_p, \rho_q, (\rho_p)_A \otimes (\rho_q)_S$ transforms into $(\rho_q)_A \otimes (\rho_p)_S$. It is important to remark that the swapping is not physical, but just logical. A possible physical way to implement a SWAP gate between magnetic qubits is by means of the Heisenberg exchange interaction, which performs this operation in a reversible way.

The S qubit starts the cycle as it finishes the last stage, so that it is in equilibrium with the thermal reservoir in a completely depolarized state $\rho_{S,0} = \frac{1}{2}\mathbb{1}$, with no applied magnetic field; the ancilla begins in a known, possibly mixed, state $\rho_{A,0}$. After the swapping, the ancilla is released in a completely depolarized state $\rho_{A,1} = \frac{1}{2}\mathbb{1}$ and the system qubit S is left in the $\rho_{A,0}$ state. Now an auxiliary magnetic field is applied to rotate the ρ_A state in order to align its spin with the axis of the coil (the z axis). This means that the S qubit exits this stage in

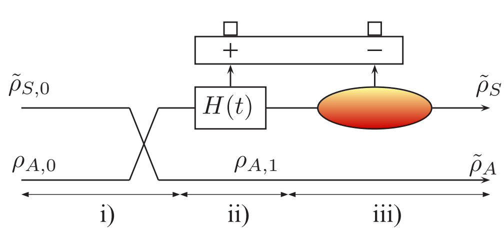

FIG. 2. (Color online) Simplified workflow of a 1MQIHE. Density matrices of qubits A and S are swapped; then a time-varying current $I_C(t)$ determines a Hamiltonian $H(t)$ which acts isolated from $R(T)$ at stage ii and in thermal contact at stage iii. The shaded oval represents evolution under thermal equilibrium with $R(T)$, and $\tilde{\rho}_S$ represents a completely depolarized state for qubit S.

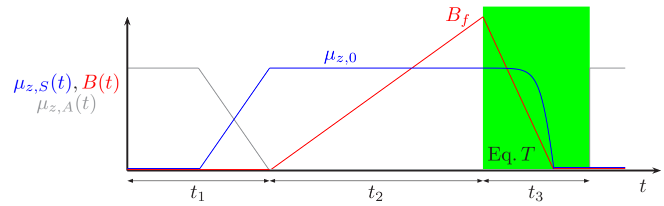

FIG. 3. (Color online) Evolution of the magnetic field $B(t)$ (red line) and the z components of the magnetic moments μ_z of qubit A (gray line) and qubit S (blue line). The shaded (green) rectangular area corresponds to thermal equilibrium with the reservoir.

the state

$$\rho_{S,1} = \tfrac{1}{2}(\mathbb{1} + c\sigma_z), \tag{1}$$

with the same entropy as the initial ancilla: $S(\rho_{A,0}) = S(\rho_{S,1})$; moreover, we can assume, without loss of generality, that $c \geqslant 0$.

So far, no energy has been interchanged with the coil.

(ii) Next, S keeps thermally isolated from $R(T)$, the current source is turned on, and the current through the coil I_C grows from 0 to a maximum value I_M. The magnetic field created by I_C, which is in the z direction, grows accordingly. At this stage, the system qubit does not change, because its state commutes with the time-varying Hamiltonian; therefore, the magnetic moment is also constant, $\mu_{z,0}$. The final value of I_C corresponds to a value B_f of B determined under the condition that the overall energy extraction is maximal. It is shown in Appendix A that the optimum value of B_f is the one that causes the S qubit state $\rho_{S,1}$ to be in equilibrium with the reservoir at temperature T.

(iii) Finally, the system qubit S is put in thermal contact with the reservoir and the current I_C is gradually lowered so that the qubit stays in equilibrium with the reservoir. At this stage the current source recovers an energy in excess of the one previously supplied. The equilibrium for ρ_S when the current is set to 0 is the completely depolarized state and the engine is ready for the next cycle.

The thermal contact is interrupted when the cycle starts once more until stage iii is reached again. Figure 3 shows the evolution of $B(t)$, $\mu_z(t)$, and $R(t)$ with all magnitudes normalized.

We claim that 1MQIHE is optimal, in the sense that it obtains the maximum theoretically allowable work from the processing of the qubit A, in the presence of a thermal reservoir at temperature T. It is well known [25,42], that the maximum amount of work that a system S can obtain from a bath is given by

$$W_{1\to 2} = -\frac{S(\rho_1||\rho_\beta(H_1)) - S(\rho_2||\rho_\beta(H_2))}{\beta}\ln 2 - \Delta F, \tag{2}$$

where ρ_1, ρ_2, H_1, H_2 are the initial and final states and Hamiltonians of qubit S, $\rho_\beta(H)$ is the state that corresponds to thermal equilibrium with the reservoir for the Hamiltonian H, ΔF is the difference of the free energies of states $\rho_\beta(H_1), \rho_\beta(H_2)$ and $S(\sigma||\tau) = \mathrm{Tr}\{\sigma\ln\sigma - \sigma\ln\tau\}$ is the relative entropy of σ with respect to τ. If $H_2 = H_1 = 0$, then $W_{1\to 2} = -\frac{S(\rho_1||\rho_\beta(H_1)) - S(\rho_2||\rho_\beta(H_2))}{\beta}\ln 2$ and if we also assume that the final state is in thermal equilibrium and the dimension of the

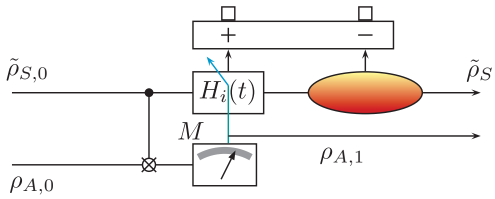

FIG. 4. (Color online) Simplified workflow of the feedback version of the 1MQIHE (see Fig. 2). The controlled-NOT(S,A) gate is followed by a measurement of A; the result determines the choice of $H_i(t)$, then it proceeds like the MQIHE.

Hilbert space is d, we have $W_{1\to 2} = \frac{\log_2 d - S(\rho_1)}{\beta} \ln 2$ which, if the system is a set of N qubits, yields

$$W_{1\to 2} = \frac{N - S(\rho_1)}{\beta} \ln 2. \tag{3}$$

A detailed deduction of the work obtained in the cycle is presented in Appendix A. The result is Eq. (A12) that represents the particularization of Eq. (3) for one qubit.

Two problems emerge from the previous consideration:

(a) is there a thermal equilibrium state unitarily equivalent to an arbitrary initial state?

(b) can the system be unitarily steered from an arbitrary initial state to thermal equilibrium?

In Sec. II C we extend the 1-qubit MQIHE to a two-qubit MQIHE, answer the previous questions and then show that with a controllable Heisenberg interaction the machine is still optimal. In Sec. II B we describe a feedback model for the 1MQIHE.

B. Feedback version of 1MQIHE

A feedback version of 1MQIHE (Fig. 4) has a slightly different cycle. At stage i, instead of logically swapping qubits A and S, a controlled-NOT(S,A) [CNOT(S,A)] gate, controlled by S, is applied. We assume that $\rho_{A,0} = \frac{1}{2}(\mathbb{1} + c\sigma_z)$; if it is not, an auxiliary field is used to rotate the state. After the CNOT(S,A) stage, the bipartite state is

$$\begin{aligned}\rho_{AS,1} = \tfrac{1}{4}[&(1+c)|\uparrow\uparrow\rangle_{AS}\langle\uparrow\uparrow| + (1-c)|\downarrow\uparrow\rangle_{AS}\langle\downarrow\uparrow| \\ &+ (1+c)|\downarrow\downarrow\rangle_{AS}\langle\downarrow\downarrow| + (1-c)|\uparrow\downarrow\rangle_{AS}\langle\uparrow\downarrow|].\end{aligned} \tag{4}$$

Next, qubit A is measured in the computational basis; the result p may be either $p = 1$ or $p = 2$, corresponding to $|\uparrow\rangle$ or $|\downarrow\rangle$, respectively, and the postmeasurement state $\rho_{i,p}$ of S is

$$\rho_{i,p} = \tfrac{1}{2}(\mathbb{1}_S \pm c\,\sigma_{z,S}), \tag{5}$$

where the plus or minus signs hold for $p = 1$ or $p = 2$, respectively. In the first case ($|A\rangle = |\uparrow\rangle$), the S qubit is in the same state as at the end of stage ii for 1MQIHE and the machine proceeds identically; otherwise ($|\downarrow\rangle$) the same analysis applies, provided that the coil current and, accordingly, the magnetic flux density B are inverted. Consequently, the energy balance is also the same.

C. Two-qubit magnetic quantum engine

Information stored in single qubits can be used to optimally extract work, as has been proved in the two previous subsections. However, information can also be stored in two-qubit systems, which may be entangled. In Sec. VI it is shown that one-qubit control Hamiltonians, e.g., Zeeman interactions with two external magnetic fields, cannot extract the optimal amount of work from a general, possibly entangled, two-qubit system. In order to open the generalization of information heat engines to many qubits we now define a system endowed with a well-known interaction: the Heisenberg two-spin interaction acting on a system consisting of two magnetic spin $\frac{1}{2}$'s.

The two-qubit MQUIHE (2MQUIHE) contains two system qubits S_1 and S_2 and two ancillae A_1 and A_2. At the beginning of each cycle S_1 and S_2 are completely depolarized, whereas qubits A_1 and A_2 enter in an arbitrary, possibly mixed, state defined by the 4×4 density matrix ρ_i. At the end of each cycle, all four qubits exit in a completely depolarized and hence disentangled state. Work is extracted from the increase in entropy of the two-qubit system A_1, A_2. It is supposed that the Hamiltonian of qubits A_1, A_2, S_1, and S_2 is the same and completely degenerate at the beginning and the end of each cycle.

The states of qubits S_1, S_2 are initially swapped with those of A_1, A_2. Qubits A_1, A_2 now exit in a completely depolarized state. Then the S_1, S_2 pair is unitarily driven by an adequate steering of the magnetic fields $\vec{B}_1, \vec{B}_2$ acting individually on S_1, S_2 and of the distance between S_1, S_2 (tuning the strength of the Heisenberg interaction) to state $\rho_\beta(H_\beta)$ in thermal equilibrium at temperature T for the Hamiltonian H_β. In Sec. IV the existence of the Hamiltonian H_β, and hence of the state $\rho_\beta(H_\beta)$, is proved and in Sec. III the possibility of unitarily driving the system from any initial state to $\rho_\beta(H_\beta)$ is established. Figure 5 shows the two stages schematically. In the first one (black arrow), the initial state ρ_i of A_1, A_2, which has been transferred to S_1, S_2, undergoes a unitary evolution

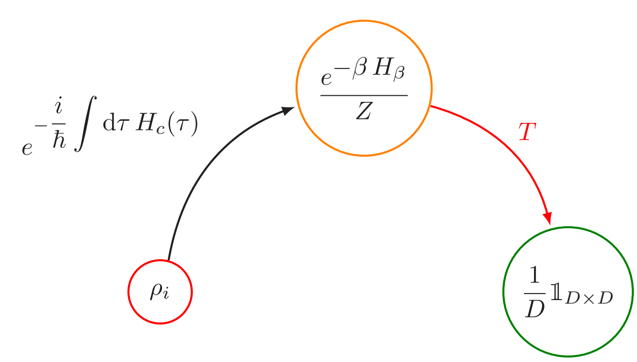

FIG. 5. (Color online) Schematic of the transformation from the input state ρ_i of the input qubits to a completely depolarized state. After transferring state ρ_i to the internal qubits, they should be driven to a thermal equilibrium state $\rho_\beta(H_\beta)$ by adequately steering the system through the Hamiltonian $H_c(t)$. This process corresponds to the left (black) arrow line. The existence of an available Hamiltonian H_β for which the equilibrium state $\rho_\beta(H_\beta)$ is unitarily reachable from ρ_i through the control Hamiltonians is a central point of this paper. The right (red) arrow line represents how, by gradually turning off the control Hamiltonians, the state becomes completely depolarized.

driven by the Hamiltonian $H_c(t)$ which results from a suitable combination of the control Hamiltonians, as explained further in Sec. III. The resulting state should be in thermal equilibrium at temperature T for one available Hamiltonian H_β. The second stage is represented by the right (red) arrow line and takes place entirely in thermal equilibrium at temperature T; the Hamiltonians are progressively switched off, leading to a completely depolarized state at the end.

Finally, the steering fields are gradually turned off and the positions of S_1,S_2 are restored to their initial locations, extracting a work given by $-\Delta F$, where F is the Helmhotz free energy of the system of qubits S_1,S_2. Under a completely degenerate Hamiltonian and being in thermal equilibrium, qubits S_1 and S_2 relax to a completely depolarized state.

An energy analysis equivalent to the one performed in Sec. II A yields a work extraction given by

$$W = k_B T\,(\ln 2)\,(2 - S(\rho_i)), \tag{6}$$

which is the optimal value consistent with the second law of thermodynamics and the information content of the processed qubits A_1 and A_2.

D. Feedback version of the 2MQIHE

The feedback version of the 2MQIHE follows immediately: instead of swapping qubits A_1,A_2 and S_1,S_2, the CNOT(S_1,A_1), CNOT(S_2,A_2) are followed by a measurement of qubits A_1,A_2 on the computational basis. Because S_1,S_2 are completely depolarized before the operation, all four possible outcomes are equally likely, implying that A_1,A_2 exit completely unpolarized. After measurement, qubits S_1,S_2 are in the initial state ρ_i of the ancillae A_1,A_2, except for a possible NOT-gate operation on each of the qubits, depending on the result of the measurement. Thus, all the controllability and thermalizability considerations previously written for the 2MQIHE also hold for its feedback version.

III. CONTROL HAMILTONIANS

In order to avoid inefficient irreversible steps, information heat engines should be able to maneuver the internal qubits in such a way that their density matrix undergoes no discontinuity when thermal equilibrium with the reservoir sets in. It is then imperative to provide an effective control system able to drive the internal qubits from the input state to another one which is in thermal equilibrium at temperature T. This process is depicted in Fig. 5.

In this section we stick to a typical quantum control scenario where a system with density matrix ρ evolves unitarily under a Hamiltonian given by a linear combination $H_c = \sum_{j=1}^{C} c_j(t) H_j$, where $\mathcal{C} = \{H_j\}$ for $j = 1,\ldots,C$ is the set of *control Hamiltonians* and the C-tuple of time functions $c = (c_1(t),\ldots,c_C(t))$ is the control vector.

A quantum system, described by a general density matrix ρ, defined on a D-dimensional Hilbert space $\mathcal{H}_N$ of N particles and having a control set $\mathcal{C}$ is said to be density matrix controllable iff, for every unitary transformation U on $\mathcal{H}_N$, there exist at least one control vector c and a time t such that

$$U = \exp\left(-\frac{i}{\hbar}\sum_{j=1}^{C}\int_0^t d\tau\, c_j(\tau) H_j\right). \tag{7}$$

Henceforth, a system exhibiting density matrix controllability will be termed a density matrix controllable (DMC) system. It is well known [33] that a system is DMC if the control Lie algebra, which is the $\mathcal{A}_L(C)$ one generated by $\mathcal{C}$, spans the special unitary algebra $\mathfrak{su}(D)$, where D is the dimension of the Hilbert space associated with the physical system.

The available control Hamiltonians in the 2MQIHE are those corresponding to one-qubit interactions with external magnetic fields,

$$H_{1\text{–}6} = \sigma_{x,y,z}^{(1,2)}, \tag{8}$$

and the Heisenberg interaction,

$$H_7 = \sum_{j=x,y,z} \sigma_j^{(1)}\sigma_j^{(2)}. \tag{9}$$

It can be shown [33] that the two-qubit system endowed with the control set

$$\begin{aligned}\mathcal{C}_2 = \{&\sigma_x^{(1)}\otimes\mathbb{1}^{(2)},\sigma_y^{(1)}\otimes\mathbb{1}^{(2)},\sigma_z^{(1)}\otimes\mathbb{1}^{(2)},\\ &\mathbb{1}^{(1)}\otimes\sigma_x^{(2)},\mathbb{1}^{(1)}\otimes\sigma_y^{(2)},\mathbb{1}^{(1)}\otimes\sigma_z^{(2)},\\ &\sigma_x^{(1)}\otimes\sigma_x^{(2)}+\sigma_y^{(1)}\otimes\sigma_y^{(2)}+\sigma_z^{(1)}\otimes\sigma_z^{(2)}\}\end{aligned} \tag{10}$$

is DMC. This implies that the system can be steered from any input state to any output state provided that their density matrices are unitarily equivalent. Two density matrices are unitarily equivalent iff their spectra share the same eigenvalues with equal multiplicity. Under these premises the system can be steered from one state to the other by a suitable choice of the control vector c.

Considering now the feedback version of the 2MQIHE (Fig. 6), no further conditions need to be met. The control set $\mathcal{C}$ may steer the system from any postmeasurement state to any other as long as both of them are unitarily equivalent. Note that the four possible outcomes determine four unitarily equivalent states which are equal up to, at most, two NOT gates. Accordingly, the controllability of feedback machines can be referred to that of swap engines.

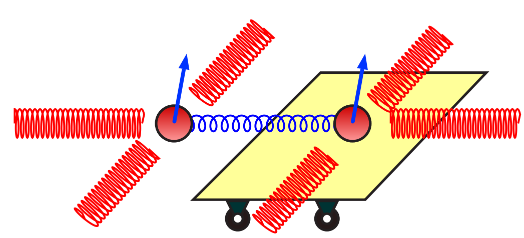

FIG. 6. (Color online) 2MQIHE engine outline: two possibly entangled qubits [joined by the loopy (blue) line] lie in the magnetic fields created by a set of control currents running through the corresponding coils. The system evolves under the action of independent magnetic fields and a Heisenberg spin interaction suitably tuned by changing the distance.

IV. CONTROLLED THERMALIZABILITY

As established in the introduction to Sec. III, it is necessary to tune the control Hamiltonians so that the state reached after the unitary evolution stage is in thermal equilibrium at temperature T. The central circle in Fig. 5 represents such a state. If the system is DMC, as established in the previous section, the input and the thermal equilibrium states should be equal up to a unitary transformation. Now the problem can be stated as, Given a control set $\mathcal{C}$ containing C Hermitian operators $H_1,\ldots,H_C$, is there a linear combination thereof whose thermal equilibrium state $\rho_\beta(H)$ is unitarily equivalent to any input state ρ_i? Henceforth this property is referred to as *controllable thermalizability* (CT). In order to answer this question the following comments are in order.

(1) The spectrum of $\rho_\beta(H)$ is the set $\mathfrak{S}=\{Z^{-1}\,e^{-\beta\lambda_\ell}\}$, where $Z=\sum_\ell e^{-\beta\lambda_\ell}$ and λ_ℓ is an eigenvalue of H.

(2) Considering that the traces of $\rho_i,\rho_\beta(H)$ are equal to 1, their spectra are equal iff those of $-\beta^{-1}\log\rho_i$ and H are shifted relative to each other by a real value, which can be either positive, negative, or 0.

As a consequence, multiples of the identity can be appended to the control set $\mathcal{C}$ or added to any of its elements without changing either the density matrix controllability or the CT. Note that both concepts have been defined for systems with any dimensionality and control set $\mathcal{C}$.

Appendix B shows that, in addition to being DMC, 2MQIHE is CT. This means that this engine can extract the maximum work $k_B T\,(\ln 2)\,[2-S(\rho_i)]$ from any input state ρ_i.

The analysis of the DMC and CT problems presented in Sec. III and Appendix B yield different results for qubit engines with $N>2$ and $N=2$. With all one-qubit Hamiltonians and a Heisenberg interaction Hamiltonian for each couple of qubits, the system is controllable, irrespective of the number N of qubits involved. However, the result for the existence of a thermal equilibrium state unitarily equivalent to ρ_i by a suitable combination of the control Hamiltonians is different. This can be inferred by noting that for large N the number of eigenvalues of ρ_i scales as 2^N, while the number of adjustable parameters for one-qubit Hamiltonians and two-qubit Heisenberg interactions scales as N^2. This consideration leads to the conclusion that no reversible process consisting of unitary and isothermal evolutions exists for systems of an arbitrary number of qubits when only the aforementioned control Hamiltonians are available. Moreover, even if not only one-qubit and Heisenberg interactions, but also any two-qubit interactions are reckoned, the same conclusion holds, as is obvious because the scaling of the independent parameters also goes as N^2. Consequently, k-body interaction control Hamiltonians are needed. Note the difference from the controllability problem, where only one- and two-body interactions are sufficient for achieving density matrix controllability, as can be easily inferred from the fact that universal sets of gates for N-qubit systems can be made of only one- and two-qubit gates [8]. On the other hand, in one-qubit systems, the σ_z control Hamiltonian guarantees CT yet fails to provide density matrix controllability. Thus, CT neither implies nor is implied by density matrix controllability.

Note that pure states can only become thermal states under Hamiltonians with infinite energy differences. For instance, in a 1MQIHE, a pure state is also a thermal state only if the magnetic field is infinite. In this paper we admit unrestricted linear combinations of control Hamiltonians and implicitly consider the limit of any succession of thermal states as a thermal state in its own right. This assumption allows pure states also to be thermal states.

V. USITIR MACHINES: QUANTUM ENGINES USING MANY-BODY STATES

An N-qubit Hilbert space is equivalent to a $D=2^N$-dimensional qudit. In order to pursue our task of extending the previous study to many-body states, we next define a more abstract QIHE. Examining the one- and two-qubit machines in Sec. II, generalization to qudits and a wider set of engines can now proceed straightforwardly. We consider a physical device with the following elements:

(a) An internal or system D-dimensional qudit S.

(b) A control set $\mathcal{C}$ containing C Hermitian operators $H_1,\ldots,H_C$ so that qudit S can be made to evolve unitarily under the Hamiltonian

$$H=\sum_{j=1}^{C}c_j(t)\,H_j+f(t)\,\mathbb{1}_{D\times D} \tag{11}$$

for any control vector $c=(c_1(t),\ldots,c_C(t))$ whose components are arbitrary functions of time. The elements of $\mathcal{C}$ play a dual role. They steer the density matrix of the system according to a definite strategy and, at the same time, provide or store the energy interchanged as work in the process. In addition, $f(t)$ is any function of time, possibly unknown, that, as will be seen, plays no role in the final energy balance.

(c) A thermal reservoir $R(T)$ at temperature T.

(d) A reversible isothermal mechanism whereby the Hamiltonian can be taken from any of the values attainable in item b to a multiple of the identity, with the extraction of work $W=-\Delta F$, where F is the Helmholtz free energy of qudit S. This process leads to a completely depolarized final state for S.

(e) A not-completely-depolarized D-dimensional qudit A, referred to as the *ancilla* or *input* qudit, with a mechanism either to be swapped with S or to hold the result of a generalized CNOT(S,A) operation [43,44], followed by a Von Neumann measurement of A in the computational basis, as explained for feedback versions in previous sections.

In addition, the engine must work in cycles, so that the initial and final states of qudit S are completely depolarized, the initial and final Hamiltonians are the same, and the final state of qudit A is also completely depolarized. Finally, we suppose that the machine can also implement an IT whereby the system qudit S with Hamiltonian H in a state $\rho\neq\rho_\beta(H)$ is put in thermal contact with $R(T)$; in this situation we assume that the equilibrium state $\rho_\beta(H)$ is reached without performing any work and keeping the Hamiltonian constant in the process.

After a swap or feedback operation, we assume a three-stage process, as depicted in Fig. 7, defined as follows.

(1) US, driven by a suitable choice of the control vector c, which takes qudit S from the initial state ρ_i to another one, ρ_c, chosen to maximize the extractable energy, as analyzed

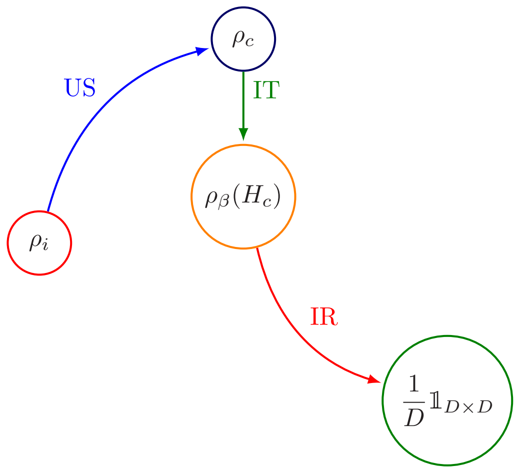

FIG. 7. (Color online) Schematic view of USITIR engine workflow. In the first stage, unitary steering (US), the input state ρ_i is unitarily steered to a state ρ_c with Hamiltonian H_c, from which an irreversible thermalization (IT) under a constant Hamiltonian and with no work extraction takes the system to the equillibrium state for Hamiltonian H_c, given by $\rho_\beta(H_c) = \frac{e^{-\beta H_c}}{Z}$. Finally, the Hamiltonian is progressively turned off in an isothermal process (IR), leading to a completely depolarized state.

in Sec. VI. If the control set allows it, ρ_c will be a thermal equilibrium state for one of the available Hamiltonians.

(2) IT, where the state is taken to thermal equilibrium without any work extraction and keeping the Hamiltonian constant. This stage is irrelevant if ρ_c is a thermal equilibrium state.

(3) IR, which takes the system reversibly and isothermally to the completely depolarized state under the initial Hamiltonian.

An information heat engine fitting this description is called a USITIR machine.

Given premises a–e (above), the three USITIR stages and their sequential order are unavoidable. Once the thermal equilibrium is reached, no work gain can be achieved by alternating unitary and isothermal steps. Unitary evolution must take the first turn and IT inevitably proceeds in between.

Qudit systems are introduced with the intention of naturally accommodating N particles, particulary, N-qubit systems with quantum statistics. One-qubit, two-qubit, or N-qubit MQIHEs of distinguishible particles fit into $D = 2$, $D = 4$, or $D = 2^N$-dimensional qudit USITIR machines, respectively. Moreover, N-bosonic qubits are naturally modeled with $D = (N+1)$–dimensional qudits, whereas a two-fermion qubit system is equivalent to a trivial one-dimensional qudit.

In statistical physics particle interactions are often neglected and all particles are assumed to interact with external Hamiltonians. For one-particle systems where the Hilbert space is d-dimensional, we define the control set $\mathcal{L}_1$ as the one formed by all $d^2 - 1$ generators of SU(d). Assuming N distinguishable particles with a d-dimensional Hilbert space for each, we define the *local independent* control set $\mathcal{L}_N$ as the union of all N control sets $\mathcal{L}_1^{(k)}$ for $k = 1, \ldots N$, each one acting on its part of the product space. We pay special attention to the case of $N = 2$ qubits, with $d = 2$, so that

$$\mathcal{L}_2 := \{\sigma_x^{(1)} \otimes \mathbb{1}^{(2)}, \sigma_y^{(1)} \otimes \mathbb{1}^{(2)}, \sigma_z^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes \sigma_x^{(2)}, \mathbb{1}^{(1)} \otimes \sigma_y^{(2)}, \mathbb{1}^{(1)} \otimes \sigma_z^{(2)}\}. \quad (12)$$

Hilbert spaces for bosonic or fermionic systems can be envisioned as subspaces of the distinguishable particles ones. The pure states that span the Hilbert subspace feature symmetry or antisymmetry with respect to two-particle swapping. On the other hand, Hamiltonians for identical particle systems without mutual interactions are always symmetric under the same operation. This implies that a bosonic or fermionic density matrix always remains bosonic or fermionic, respectively, under any unitary transformation built from the available control set. Consequently, for N identical bosonic or fermionic USITIR engines it may prove practical to describe the control set or the density matrices using a basis for the larger system, i.e., the one which corresponds to the same number of distinguishable particles.

As all identical particles are assumed to be indistinguishable, they experience the same Hamiltonians. We are especially interested in noninteracting situations and define a special control set $\mathcal{G}_N$ for systems of N identical particles, each having an individual d-dimensional Hilbert space. Working in the Nth tensor power of the one-particle Hilbert space, $\mathcal{G}_N$ is the *local common* control set of $d^2 - 1$ elements, each one being the sum $\sum_{k=1}^N \gamma_j^{(k)}$, where $\gamma_j^{(k)}$ is the jth generator of SU(d_1) acting on the k-factor space. For two-qubit systems, we have

$$\mathcal{G}_2 := \{\sigma_x^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \sigma_x^{(2)}, \sigma_y^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \sigma_y^{(2)}, \sigma_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \sigma_z^{(2)}\}. \quad (13)$$

A final type of control set for N-qubit systems is given by the singleton,

$$\mathcal{F}_N := \left\{\sum_{k=1}^N \sigma_z^{(k)}\right\}, \quad (14)$$

where $\sigma_z^{(k)}$ is the operator that performs a σ_z in the kth factor subspace. This control set will prove useful when considering SZEs, as explained in Sec. VI. We use the control sets $\mathcal{G}_N, \mathcal{F}_N$ in USITIR engines for both distinguishable and identical particles.

In the particular case where DMC and CT are fulfilled, that is, the control set $\mathcal{C}$ allows the engine to be steered to a thermal equilibrium state in the US stage, the machine is able to extract the maximum work compatible with thermodynamics and is said to be *optimal*; it will obtain a work given by the difference between the energy decrease in the US process and the free energy increase in the IR stage. Thus, the work obtained, bearing in mind that the increment of the energy in a full cycle must be null, reads

$$W = k_B T (\ln 2)(\log_2 D - S(\rho_i)), \quad (15)$$

where ρ_i is the initial state of A. According to the previous discussion, 1MQIHE, 2MQIHE, and, more generally, all DMC and CT machines are optimal. A general expression for the extractable work in nonoptimal USITIR engines is given in the next section.

VI. EXTRACTABLE WORK AND UNCONTROLLABLE ENTROPIES

As has been shown previously, the availability of control Hamiltonians that allow us to accomplish both DMC and CT guarantees the possibility of the extraction of an amount of work given by Eq. (15). However, this may not always be the case. It could happen that the control Hamiltonians fail to provide a reversible way to a thermal state for input density matrix ρ_i. We assume then that, at some point in the process, there must be an IT step with neither work extraction nor a change in the Hamiltonian. This irreversible stage contributes a work penalty W_p given by the work performed by a reversible process between the same initial and final states and the Hamiltonians, which is [25]

$$W_p = k_B T\,(\ln 2)\,S(\rho_1\,||\,\rho_2), \tag{16}$$

where ρ_1 and ρ_2 are the density matrices right before and immediately after the IT step, respectively. We may let ρ_1 take any value reachable from the input density matrix ρ_i with the available control set $\mathcal{C}$ and let ρ_2 vary along all thermal states defined with such control Hamiltonians. Let us then define the *uncontrollable entropy* of ρ_i under $\mathcal{C}$ by

$$S_u(\rho_i,\mathcal{C}) := \min_{\rho_1,\rho_2} S(\rho_1\,||\,\rho_2), \tag{17}$$

where minimization is taken over the previously defined ranges for ρ_1,ρ_2. This magnitude, always non-negative, is null for the 1MQIHE and 2MQIHE described in Sec. II C, on account of its DMC (see Sec. III) and CT (see Sec. IV).

According to definition (17), the maximum extractable work obtainable from an USITIR engine is

$$W(\rho_i,\mathcal{C}) = k_B T\,(\ln 2)\,(\log_2 D - S(\rho_i) - S_u(\rho_i,\mathcal{C})). \tag{18}$$

Next we analyze a control set $\mathcal{C}$ acting on a two-qubit system in two particular cases: (a) $\mathcal{C} = \mathcal{L}_2$ and (b) $\mathcal{C} = \mathcal{G}_2$.

In case a it is evident that the range for ρ_2 is the set of factorizable states and thus the uncontrollable entropy is the sum of the entropies of the reduced states minus that of the initial state (see Appendix C for extended calculation),

$$S_u(\rho_i,\mathcal{L}) = S\big(\rho_i^{(1)}\big) + S\big(\rho_i^{(2)}\big) - S(\rho_i), \tag{19}$$

where $\rho_i^{(1)}$, $\rho_i^{(2)}$ are the reduced states of qubits A_1,A_2 in the initial bipartite state ρ_i. For case b, ρ_2 ranges over all factorizable states with equal factors and thus the uncontrollable entropy is the sum of the entropies of the averaged reduced states minus that of the initial state (see Appendix C for extended calculation),

$$S_u(\rho_i,\mathcal{G}) = 2\,S\left(\frac{\rho_i^{(1)} + \rho_i^{(2)}}{2}\right) - S(\rho_i). \tag{20}$$

In feedback USITIR engines the situation is somewhat different, since the problems of controllability and thermalizability appear for each of the equally likely P possible outcomes $\rho_{i,\,p}$ $p = 1,\ldots P$ of the measurement, where $P = D$ if the state of qudit S before performing the CNOT(S,A) operation is completely depolarized. The analysis leads to several problems, each of them being a particular case of the swap machine situation. The extractable work results in an average of the results for the P possible outcomes, given by

$$W_{\rm Fb}(\rho_i,\mathcal{C}) = k_B T\,(\ln 2)\left(\log_2 D - S(\rho_i) - \frac{1}{P}\sum_{p=1}^{P} S_u(\rho_{i,\,p},\mathcal{C})\right). \tag{21}$$

Using Eq. (C17) in Appendix C for the uncontrollable entropy, one can conclude that when many-qubit machines have only local independent control Hamiltonians, i.e., the control set is $\mathcal{L}_N$, then they are optimal only if

$$S(\rho_i) = \sum_{k=1}^{N} S\big(\rho_i^{(k)}\big), \tag{22}$$

which is fulfilled only for a particular class of input sates, namely, those factorizable in their N partial states. The same result is shared by swap and feedback machines, since in the latter the factorizability is not affected by the possible NOT gates that link the measurement outcomes.

If, besides being single qubit, the control Hamiltonians are common to all qubits, i.e., the control set is $\mathcal{G}_N$, then according to the uncontrollable entropy given by Eq. (C19), the engine is optimal only if

$$S(\rho_i) = N\,S\left(\frac{1}{N}\sum_{k=1}^{N}\rho_i^{(k)}\right), \tag{23}$$

which, on account of the concavity of Von Neuman entropies, is satisfied only if ρ_i represents a factorizable state of equal components. For feedback machines this is impossible, because the internal qubits start from a completely depolarized state, and consequently, all postmeasurement states are equally likely.

The previous definitions of extractable work and uncontrollable entropy also apply to N-particle systems with quantum statistics, provided they are treated within the correct Hilbert space. Systems of N fermionic qubits are not very interesting: for $N = 1$ statistics play no role; and for $N = 2$ the Hilbert space is trivially unidimensional and no systems exist for $N > 2$. Accordingly, further attention is paid to the N-qubit bosonic cases only, which are naturally accommodated in $D = (N+1)$–dimensional qudit USITIR engines. In addition, a $\mathcal{F}_N$ control set is assumed. The relevant action of the US stage reduces to driving the energy difference between the two levels from an initial value to another one from where the IT process is launched. A derivation of the extractable work for N-bosonic qubits is given in Appendix E.

So far, we have considered that the quantum many-body states in the system of the USITIR engine are particle distinguishable. Now, we are going to derive the corresponding formulas of the optimal extractable work for identical particles in SZE engines and compare them with recent results [27,28]. Indeed, SZEs are the best-known information heat engines. A one-particle SZE is basically a hollow cylinder whose inner space is divided into two parts by a barrier B. The motion of B is externally controlled according to the outcomes of a measurement of the positions of the particles. The displacement of the barrier is a unitary process which starts at the center of the cylinder and ends at a suitably chosen position. Then, after the particles are allowed to redistribute, the barrier is restored to its initial position and a new cycle begins. Focusing on the unitary displacement of B, the temperature is presumed to be low enough to assume that the particle is in a superposition of the fundamental left and right states, $|L\rangle$ and $|R\rangle$, or a density matrix built on the space spanned by them. Thus, the system is reduced to a qubit for one particle or, more generally, to N qubits if there are N particles in the cylinder. As the barrier moves, the energies of $|L\rangle, |R\rangle$ change, the Hamiltonian of the system being a linear combination of the identity and σ_z operators. Moreover, the position of B is the only parameter that determines the Hamiltonian of the system. Therefore, this stage is equivalent to the US step of an N-qubit, $\mathcal{F}_N$ control set, USITIR engine. The last stage is an isothermal reversible restoration of the barrier to its initial stand, which matches exactly the IR step for the N-qubit, $\mathcal{F}_N$ USITIR machine. The transition from the first to the last stage may always proceed reversibly for the case where $N = 1$ particle. As for USITIR engines, every cycle yields an optimal $k_B T \ln 2$ work output. For more than one particle, if no further steering resources are available, the uncontrollable entropies computed in Appendixes D and E show that there must be another stage, called IT, in the USITIR setup, between US and IR, that contributes a work penalty that is also obtained in Appendixes D and E.

In order to draw some results for particular examples, we next consider the qubit feedback engine with $N = 2$ qubits, and with the $\mathcal{F}_2$ control set, under distinguishable bosonic and fermionic statistics. In feedback engines, if the particles are distinguishable, Eq. (21) for the obtainable work reads

$$W_{\rm Fb}(\rho_i,\mathcal{F}_2) = k_B T\,(\ln 2)\left(2 - \frac{1}{4}\sum_{p=1}^{4} S_u(\rho_{i,\,p},\mathcal{F}_2)\right). \quad (24)$$

In order to compute the uncontrollable entropy, we use the general expression given by Eq. (D7), obtained in Appendix D,

$$S_u(\rho_{i,\,p},\mathcal{F}_2) = 2\,S\left(\frac{\rho^{(1)}_{i,d,p} + \rho^{(2)}_{i,d,p}}{2}\right), \quad (25)$$

where $\rho^{(k)}_{i,d,p}$ is the kth reduced matrix corresponding to the pth measurement outcome, decohered[1] in the computational basis. Considering pure-state ancillae, which imply that after measuring qudit A, qudits A and S are in the same pure state, Eq. (25) evaluates to 0 when $\rho_{i,p}$ represents postmeasurement states in which the two particles are on the same side and to 2 when they are on different sides. Substituting into Eq. (24), we obtain

$$W_{\rm Fb}(\rho_i,\mathcal{F}_2) = k_B T\,(\ln 2), \quad (26)$$

in agreement with Ref. [27].

The fermionic case is trivial and the bosonic one can be easily discussed taking into account the results in Appendix E. According to Eq. (21) and reading from Table II, the maximum extractable work in the bosonic two-qubit system, when it is fed with pure-state ancillae, is

$$W_{\rm Fb}(\rho_i,\mathcal{F}_2) = \tfrac{2}{3}\,k_B T\,(\ln 3), \quad (27)$$

again in agreement with Ref. [27]. If more control Hamiltonians were available, the uncontrollable entropy could be made null and more work could be extracted. Assuming DMC and CT, the quantum SZE would yield the optimum value given by Eq. (15), which, if ancillas are fed in a pure state, reads

$$W_{\rm Fb}(\rho_i,\mathcal{F}_2) = k_B T\ \ln D, \quad (28)$$

where $D = 4$, 3, and 1 for distinguishable, bosonic, and fermionic two-qubit systems, respectively, in agreement with Ref. [28].

VII. CONCLUSIONS

This paper has presented a concept model for an MQIHE. As such, it extracts heat from a thermal reservoir and converts it into electrical work at the expense of increasing the entropy of input qubits. In addition, it features some interesting properties:

(1) It is scalable in the sense that it deals with information contained in any number of arbitrarily entangled qubits. For two-qubit systems a physical model has been proposed that saturates the thermodynamic limit. For a higher number of qubits it has been proved that many-body interactions are needed.

(2) The workflow for the swap model does not include any measurement or feedback system, unlike most modern descriptions of information heat engines. Nevertheless, a feedback version has been described to show that its study refers to the one without feedback.

(3) A generalization, called the USITIR engine, has been described that includes other well-known models.

In addition, we have described a general framework for analyzing the performance that can be expected from a wide class of QIHEs. It focuses on the set of control Hamiltonians that must be studied in two particular problems. The first one is assuring its DMC. The second one is more original and has been named CT. It refers to the search for a Hamiltonian, within the set of allowable ones, that defines a thermal equilibrium state at temperature T which is unitarily equivalent to the input state of the ancilla. It turns out that neither of the properties, CT and DMC, implies the other, as shown in Sec. V.

For nonoptimal machines, a measure of the penalty in the extractable work has been formulated as a function of a newly defined quantity: the *uncontrollable entropy*. The generalization presented in Sec. V predicts the limitation of engines with only one-qubit independent Hamiltonians or one-qubit common ones. Different kinds of quantum statistics effects have been shown to fit nicely into the general model. Detailed calculations developed in Appendixes C–E quantify their maximum extractable work and identify the cause of their limitation for N-qubit engines under several control sets and for different quantum statistics. The conclusion, as formulated at the end of Sec. VI, is that neither independent nor common local Hamiltonians allow optimal work extraction from arbitrary N-qubit inputs. Only for particular classes of inputs, as described in Sec. VI, can swap and feedback engines be optimal. However, when control Hamiltonians are common,

[1]The density matrix ρ *decohered* in the computational basis refers to the state $\rho_d := \sum_{k=1}^{D} |k\rangle\langle k|\,\rho\,|k\rangle\langle k|$, that is, the same ρ with all nondiagonal terms removed.

irrespective of the input states, no feedback engine can be optimal. This type of engine represents the case of many magnetic qubits in a common induction field or many particles in a cylinder, separated by the same barrier.

ACKNOWLEDGMENTS

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APPENDIX A: WORK EXTRACTION CALCULATION FOR A 1MQIHE

In this Appendix we proceed to obtain an expression for the electric power delivered to a battery in the different stages of a 1MQIHE. Then we use it to draw a final balance of energy in a full cycle.

We start from the reciprocity theorem [45], as is often done in the nuclear magnetic resonance literature. Let $\vec{B}_1 = B_1\vec{u}_z$ be the magnetic induction generated at the site of the spin atom, assuming that the qubit S is an atomic spin, by a unit current running through C, so that the magnetic field $\vec{B}$ at S reads $\vec{B} = I_C B_1 \vec{u}_z$. Then the electromotive force $\mathcal{E}'$ induced on C by the magnetic moment $\vec{\mu}$ of the atom is

$$\mathcal{E}' = -\frac{d(\vec{B}_1\cdot\vec{\mu})}{dt} = -B_1\frac{d\mu_z}{dt}, \tag{A1}$$

and the total electromotive force for circuit C is

$$\mathcal{E} = -L\frac{dI_C}{dt} + \mathcal{E}' \tag{A2}$$

Thus, the electric power supplied to the battery now reads

$$P_s = I_C\mathcal{E} = -LI_C\frac{dI_C}{dt} - I_C B_1\frac{d\mu_z}{dt} \tag{A3}$$

and taking into account that $B = I_C B_1$,

$$P_s = -\frac{dR}{dt} + \mu_z\frac{dB}{dt} \text{ with } R = \frac{1}{2}LI_C^2 + \mu_z B. \tag{A4}$$

According to Eq. (A4), the electric battery connected to the source keeps an energy in excess of its initial value given by

$$E_s = \int_0^t d\tau\, P_s(\tau) = -\Delta R + \int_0^B dB\,\mu_z(B), \tag{A5}$$

and in a cycle,

$$E_s = \oint dB\,\mu_z(B). \tag{A6}$$

Figure 8 shows the energy gain, which corresponds to the hatched (green) area, when the ancilla is fed in a state $\frac{1}{2}(\mathbb{1} + c\sigma_z)$, so that, after swapping, qubit S has a magnetic moment $\mu_{z,0} = c\mu_M$. Now we apply these expressions to the three stages described previously.

(i) At this stage, the I_C, B are constant and so P_s is 0. There is no power flowing from or into the battery and the function R is also 0.

(ii) Now the R function increases, implying a negative energy flow into the battery. There is also a positive contribution, $\mu_z dB$, which does not compensate for the negative one. The energy of the battery is incremented by $\Delta E_2 = -\Delta R + \mu_{z,0}B_f$.

(iii) At this stage the R component falls back to 0, restoring the negative contribution of the previous stage. There is also a negative term coming from the negative value of dB. As there is thermal equilibrium, μ_z depends on B. The state of qubit S is

$$\rho_\beta(-\mu_M B\,\sigma_z) = \frac{e^{\beta\mu_M B}\,|\uparrow\rangle\langle\uparrow| + e^{-\beta\mu_M B}\,|\downarrow\rangle\langle\downarrow|}{e^{-\beta\mu_M B} + e^{\beta\mu_M B}}, \tag{A7}$$

so that $\mu_z = \mu_M \tanh\beta\mu_M B$ and the energy stored in the battery is incremented by $\Delta E_3 = \Delta R - \frac{1}{\beta}\ln\cosh\beta\mu_M B_f$.

Adding all contributions, we have

$$\Delta E_s = \mu_{z,0}B_f - \frac{1}{\beta}\ln\cosh\beta\mu_M B_f. \tag{A8}$$

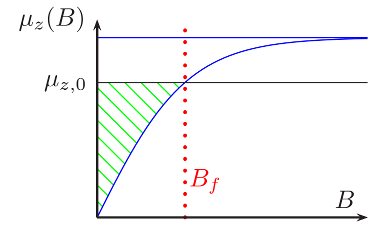

FIG. 8. (Color online) Brillouin magnetization curve for qubit S. The cycle of the 1MQIHE corresponds to a clockwise loop around the hatched (green) zone, whose area represents the extracted work.

In order to find the optimum value for B_f we derive the expression and immediately find that $\mu_z = \mu_M\tanh\beta\mu_M B_f$, so that the value of B_f is the one that causes the S qubit state $\rho_{S,1}$ to be in equilibrium with the reservoir at temperature T. This result could have been anticipated, as it is the only one that makes the process reversible. The resulting energy is

$$\Delta E_s = \mu_M B_f\,\tanh(\beta\mu_M B_f) - \frac{1}{\beta}\ln\cosh(\beta\mu_M B_f). \tag{A9}$$

Now we explore the relation of (A9) to the initial entropy of the ancilla, which is the same as that of $\rho_{S,1}$. Using the fact that B_f causes $\rho_{S,1}$ to be in thermal equilibrium with a reservoir at temperature T, we can express the entropy $S(\rho_{S,1})$ as a function of B_f,

$$S(\rho_{S,1}) = \frac{e^{\beta\mu_M B_f}}{Z\ln 2}(-\beta\mu_M B_f + \ln Z) + \frac{e^{-\beta\mu_M B_f}}{Z\ln 2}(\beta\mu_M\,B_f + \ln Z), \tag{A10}$$

where $Z = 2\cosh\beta\mu_M B_f$. Further calculations yield

$$S(\rho_{S,1}) = \frac{\ln(2\cosh\beta\mu_M B_f) - \beta\mu_M B_f\,\tanh\beta\mu_M B_f}{\ln 2}. \tag{A11}$$

Combining (A9) and (A11), we arrive at

$$\Delta E_s = \frac{\ln 2}{\beta}[1 - S(\rho_{S,1})], \tag{A12}$$

which is equal to $k_B T \ln 2$ times the entropy decrease in the reservoir, assuming that it is measured in bits.

APPENDIX B: CONTROLLABLE THERMALIZABILITY FOR A 2MQIHE

In this Appendix we show that the control set $\mathcal{C}_2$, defined in Eq. (10), acting on two-qubit engines, features CT, in addition to DMC. Considering that the spectrum of $\rho_\beta(H)$ is invariant under any transformation belonging to $\mathrm{SU}(2)^{(1)} \otimes \mathrm{SU}(2)^{(2)}$ and that any linear combination of the first (second) three operators of $\mathcal{C}_2$ can be reduced to a multiple of $H_3 = \sigma_z^{(1)} \otimes \mathbb{1}^{(2)}$ ($H_6 = \mathbb{1}^{(1)} \otimes \sigma_z^{(2)}$) by some such transformation, we can limit our search to linear combinations of H_3, H_6, H_7 that match the spectrum of the initial density matrix ρ_i up to an additive constant. This is equivalent to finding a linear combination of $H_3, H_6, H_7, \mathbb{1}_{4\times4}$ which is unitarily equivalent to ρ_i.

It proves useful to write the Hamiltonian as

$$H = c_1 \frac{\mathbb{1}_{4\times4} - H_3}{2} + c_2 \frac{\mathbb{1}_{4\times4} - H_6}{2} + c_3 \frac{\mathbb{1}_{4\times4} - H_7}{2} + c_4 \mathbb{1}_{4\times4} \tag{B1}$$

and solve for c_1, c_2, c_3, c_4 under the condition that H and ρ_i share the same spectrum. In matrix form, the Hamiltonian reads

$$H = c_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B2}$$

whose eigenvalues are

$$c_4, c_4 + c_1 + c_2, \quad c_4 + \frac{c_1 + c_2 + 2c_3}{2} \pm \frac{\sqrt{(c_1 + c_2 - 2c_3)^2 - 4c_1 c_2}}{2}, \tag{B3}$$

henceforth referred to as $\lambda_1(c)$, $\lambda_2(c)$, $\lambda_3(c)$, $\lambda_4(c)$, respectively, with $c = (c_1, c_2, c_3, c_4)$. They should match the eigenvalues of the density matrix ρ_i, which are denoted $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4$, ordered as $\mathfrak{r}_1 \leqslant \mathfrak{r}_2 \leqslant \mathfrak{r}_3 \leqslant \mathfrak{r}_4$ or any permutation thereof. Next we show that there always exists at least one real solution for c_1, c_2, c_3, c_4 in the system of equations defined by

$$\lambda_1(c) = \mathfrak{r}_2, \quad \lambda_2(c) = \mathfrak{r}_3, \quad \lambda_3(c) = \mathfrak{r}_1, \quad \lambda_4(c) = \mathfrak{r}_4. \tag{B4}$$

This can be inferred by considering the solution $c_4 = \mathfrak{r}_2$, $c_3 = \frac{\mathfrak{r}_1 + \mathfrak{r}_4 - \mathfrak{r}_2 - \mathfrak{r}_3}{2}$, and c_1, c_2 determined by the system:

$$c_1 + c_2 = \mathfrak{r}_3, \quad c_1 - c_2 = \pm\sqrt{(c_4 - c_1)^2 - (\mathfrak{r}_1 + \mathfrak{r}_4 - \mathfrak{r}_2 - \mathfrak{r}_3)^2}. \tag{B5}$$

Equation (B5) defines a real solution for c_1, c_2, iff

$$(\mathfrak{r}_1 - \mathfrak{r}_4)^2 \geqslant (\mathfrak{r}_1 + \mathfrak{r}_4 - \mathfrak{r}_2 - \mathfrak{r}_3)^2, \tag{B6}$$

that, taking into account the order $\mathfrak{r}_1 \leqslant \mathfrak{r}_2 \leqslant \mathfrak{r}_3 \leqslant \mathfrak{r}_4$, is always fulfilled, proving the CT of the swap version of the 2MQIHE.

It is straightforward that the previous result extends also to the feedback version of the 2MQIHE, on account of its DMC and realizing that all four postmeasurement states are unitarily equivalent.

APPENDIX C: UNCONTROLLABLE ENTROPIES FOR SYSTEMS WITH LOCAL CONTROL HAMILTONIANS

In this Appendix we present a derivation of the uncontrollable entropies for two-qubit systems under the control sets $\mathcal{L}_2, \mathcal{G}_2$, as defined in Sec. VI. A generalization to N-qubit systems follows immediately.

First, we study the case $\mathcal{C} = \mathcal{L}_2$ in a USITIR engine. At the IT stage no work is assumed to be interchanged. Thus, work is only accounted for in the first and the last stages. For the first one we can write

$$W_1 = \mathrm{Tr}\,\{\rho_i H_i - \rho_c H_c\}, \tag{C1}$$

where ρ_c and H_c are the density matrix and the Hamiltonian at the end of the first stage. Recalling that the Hamiltonians allowed by $\mathcal{L}_2$ can be decomposed as

$$H = H^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes H^{(2)}, \tag{C2}$$

we can rewrite Eq. (C1) as

$$W_1 = \mathrm{Tr}\left\{\rho_i H_i - \rho_c^{(1)} H_c^{(1)} - \rho_c^{(2)} H_c^{(2)}\right\}. \tag{C3}$$

The work extracted at the last stage, taking into account that it is an isothermal process and decomposition, Eq. (C2), is

$$W_2 = \mathrm{Tr}\left\{\rho_\beta\big(H_c^{(1)}\big) H_c^{(1)} + \rho_\beta\big(H_c^{(2)}\big) H_c^{(2)} - \rho_i H_i\right\} + k_B T\,(\ln 2)\,[S(\rho_f) - S(\rho_d)], \tag{C4}$$

where $\rho_d = \rho_\beta(H_c^{(1)}) \otimes \rho_\beta(H_c^{(2)})$ and $\rho_f = \frac{1}{4}\,\mathbb{1}^{(1)} \otimes \mathbb{1}^{(2)}$ are the density matrices at the beginning and the end of the last stage, respectively.

Adding Eq. (C3) and Eq. (C4) we obtain the total work:

$$W = \mathrm{Tr}\left\{\big[\rho_\beta\big(H_c^{(1)}\big) - \rho_c^{(1)}\big]H_c^{(1)} + \big[\rho_\beta\big(H_c^{(2)}\big) - \rho_c^{(2)}\big]H_c^{(2)}\right\} + k_B T\big[S\big(\rho_f^{(1)}\big) + S\big(\rho_f^{(2)}\big) - S\big(\rho_\beta\big(H_c^{(1)}\big)\big) - S\big(\rho_\beta\big(H_c^{(2)}\big)\big)\big]\ln 2. \tag{C5}$$

Note the symmetry for the first and second qubits. Next we elaborate further on the part which refers to the first qubit,

$$W^{(1)} = \mathrm{Tr}\left\{\big[\rho_\beta\big(H_c^{(1)}\big) - \rho_c^{(1)}\big] H_c^{(1)}\right\} + k_B T\big[S\big(\rho_f^{(1)}\big) - S\big(\rho_\beta\big(H_c^{(1)}\big)\big)\big]\ln 2, \tag{C6}$$

where, considering that $H_c^{(1)} = -k_B T(\log_2 Z_1 \mathbb{1}^{(1)} + \log_2 \rho_\beta(H_c^{(1)}) \ln 2$, we arrive at

$$\frac{W^{(1)}}{k_B T \ln 2} = \mathrm{Tr}\left\{\big[-\rho_\beta\big(H_c^{(1)}\big) + \rho_c^{(1)}\big] \log_2 \rho_\beta\big(H_c^{(1)}\big)\right\} - \mathrm{Tr}\big\{\big[\rho_f^{(1)} \log_2\big(\rho_f^{(1)}\big) - \rho_\beta\big(H_c^{(1)}\big) \log_2 \rho_\beta\big(H_c^{(1)}\big)\big]\big\}, \tag{C7}$$

which can be written as

$$\frac{W^{(1)}}{k_B T \ln 2} = S\big(\rho_f^{(1)}\big) - S\big(\rho_i^{(1)}\big) - S\big(\rho_c^{(1)}||\rho_\beta\big(H_c^{(1)}\big)\big). \quad \text{(C8)}$$

Maximization of $W^{(1)} + W^{(2)}$ can proceed independently for each term. Considering that $\mathcal{L}_2$ allows complete controllability over each of the qubits independently and that relative entropies are non-negative, the maximum work is

$$W(\rho_i,\mathcal{L}_2) = k_B T\,(\ln 2)\big[2 - S\big(\rho_i^{(1)}\big) - S\big(\rho_i^{(2)}\big)\big], \quad \text{(C9)}$$

which determines the uncontrollable entropy:

$$S_u(\rho_i,\mathcal{L}_2) = S\big(\rho_i^{(1)}\big) + S\big(\rho_i^{(2)}\big) - S(\rho_i). \quad \text{(C10)}$$

When the control set is $\mathcal{G}_2$, as defined in Eq. (13), all the previous derivation holds until Eq. (C8). Now maximization must be made for the whole work $W = W^{(1)} + W^{(2)}$. With this goal we write the expression for it:

$$\frac{W}{k_B T \ln 2} = S(\rho_f) - S\big(\rho_i^{(1)}\big) - S\big(\rho_i^{(2)}\big) - S\big(\rho_c^{(1)}\big|\big|\rho_\beta(H_c)\big) - S\big(\rho_c^{(2)}\big|\big|\rho_\beta(H_c)\big), \quad \text{(C11)}$$

where any superindex for a magnitude which is equal for both subsystems has been suppressed. After expansion of the expressions for the relative entropies and simplification we arrive at

$$\frac{W}{k_B T \ln 2} = S(\rho_f) + \mathrm{Tr}\,\big\{\big[\rho_c^{(1)} + \rho_c^{(2)}\big]\log_2 \rho_\beta(H_c)\big\}, \quad \text{(C12)}$$

which can be rewritten as

$$\frac{W}{2k_B T \ln 2} = S\big(\rho_f^{(1)}\big) - S(\rho_a) - S(\rho_a||\rho_\beta(H_c)), \quad \text{(C13)}$$

where $\rho_a = \frac{1}{2}(\rho_c^{(1)} + \rho_c^{(2)})$. Under $\mathcal{G}_2$, we have complete control over $\frac{1}{2}(\rho^{(1)} + \rho^{(2)})$, and consequently, the relative entropy can be made null. The extractable work can thus be written as

$$W(\rho_i,\mathcal{G}_2) = 2\,k_B T\,(\ln 2)\left[1 - S\left(\frac{\rho_i^{(1)} + \rho_i^{(2)}}{2}\right)\right], \quad \text{(C14)}$$

which determines the uncontrollable entropy:

$$S_u(\rho_i,\mathcal{G}_2) = 2\,S\left(\frac{\rho_i^{(1)} + \rho_i^{(2)}}{2}\right) - S(\rho_i). \quad \text{(C15)}$$

Following the previous derivation, it is straightforward to extend these results to the case of N qubits under a control set of local independent $\mathcal{L}_N$ or common $\mathcal{G}_N$ control sets. We write the results directly:

$$W(\rho_i,\mathcal{L}_N) = k_B T\,(\ln 2)\left[N - \sum_{k=1}^{N} S\big(\rho_i^{(k)}\big)\right], \quad \text{(C16)}$$

$$S_u(\rho_i,\mathcal{L}_N) = -S(\rho_i) + \sum_{k=1}^{N} S\big(\rho_i^{(k)}\big), \quad \text{(C17)}$$

$$W(\rho_i,\mathcal{G}_N) = N\,k_B T\,(\ln 2)\left[1 - S\left(\frac{1}{N}\sum_{k=1}^{N}\rho_i^{(k)}\right)\right], \quad \text{(C18)}$$

$$S_u(\rho_i,\mathcal{G}_N) = N\,S\left(\frac{1}{N}\sum_{k=1}^{N}\rho_i^{(k)}\right) - S(\rho_i). \quad \text{(C19)}$$

APPENDIX D: UNCONTROLLABLE ENTROPIES FOR N DISTINGUISHABLE QUBITS UNDER $\mathcal{F}_N$ CONTROL SETS

In this Appendix we assume that a USITIR engine works with a system of N distinguishable qubits, under the control set $\mathcal{F}_N$ defined in Eq. (14). We deliberately dedicate an Appendix to this problem to facilitate comparison with the bosonic case, treated in the next section. We work in the N^2-dimensional Hilbert space spanned by the tensor product of the $\sigma_z^{(k)}$, $k = 1,\dots,N$, operator eigenstates.

We intend to find the uncontrollable entropy $S_u(\rho_i,\mathcal{F}_N)$ as defined in Sec. VI. Accordingly, we have to find the minimum of the relative entropy of ρ_1 with respect to ρ_2, where ρ_1 ranges over all possible states connected to ρ_i via an unitary operator U_1 built from the control set $\mathcal{F}_N = \{\sum_k \sigma_z^{(k)}\}$, while ρ_2 may vary over the Gibbs state $\rho_2(h) := e^{F_N h}\,Z^{-1}(h)$, where h is a real constant and $Z(h) = (e^h + e^{-h})^N$. The function to be minimized is

$$S(\rho_2||\rho_1) = -S(U_1\rho_i U_1^\dagger) - \mathrm{Tr}\,\{U_1\rho_i U_1^\dagger\,\log_2\,\rho_2(h)\}. \quad \text{(D1)}$$

The first term is the Von Neumann entropy of ρ_i and does not change under unitary transformations. In addition, U_1 and the $\rho_2(h)$ Gibbs state commute, so that, considering the cyclicity of the trace operation, we conclude that the result does not depend on U_1. Therefore, we are left with the maximization of the term given by

$$J(h) := \mathrm{Tr}\,\{\rho_i\,\log_2\,\rho_2(h)\}, \quad \text{(D2)}$$

on account of the factorizability of $\rho_2(h)$; it follows that

$$J(h) = \sum_{k=1}^{N}\mathrm{Tr}\,\big\{\rho_i^{(k)}\,\log_2\,\rho_2^{(k)}(h)\big\}, \quad \text{(D3)}$$

where $\rho_2^{(k)}(h) = Z(h)^{1/N}\,e^{h\sigma_z}$ does not depend on k. Moreover, $\rho_2^{(k)}(h)$ is diagonal in the computational basis, so that only the decohered $\rho_{i,d}^{(k)}$ part of $\rho_i^{(k)}$ contributes, where $(\rho_{i,d}^{(k)})_{m,n} := \delta_{m,n}\,(\rho_i^{(k)})_{m,n}$. With these considerations, $J(h)$ reads

$$J(h) = N\mathrm{Tr}\,\big\{\rho_{i,d}\,\log_2\,\rho_2^{(k)}(h)\big\}, \quad \text{(D4)}$$

where

$$\rho_{i,d} := \frac{1}{N}\sum_{k=1}^{N}\rho_{i,d}^{(k)}, \quad \text{(D5)}$$

which allows us to rewrite $J(h)$ as

$$J(h) = -NS\big(\rho_{i,d}||\rho_2^{(1)}(h)\big) - S(\rho_{i,d}), \quad \text{(D6)}$$

so that the relative entropy, which is always non-negative, can be made null. The right strategy is tuning the h parameter to

TABLE I. Values for the uncontrollable entropy $S_u(\rho_i,\mathcal{F}_2)$ and maximum extractable work $W(\rho_i,\mathcal{F}_2)$ in a distinguishable two-qubit system for different input states ρ_i.

ρ_i	$S_u(\rho_i,\mathcal{F}_2)$	$W(\rho_i,\mathcal{F}_2)$
$\|00\rangle\langle 00\|$	0	$k_B T \ln 4$
$\|10\rangle\langle 10\|$	$k_B T \ln 4$	0
$\|01\rangle\langle 01\|$	$k_B T \ln 4$	0
$\|11\rangle\langle 11\|$	0	$k_B T \ln 4$

match the expected value of F_N for the input and the Gibbs states. The uncontrollable entropy then results,

$$S_u(\rho_i,\mathcal{F}_N) = N S(\rho_{i,d}) - S(\rho_i), \tag{D7}$$

and the maximum extractable work is

$$W = k_B T\,(\ln 2)\,N\,(1 - S(\rho_{i,d})). \tag{D8}$$

Some values of uncontrollable entropies and maximum extractable works for $N = 2$ that are used in this paper are listed in Table I.

APPENDIX E: UNCONTROLLABLE ENTROPIES FOR N BOSONIC QUBITS

In this Appendix we assume that a USITIR engine works with a system of N indistinguishable qubits, obeying bosonic statistics, under the control set $\mathcal{F}_N$ defined in Eq. (14). Much of the exposition goes along the same lines as in Appendix D, but we prefer to repeat some considerations in order to improve readability. We work in the $(1+N)-$dimensional Hilbert space spanned by the basis $\mathcal{B}_N = \{|0\rangle,|1\rangle,\ldots,|N\rangle\}$, whose vectors represent states with well-defined values of the occupation numbers in the σ_z operator eigenstates. That is, $|n\rangle$ is the state that corresponds to n bosons in the $\sigma_z = 1$ subspace and $N - n$ in the $\sigma_z = -1$. In the $\mathcal{B}_N$ basis the matrix form of the only operator F_N in the control set reads

$$F_N := \begin{pmatrix} -N & 0 & \cdots & 0 \\ 0 & -(N-2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & N \end{pmatrix}. \tag{E1}$$

Our first task is to find the uncontrollable entropy $S_u(\rho_i,\mathcal{F}_N)$ as defined in Sec. VI. Accordingly, we have to find the minimum of the relative entropy of ρ_1 with respect to ρ_2, where ρ_1 ranges over all possible states connected to ρ_i via a unitary operator U_1 built from $\mathcal{F}_N$ and ρ_2 may vary over the Gibbs

TABLE II. Values for the uncontrollable entropy $S_u(\rho_i,\mathcal{F}_2)$ and maximum extractable work $W(\rho_i,\mathcal{F}_2)$ in two-qubit bosonic systems for different input states ρ_i expressed in the occupation number basis.

ρ_i	$S_u(\rho_i,\mathcal{F}_2)$	$W(\rho_i,\mathcal{F}_2)$
$\|0\rangle\langle 0\|$	0	$k_B T\ \ln 3$
$\|1\rangle\langle 1\|$	$k_B T\ \ln 3$	0
$\|2\rangle\langle 2\|$	0	$k_B T\ \ln 3$

state $e^{F_N h} Z^{-1}(h)$, where h is a real constant. The function to be minimized is

$$S(\rho_2||\rho_1) = -S(U_1\rho_i U_1^\dagger) - \mathrm{Tr}\left\{U_1\rho_i U_1^\dagger \log_2 \frac{e^{F_N h}}{Z(h)}\right\}. \tag{E2}$$

The first term is the Von Neumann entropy of ρ_i and cannot be minimized. In addition, U_1 and the ρ_2 Gibbs state commute, so that, considering the cyclicity of the trace operation, we conclude that the result does not depend on U_1. Therefore, we are left with the maximization of the term given by

$$J(h) := \mathrm{Tr}\left\{\rho_i \log_2 \frac{e^{F_N h}}{Z(h)}\right\}, \tag{E3}$$

which can be easily evaluated in the $\mathcal{B}_N$ basis. If ρ_{jk} are the the components of ρ_i in $\mathcal{B}_N$, $J(h)$ reads

$$J(h) = -\log_2 Z(h) + \frac{1}{\ln 2}\sum_{j=1}^{N+1}\rho_{jj}F_{N,jj}h. \tag{E4}$$

Now we equate its derivative to 0 in order to find the optimum h^* value for h,

$$\sum_{j=1}^{N+1}\rho_{jj}F_{N,jj} = \frac{Z'(h^*)}{Z(h^*)}, \tag{E5}$$

which, in other words, states that the system must evolve unitarily until reaching a Hamiltonian for which the expected occupation numbers of the Gibbs and the input states are equal.

As a function of h^*, the uncontrollable entropy is

$$S_u(\rho_i,\mathcal{F}_N) = -J(h^*) - S(\rho_i), \tag{E6}$$

and the maximum extractable work is

$$W = k_B T\,(\ln 2)\,(\log_2(N+1) + J(h^*)). \tag{E7}$$

Some values of uncontrollable entropies and maximum extractable works for $N = 2$ that are used in the paper are listed in Table II.

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