

Entanglement in fermion systems and quantum metrologyF. Benatti,^{1,2} R. Floreanini,² and U. Marzolino³¹*Dipartimento di Fisica, Università di Trieste, I-34151 Trieste, Italy*²*Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34151 Trieste, Italy*³*Department of Physics, University of Ljubljana, 1000 Ljubljana, Slovenia*

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Entanglement in fermion many-body systems is studied using a generalized definition of separability based on partitions of the set of observables, rather than on particle tensor products. In this way, the characterizing properties of nonseparable fermion states can be explicitly analyzed, allowing a precise description of the geometric structure of the corresponding state space. These results have direct applications in fermion quantum metrology: Sub-shot-noise accuracy in parameter estimation can be obtained without the need of a preliminary state entangling operation.

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I. INTRODUCTION

In trying to apply the standard definition of separability and entanglement to systems of identical particles, one immediately faces a problem: The indistinguishability of the system constituents conflicts with Hilbert space tensor product structures on which these notions are based. The point is that the particles are identical and therefore they can be neither singly addressed, nor their individual properties measured: Only collective, global system operators are in fact admissible, experimentally accessible observables [1,2].

This observation unavoidably leads to a radical change in perspective concerning the attitude towards the notion of entanglement in general: The presence of quantum correlations in any physical system is less signaled by *a priori* properties of the system states, than by those of the algebra of the system observables and by the behavior of the associated correlation functions. In other terms, the usually adopted definition of separability based on the particle aspect of first quantization appears to be too restrictive, leading possibly to misleading results; rather, it should be replaced by one directly emerging from the second quantized description, usually adopted for studying many-body systems.¹

This new approach to separability and entanglement has been advocated before [26–29], but formalized only recently [30–36]. So far the focus has been on bosonic systems, with particular attention to bipartite entanglement, aiming at specific applications to quantum metrology. Suitable criteria able to detect nonclassical correlations in systems with a fixed number of elementary bosonic constituents have been discussed. In particular, it has been found that in general the operation of partial transposition [37,38] gives rise to a much more exhaustive criterion for detecting bipartite entanglement than in the case of distinguishable particles [32,33]. This allows obtaining a rather complete classification of the structure of bipartite entangled states in systems composed by a fixed number of bosons [33,34]. Furthermore, entangled bosonic

states turn out to be much more robust than distinguishable particle ones against mixing with other states and an explicit expression for the so-called “robustness” [39,40] has been derived [34]. In this way, a general characterization of the geometry of the space of bosonic states can be given, that is indeed much richer than in the case of systems of distinguishable constituents.

In the following, we shall extend the study of the notions of separability and entanglement to the case of systems composed of fermions following the lines previously adopted for bosons. In this case, the elementary creation and annihilation operators associated with the fermion system constituents satisfy an algebra given in terms of anticommutation relations: This poses new questions regarding the connection between the properties of locality and commutativity of the system observables, making the theory of fermion entanglement even richer than in the case of bosonic systems.

Application to quantum metrology using fermion systems will also be discussed: As in the case of bosonic systems, also in the case of fermion ones it will be explicitly shown that sub-shot-noise accuracy in parameter estimation can be achieved without the need of feeding the measuring apparatus with entangled states; the required nonlocality can be provided by the apparatus itself. These results may have practical implications in interferometric experiments using ultracold fermion gases.

II. ENTANGLEMENT IN MULTIMODE FERMION SYSTEMS

We shall consider generic fermion many-body systems made of N elementary constituents that can occupy M different orthogonal states or modes, $N < M$. This is a quite general model that can accommodate various physical situations in atomic and condensed matter physics; in particular, it can be used to describe the behavior of ultracold gases confined in multisite optical lattices, that are becoming so relevant in the study of quantum many-body phenomena (e.g., see [20–23,41–51], and references therein).

A many-body system made of identical particles is usually described by means of creation and annihilation operators, a_i^\dagger , a_i , for each of the M modes that the particles can occupy, $i = 1, 2, \dots, M$ [52,53]. Fermion particles are characterized

¹Entanglement in many-body systems has been widely discussed in the recent literature, e.g., see [3–25]; however, for the reasons just pointed out, only a limited part of those results are really applicable to the case of identical particle systems.

by the fact that the operators a_i^\dagger, a_i , obey the canonical anticommutation relations,

$$\{a_i, a_j^\dagger\} \equiv a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \quad (1)$$

The total Hilbert space \mathcal{H} of the system is then spanned by the many-body Fock states, obtained by applying creation operators to the vacuum:

$$|n_1, n_2, \dots, n_M\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_M^\dagger)^{n_M} |0\rangle, \quad (2)$$

the integers n_1, n_2, \dots, n_M representing the occupation numbers of the different modes; due to (1), they can take only the two values 0 or 1. Since the number of fermions N is fixed, the total number operator $\sum_{i=1}^M a_i^\dagger a_i$ is a conserved quantity and the occupation numbers must satisfy the additional constraint $\sum_{i=1}^M n_i = N$; in other words, all states must contain exactly N particles and the dimension D of the system Hilbert space \mathcal{H} is then $D = \binom{M}{N}$. In addition, the set of polynomials in all creation and annihilation operators, $\{a_i^\dagger, a_i \mid i = 1, 2, \dots, M\}$, form an algebra that, together with its norm closure, coincides with the algebra $\mathcal{A}(\mathcal{H})$ of bounded operators acting on \mathcal{H} ; the observables of the systems are part of this algebra.

As mentioned in the introductory remarks, in this framework the natural interpretation of entanglement in terms of particle correlations has to be rethought. For instance, in the case of a system composed by two standard, distinguishable qubits, the natural Hilbert space product structure $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and the corresponding algebraic product structure for the space of the associated observables $\mathcal{A} = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, with $M_2(\mathbb{C})$ the set of 2×2 complex matrices, immediately identify the local observables as the one taking the form,

$$A \otimes B = (A \otimes 1)(1 \otimes B), \quad (3)$$

where A is an observable of the first qubit, while B that for the second one. In other terms, local observables for the two-qubit systems are characterized by being tensor products of observables pertaining each to one of the two parties: They commute and are thus algebraically independent.

Consider instead a system composed by two fermions that can occupy two modes, and thus described by the set of operators $(a_1, a_1^\dagger, a_2, a_2^\dagger)$: The single particle Hilbert space is still \mathbb{C}^2 ; the difference with respect to the qubit case is that the total Hilbert space \mathcal{H} is now one-dimensional, containing just one Fock vector, namely $a_1^\dagger a_2^\dagger |0\rangle$. In the language of first quantization, this corresponds to the fact that only antisymmetric states are allowed due to the Fermi statistics, and this is automatically enforced in the second quantized language due to the algebra in (1). Furthermore, the algebra \mathcal{A} of operators is linearly generated by the identity together with at most second-order monomials in a_1, a_1^\dagger and a_2, a_2^\dagger . In this case, the particle Hilbert space tensor product structure is lost, reflecting the fact that the two particles are indistinguishable. Similarly, also the usual notion of local observables, the one based on the tensor product structure as in (3), is no longer available and needs to be reformulated.

In dealing with systems of identical particles, it is natural to define the notion of bipartite entanglement by the presence of nonclassical correlations among averages of operators. It is

then convenient to start with the following general definitions, valid for both boson and fermion systems.

Definition 1. An **algebraic bipartition** of the operator algebra $\mathcal{A}(\mathcal{H})$ is any pair $(\mathcal{A}_1, \mathcal{A}_2)$ of subalgebras of $\mathcal{A}(\mathcal{H})$ generated by disjoint subsets of modes, namely $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}(\mathcal{H})$, $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathbf{1}$.

In general the two subalgebras \mathcal{A}_1 and \mathcal{A}_2 need not reproduce the whole algebra $\mathcal{A}(\mathcal{H})$, i.e., $\mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{A}(\mathcal{H})$; however, in the cases of partitions defined in terms of modes, as discussed below, one has $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}(\mathcal{H})$.

Any algebraic bipartition encodes in a natural way the definition of the system local observables.

Definition 2. An element (operator) of $\mathcal{A}(\mathcal{H})$ is said to be $(\mathcal{A}_1, \mathcal{A}_2)$ **local**, i.e., local with respect to a given bipartition $(\mathcal{A}_1, \mathcal{A}_2)$, if it is the product $A_1 A_2$ of an element A_1 of \mathcal{A}_1 and another A_2 in \mathcal{A}_2 .

From this notion of operator locality, a natural definition of state separability and entanglement follows [30].

Definition 3. A state ω on the algebra $\mathcal{A}(\mathcal{H})$ will be called **separable** with respect to the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ if the expectation $\omega(A_1 A_2)$ of any local operator $A_1 A_2$ can be decomposed into a linear convex combination of products of expectations:

$$\omega(A_1 A_2) = \sum_k \lambda_k \omega_k^{(1)}(A_1) \omega_k^{(2)}(A_2),$$

$$\lambda_k \geq 0, \quad \sum_k \lambda_k = 1, \quad (4)$$

where $\omega_k^{(1)}$ and $\omega_k^{(2)}$ are given states on $\mathcal{A}(\mathcal{H})$; otherwise the state ω is said to be **entangled** with respect the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$.²

Remark 1. (i) This generalized definition of separability can be easily extended to the case of more than two partitions, by an appropriate, straightforward generalization; specifically, in the case of an n partition, Eq. (4) would extend to

$$\omega(A_1 A_2 \dots A_n) = \sum_k \lambda_k \omega_k^{(1)}(A_1) \omega_k^{(2)}(A_2) \dots \omega_k^{(n)}(A_n),$$

$$\lambda_k \geq 0, \quad \sum_k \lambda_k = 1. \quad (5)$$

(ii) As already observed before, in systems of identical particles there is no *a priori* given, natural partition to be used for the definition of separability; therefore, issues about entanglement and nonlocality are meaningful *only* with reference to a choice of a specific partition in the operator algebra [26–35]; this general observation, often overlooked, is at the origin of much confusion in the recent literature. ■

In the language of second quantization introduced before for the description of fermion systems, these general definitions can be made more explicit. A bipartition of the M modes fermion algebra $\mathcal{A}(\mathcal{H})$ can be given by splitting

²In general, a state ω is a normalized, positive, linear functional on $\mathcal{A}(\mathcal{H})$, such that the average of any observable \mathcal{O} can be expressed as the value taken by ω on it, $\langle \mathcal{O} \rangle = \omega(\mathcal{O})$; a standard representation of this expectation value map is given by the trace operation over density matrices.

the collection of creation and annihilation operators into two disjoint sets, $\{a_i^\dagger, a_i \mid i = 1, 2, \dots, m\}$ and $\{a_j^\dagger, a_j \mid j = m + 1, m + 2, \dots, M\}$; it is thus uniquely determined by the choice of the integer m , with $0 \leq m \leq M$.³

All polynomials in the first set (together with their norm closures) form a subalgebra \mathcal{A}_1 , while the remaining set analogously generates a subalgebra \mathcal{A}_2 . Due to the anticommutation relations (1), the two subalgebras $\mathcal{A}_1, \mathcal{A}_2$ do not in general commute. Nevertheless, from the algebraic relations,

$$[AB, C] = A\{B, C\} - \{A, C\}B,$$

it follows that all even powers of elements in \mathcal{A}_1 (\mathcal{A}_2) commute with all elements of \mathcal{A}_2 (\mathcal{A}_1). It is then convenient to introduce the following definition:

Definition 4. Let Θ be the automorphism on the Fermi algebra \mathcal{A} defined by $\Theta(a_i) = -a_i$, $\Theta(a_i^\dagger) = -a_i^\dagger$ for all $a_i, a_i^\dagger \in \mathcal{A}(\mathcal{H})$.⁴ The even component \mathcal{A}^e of \mathcal{A} is the subset of elements $A^e \in \mathcal{A}$ such that $\Theta(A^e) = A^e$, while the odd component \mathcal{A}^o of \mathcal{A} consists of those elements $A^o \in \mathcal{A}$ such that $\Theta(A^o) = -A^o$.

Notice that the even component \mathcal{A}^e is the algebra generated by even polynomials in creation and annihilation operators, while the odd component \mathcal{A}^o is just a linear space, but not an algebra, since the product of two odd elements is even.

Similarly, given the algebraic bipartition $(\mathcal{A}_1, \mathcal{A}_2)$, one can define the even \mathcal{A}_i^e and odd \mathcal{A}_i^o components of the two subalgebras \mathcal{A}_i , $i = 1, 2$. Only the operators of the first partition belonging to the even component \mathcal{A}_1^e commute with any operator of the second partition and, similarly, only the even operators of the second partition commute with the whole subalgebra \mathcal{A}_1 .

Coming now back to the notion of separability introduced in Definition 3, one may notice that there is a difference between bosonic and fermionic systems. In the bosonic case, the two subalgebras $\mathcal{A}_1, \mathcal{A}_2$ defining the algebraic bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ naturally commute, i.e., that each element A_1 of the operator algebra \mathcal{A}_1 commutes with any element A_2 in \mathcal{A}_2 . Instead, in the case of fermion systems, as already observed the two subalgebras $\mathcal{A}_1, \mathcal{A}_2$ do not in general commute. Nevertheless, in such systems only self-adjoint operators belonging to the even components $\mathcal{A}_1^e, \mathcal{A}_2^e$ qualify as physical observables and these do commute.

At this point, two different attitudes are possible regarding the definition of separability expressed by the condition (4): (i) use in it all operators from the two subalgebras $\mathcal{A}_1, \mathcal{A}_2$, as assumed in Definition 3 above; (ii) restrict all considerations to observables only. The first approach is in line with the notion of ‘‘microcausality’’ adopted in *constructive* quantum field theory [54,55], where the emphasis is on quantum fields, which are required either to commute (boson fields) or anticommute (fermion fields) if defined on (causally) disjoint regions. On the other hand, the second point of view reminds one of the notion

³There is no loss of generality in assuming the modes forming the two partitions to be contiguous; if in the chosen bipartition this is not the case, one can always relabel the modes in such a way to achieve this convenient ordering.

⁴In other terms, Θ is a linear map on \mathcal{A} preserving the algebra relations, i.e., $\Theta(AB) = \Theta(A)\Theta(B)$.

of ‘‘local commutativity’’ in *algebraic* quantum field theory [56,57], where only observables are considered, assumed to commute if localized in disjoint regions.

These two points of view are not equivalent, as it can be appreciated by the following simple example. Let us consider the system consisting of just one fermion that can occupy two modes, i.e., $N = 1, M = 2$, with the bipartition defined by the two modes. The following state:

$$\omega = |\phi\rangle\langle\phi|, \quad |\phi\rangle = \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}}, \quad (6)$$

where the combination of the two Fock basis states $|1,0\rangle, |0,1\rangle$ introduced in (2) appears to be entangled.⁵ Nevertheless, in the second approach mentioned above, it is found to satisfy the condition (4), hence to be separable. Indeed, only observables, i.e., self-adjoint, even operators, can be used in this case as \mathcal{A}_1 and \mathcal{A}_2 ; in practice, only the two partial number operators $a_1^\dagger a_1$ and $a_2^\dagger a_2$ together with the identity are admissible, and for these observables the state (6) behaves as the separable state $(|0,1\rangle\langle 0,1| + |1,0\rangle\langle 1,0|)/2$. Different is the situation within the first approach: In this case, all operators are admissible, for instance, $\mathcal{A}_1 = a_1^\dagger$ and $\mathcal{A}_2 = a_2$, which indeed prevent the separability condition (4) to be satisfied.

In view of this, here we advocate and adopt the first point of view, i.e., point (i) above: It gives a more general and physically complete treatment of fermion entanglement. Nevertheless, it should be stressed that the fermion algebra put stringent constraints on the form of the fermion states that can be represented as a product of other states, as the ones appearing in the decomposition (4). Specifically, any product $\omega_k^{(1)}(\mathcal{A}_1)\omega_k^{(2)}(\mathcal{A}_2)$ vanishes whenever \mathcal{A}_1 and \mathcal{A}_2 both belong to the odd components of their respective subalgebras. This fact comes from the following result [58], whose proof we explicitly give for it is a direct illustration of the effects of the anticommutative character of the fermion algebra.

Lemma 1. Consider a bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ of the fermion algebra \mathcal{A} and two states ω_1, ω_2 on \mathcal{A} . Then, the linear functional ω on \mathcal{A} defined by $\omega(A_1 A_2) = \omega_1(A_1)\omega_2(A_2)$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ is a state on \mathcal{A} only if at least one ω_i vanishes on the odd component of \mathcal{A}_i .

Proof. Suppose both states ω_1 and ω_2 do not vanish when acting on the odd components $\mathcal{A}_{1,2}^o$. Then, there exist odd elements $A_i^o \in \mathcal{A}_i^o$, such that $\omega_i(A_i^o) \neq 0$, $i = 1, 2$. The same is true for the self-adjoint combinations $(A_i^o + (A_i^o)^\dagger)/2$ and $(A_i^o - (A_i^o)^\dagger)/(2i)$: We can then assume $(A_i^o)^\dagger = A_i^o$, so that $\overline{\omega_i(A_i^o)} = \omega_i(A_i^o) \neq 0$, where the overline signifies complex conjugation. But then, due to the anticommutativity of the odd elements A_i^o , one finds

$$\overline{\omega(A_1^o A_2^o)} = \omega(A_2^o A_1^o) = -\omega(A_1^o A_2^o) = \omega_1(A_1^o)\omega_2(A_2^o) \neq 0,$$

which is a contradiction. ■

In other terms, given a mode bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ of the fermion algebra \mathcal{A} , i.e., a decomposition of \mathcal{A} in the subalgebra \mathcal{A}_1 generated by the first m modes and the

⁵As we shall see in Sec. IV, an N -fermion generalization of this state can be used in quantum metrology to achieve sub-shot-noise accuracy in parameter estimation.

subalgebra \mathcal{A}_2 , generated by the remaining $M - m$ ones, the decomposition (4) is meaningful only for local operators $A_1 A_2$ for which $[A_1, A_2] = 0$, so that, at this stage, the definition of separability that it encodes appears similar to the resulting one for bosonic systems.

As a further consequence of Lemma 1, the following criterion for entanglement holds:⁶

Corollary 1. Given the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ of the fermion algebra \mathcal{A} , if a state ω is nonvanishing on a local operator $A_1^o A_2^o$, with the two components $A_1^o \in \mathcal{A}_1^o$, $A_2^o \in \mathcal{A}_2^o$ both belonging to the odd part of the two subalgebras, then ω is entangled.

Indeed, if $\omega(A_1^o A_2^o) \neq 0$, then, by Lemma 1, ω cannot be written as in (4), and therefore it is nonseparable.

The case of pure states, i.e., states that cannot be written as a convex combination of other states, deserves a separate discussion. Indeed, in this case the separability condition (4) simplifies and the following result can be proven.

Lemma 2. Pure states ω on the fermion algebra \mathcal{A} are separable with respect to a given bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ if and only if

$$\omega(A_1 A_2) = \omega(A_1) \omega(A_2), \quad (7)$$

for all local operators $A_1 A_2$.

Proof. The *if* part of the proof is trivial: According to Definition 3, states as above are manifestly $(\mathcal{A}_1, \mathcal{A}_2)$ separable, since they are of the form (4) with just one element in the convex sum.

For the *only if* part of the proof, recall that, in the case of mode bipartition we are considering, the $(\mathcal{A}_1, \mathcal{A}_2)$ -local operators generate the whole fermion algebra \mathcal{A} . Therefore, any element $A \in \mathcal{A}$ can be written as a combination of local operators, $A = \sum_{ij} C_{ij} A_1^{(i)} A_2^{(j)}$, with $A_1^{(i)} \in \mathcal{A}_1$ and $A_2^{(j)} \in \mathcal{A}_2$. As a consequence, if by hypothesis a state ω is separable, i.e., it can be written as in (4) on all $(\mathcal{A}_1, \mathcal{A}_2)$ -local operators, then one has

$$\omega(A) = \sum_{ij} C_{ij} \sum_k \lambda_k \omega_k^{(1)}(A_1^{(i)}) \omega_k^{(2)}(A_2^{(j)}) = \sum_k \lambda_k \Omega_k(A),$$

in terms of other states, defined on the whole algebra \mathcal{A} through the relation: $\Omega_k(A) = \sum_{ij} C_{ij} \omega_k^{(1)}(A_1^{(i)}) \omega_k^{(2)}(A_2^{(j)})$.⁷ But since ω is pure by hypothesis, only one term in the above convex combination must be different from zero. By dropping the now superfluous label k , we have then found that $\omega(A_1 A_2) = \omega^{(1)}(A_1) \omega^{(2)}(A_2)$. The final form (7) is obtained by separately taking A_1 and A_2 to coincide with the identity operator. ■

III. STRUCTURE OF ENTANGLED FERMION STATES

The above discussion shows that, for many-body systems formed by N fermions that can occupy M modes, the notion of entanglement cannot be given once for all, but needs to be referred to the choice of a partition of the modes into two disjoint

sets, the first containing the first m modes, while the second the remaining $M - m$ ones. In short, we will henceforth refer to such a choice as the $(m, M - m)$ partition. It turns out that, once the partition $(m, M - m)$ is fixed, the general structure of entangled N -fermion states can be explicitly described.

Let us first consider the case of pure states; their complete characterization is given by the following.

Proposition 1. A pure state $|\psi\rangle$ in the fermion Hilbert space \mathcal{H} is $(m, M - m)$ separable if and only if it is generated out of the vacuum state by a $(m, M - m)$ -local operator, i.e., it can be written in the form,

$$|\psi\rangle = \mathcal{P}(a_1^\dagger, \dots, a_m^\dagger) \cdot \mathcal{Q}(a_{m+1}^\dagger, \dots, a_M^\dagger) |0\rangle, \quad (8)$$

where \mathcal{P}, \mathcal{Q} are polynomials in the creation operators relative to the first m modes and the last $M - m$ modes, respectively. Otherwise, the state is entangled.

Proof. First of all, recall that in the present situation the condition of separability reduces to the simpler expression (7); clearly, the state in (8) satisfies it by taking for the expectation value of a generic fermion operator $A \in \mathcal{A}$ the usual state average, $\omega(A) \equiv \langle \psi | A | \psi \rangle$.

In order to prove the converse, i.e., that from the separability condition (7) the expression (8) follows, we start decomposing $|\psi\rangle$ in the Fock basis given in (2); taking into account the $(m, M - m)$ bipartition of the modes, one can write

$$|\psi\rangle = \sum_{\{k\}, \{\alpha\}} C_{\{k\}, \{\alpha\}} |k_1, \dots, k_m; \alpha_{m+1}, \dots, \alpha_M\rangle, \quad (9)$$

$$\sum_{\{k\}, \{\alpha\}} |C_{\{k\}, \{\alpha\}}|^2 = 1,$$

where $\{k\} = (k_1, k_2, \dots, k_m)$, respectively, $\{\alpha\} = (\alpha_{m+1}, \alpha_{m+2}, \dots, \alpha_M)$, is the vector of occupation numbers of the first m , respectively, second $M - m$ modes, and

$$|k_1, \dots, k_m; \alpha_{m+1}, \dots, \alpha_M\rangle = (\hat{a}_1^\dagger)^{k_1} \dots (\hat{a}_m^\dagger)^{k_m} (\hat{a}_{m+1}^\dagger)^{\alpha_{m+1}} \dots (\hat{a}_M^\dagger)^{\alpha_M} |0\rangle.$$

The condition of separability of Lemma 2 assures that $\langle \Psi | A_1 A_2 | \Psi \rangle = \langle \Psi | A_1 | \Psi \rangle \langle \Psi | A_2 | \Psi \rangle$ for all fermion operators A_1, A_2 belonging to the first, second partition, respectively.⁸ We will use this request to force the coefficients $C_{\{k\}, \{\alpha\}}$ to be in product form, $C_{\{k\}, \{\alpha\}} = C_{\{k\}} C_{\{\alpha\}}$, through suitable choice of A_1 and A_2 . To this aim, let us consider the following operators:

$$A_1 = (a_1^\dagger)^{p'_1} \dots (a_m^\dagger)^{p'_m} \left(\frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z - N_1} \right) a_m^{p_m} \dots a_1^{p_1}, \quad (10)$$

$$A_2 = (a_{m+1}^\dagger)^{\beta'_{m+1}} \dots (a_M^\dagger)^{\beta'_M} \left(\frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z - N_2} \right) a_M^{\beta_M} \dots a_{m+1}^{\beta_{m+1}}, \quad (11)$$

$$A_1 A_2 = (a_1^\dagger)^{p'_1} \dots (a_m^\dagger)^{p'_m} (a_{m+1}^\dagger)^{\beta'_{m+1}} \dots (a_M^\dagger)^{\beta'_M} \times \left(\frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z - N} \right) a_M^{\beta_M} \dots a_{m+1}^{\beta_{m+1}} a_m^{p_m} \dots a_1^{p_1}, \quad (12)$$

⁶This criterion precisely detects the entanglement of the state $|\Phi\rangle$ in (6); indeed, with the odd elements $A_1^o = a_1^\dagger$ and $A_2^o = a_2$, one has $\langle \Phi | A_1^o A_2^o | \Phi \rangle = 1/2$.

⁷Notice that, by Lemma 1, $\Omega_k(A_1 A_2)$ is different from zero only when A_1 and A_2 are not both odd elements.

⁸Note again that, due to Lemma 1, at least one of the two operators A_1, A_2 needs to be even.

where $p_i, p'_i, \beta_j, \beta'_j$ are either 0 or 1, $N_1 = \sum_{k=1}^m a_k^\dagger a_k$, $N_2 = \sum_{j=m+1}^M a_j^\dagger a_j$ are the number operators relative to the two subsets of modes, while $N = N_1 + N_2$ is the total number of fermions in the system; furthermore, Γ is a contour around $z = 0$ excluding all other integers. The choice of contour forces the three integrals above to vanish unless $z = 0$, whence the first two project onto the subspaces with no particles in the first, second partition, respectively, while the third one projects onto the vacuum.⁹ With a slight abuse of language, the three operators above can be represented in short as $A_1 = |\{p\}\rangle\langle\{p'\}|$, $A_2 = |\{\beta\}\rangle\langle\{\beta'\}|$, and $A_1 A_2 = |\{p\}, \{\beta\}\rangle\langle\{p'\}, \{\beta'\}|$. Then, using (9), one easily obtains

$$\langle\Psi|A_1|\Psi\rangle = \sum_{\{\alpha\}} \bar{C}_{\{p'\}, \{\alpha\}} C_{\{p\}, \{\alpha\}}, \quad (13)$$

$$\langle\Psi|A_2|\Psi\rangle = \sum_{\{k\}} \bar{C}_{\{k\}, \{\beta'\}} C_{\{k\}, \{\beta\}}, \quad (14)$$

$$\langle\Psi|A_1 A_2|\Psi\rangle = \bar{C}_{\{p'\}, \{\beta'\}} C_{\{p\}, \{\beta\}}, \quad (15)$$

and the assumed separability of $|\psi\rangle$ yields the condition:

$$\begin{aligned} & \bar{C}_{\{p'\}, \{\beta'\}} C_{\{p\}, \{\beta\}} \\ &= \left(\sum_{\{\alpha\}} \bar{C}_{\{p'\}, \{\alpha\}} C_{\{p\}, \{\alpha\}} \right) \left(\sum_{\{k\}} \bar{C}_{\{k\}, \{\beta'\}} C_{\{k\}, \{\beta\}} \right). \end{aligned} \quad (16)$$

For $p' = p$ and $\beta' = \beta$ this expression becomes

$$|C_{\{p\}, \{\beta\}}|^2 = \left(\sum_{\{\alpha\}} |C_{\{p\}, \{\alpha\}}|^2 \right) \left(\sum_{\{k\}} |C_{\{k\}, \{\beta\}}|^2 \right).$$

Setting $D_{\{p\}} = \sum_{\{\alpha\}} |C_{\{p\}, \{\alpha\}}|^2$ and $D'_{\{\beta\}} = \sum_{\{k\}} |C_{\{k\}, \{\beta\}}|^2$, one can rewrite

$$C_{\{p\}, \{\beta\}} = \sqrt{D_{\{p\}}} \sqrt{D'_{\{\beta\}}} e^{i\theta_{\{p\}\{\beta\}}}. \quad (17)$$

Inserting this expression in (16), we obtain

$$\begin{aligned} e^{i(\theta_{\{p'\}\{\beta'\}} - \theta_{\{p\}\{\beta\}})} &= \sum_{\{\alpha\}} D'_{\{\alpha\}} e^{i(\theta_{\{p\}\{\alpha\}} - \theta_{\{p'\}\{\alpha\}})} \\ &\times \sum_{\{k\}} D_{\{k\}} e^{i(\theta_{\{k\}\{\beta\}} - \theta_{\{k\}\{\beta'\}})}. \end{aligned}$$

Since due to the state normalization condition $\sum_{\{p\}} D_{\{p\}} = 1 = \sum_{\{\beta\}} D'_{\{\beta\}}$, by setting $\beta' = \beta$ one sees that $\theta_{\{p\}\{\beta\}} - \theta_{\{p'\}\{\beta\}} = \phi_{pp'}$ for all β , i.e., this phase difference is a function of the set of indices p and p' , but not of β . Fixing an arbitrary p' and inserting this expression into (17) yields

$$C_{\{p\}, \{\beta\}} = \sqrt{D_{\{p\}}} e^{i\phi_{pp'}} \sqrt{D'_{\{\beta\}}} e^{i\theta_{\{p'\}\{\beta\}}}, \quad (18)$$

which is of the required form. ■

⁹Since the number operators are sums of quadratic monomials, by series expansion the three integrals provide operators that are elements of even subalgebras.

Remark 2. In the proof of the previous proposition nothing depended on having a finite number m of modes in the first partition and a finite number $M - m$ of modes in the second partition. The result thus extends to the case of infinite disjoint sets of modes for all normalized pure states $|\psi\rangle$. ■

Examples of N fermion pure separable states are the Fock states; indeed, recalling (2), they can be recast in the form (8),

$$\begin{aligned} |k_1, \dots, k_m; \alpha_{m+1}, \dots, \alpha_M\rangle &= [(\hat{a}_1^\dagger)^{k_1} \dots (\hat{a}_m^\dagger)^{k_m}] \\ &\times [(\hat{a}_{m+1}^\dagger)^{\alpha_{m+1}} \dots (\hat{a}_M^\dagger)^{\alpha_M}] |0\rangle, \end{aligned} \quad (19)$$

where \mathcal{P} and \mathcal{Q} are now monomials in the creation operators of the two partitions. By varying $k_i, \alpha_j \in \{0, 1\}$ and the integer $k = \sum_{i=1}^m k_i$, such that $0 \leq k \leq N$, these states generate the whole Hilbert space \mathcal{H} . This basis state can be relabeled in a different, more convenient way as

$$\begin{aligned} |k, \sigma; N - k, \sigma'\rangle, \quad \sigma = 1, 2, \dots, D_k &\equiv \binom{m}{k}, \\ \sigma' = 1, 2, \dots, D'_{N-k} &\equiv \binom{M - m}{N - k}; \end{aligned} \quad (20)$$

the integer k gives the number of fermions occupying the first m modes, $k \leq m$, while σ counts the different ways in which those particles can fill those modes; similarly, σ' labels the ways in which the remaining $N - k$ fermions can occupy the other $M - m$ modes.¹⁰ In this new labeling, the property of orthonormality of the states in (20) simply becomes $\langle k, \sigma; N - k, \sigma' | l, \tau; N - l, \tau' \rangle = \delta_{kl} \delta_{\sigma\tau} \delta_{\sigma'\tau'}$.

For fixed k , the basis vectors $\{|k, \sigma; N - k, \sigma'\rangle\}$ span a subspace \mathcal{H}_k of dimension $D_k D'_{N-k}$; the union of all these orthogonal subspaces give the whole fermion Hilbert space \mathcal{H} , recovering its dimension D [59]:

$$\sum_{k=0}^N D_k D'_{N-k} = D = \binom{M}{N}. \quad (21)$$

Remark 3. Note that the space \mathcal{H}_k is naturally isomorphic to the tensor product space $\mathbb{C}^{D_k} \otimes \mathbb{C}^{D'_{N-k}}$; through this isomorphism, the states $|k, \sigma; N - k, \sigma'\rangle$ can then be identified with the corresponding basis states of the form $|k, \sigma\rangle \otimes |N - k, \sigma'\rangle$. This observation will be useful below in the classification of entangled fermion states. ■

Using the above notation, a generic fermion mixed state ρ can then be written as

$$\begin{aligned} \rho &= \sum_{k, l=N_-}^{N_+} \sum_{\sigma, \sigma', \tau, \tau'} \rho_{k\sigma\sigma', l\tau\tau'} |k, \sigma; N - k, \sigma'\rangle \langle l, \tau; N - l, \tau'|, \\ \sum_{k=N_-}^{N_+} \sum_{\sigma, \sigma'} \rho_{k\sigma\sigma', k\sigma\sigma'} &= 1, \end{aligned} \quad (22)$$

¹⁰In order to completely identify the basis states, two extra labels σ and σ' are needed for each value of k , so that these labels (and the range of values they take) are in general k dependent: In order to keep the notation as simple as possible, in the following these dependencies will be tacitly understood.

where $N_- = \max\{0, N - M + m\}$ and $N_+ = \min\{N, m\}$ are the minimum and maximum number of fermions that the first partition can contain, due to the exclusion principle.

The set of all states form a convex set, whose extremals are given by the pure ones. Proposition 1 can then be used to characterize separable mixed states.

Corollary 2. A mixed state ρ as in (22) is $(m, M - m)$ separable if and only if it is the convex combination of projectors on pure $(m, M - m)$ -separable states; otherwise, the state ρ is $(m, M - m)$ entangled.

In general, to determine whether a given density matrix ρ can be written in separable form is a hard task and one is forced to rely on suitable separability tests, that, however, are in general not exhaustive.

One of such tests has already been introduced through Corollary 1, and is peculiar to fermion systems: If there exists a local operator $A_1^o A_2^o$, with both components odd under the action of the involution Θ (cf. Definition 4), such that the expectation value $\langle A_1^o A_2^o \rangle = \text{Tr}[\rho A_1^o A_2^o]$ is nonvanishing, then the state ρ is surely entangled. This entanglement criterion turns out to be exhaustive in the case of bipartitions of type $(1, M - 1)$, i.e., when the first partition contains just one mode, while the second the remaining $M - 1$ ones.

Proposition 2. An M -mode state of an N -fermion system is entangled with respect to the bipartition into one mode and the rest if and only if its expectation value is nonvanishing on a local operator whose components both belong to the odd part of the corresponding operator algebras.

Proof. A generic state of the system can be written in as in (22), though dropping the unnecessary primed Greek labels:

$$\rho = \sum_{k,l=0}^1 \sum_{\sigma,\tau} \rho_{k\sigma,l\tau} |k; N - k, \sigma\rangle \langle l; N - l, \tau|, \quad (23)$$

$$\sum_{k=0}^1 \sum_{\sigma} \rho_{k\sigma,k\sigma} = 1.$$

It can be further decomposed into a “diagonal” and “off-diagonal” part,

$$\rho = \rho_d + \eta, \quad (24)$$

with

$$\rho_d = \sum_{k=0}^1 \sum_{\sigma,\tau} \rho_{k\sigma,k\tau} |k; N - k, \sigma\rangle \langle k; N - k, \tau|, \quad (25)$$

$$\eta = \sum_{k=0}^1 \sum_{\sigma,\tau} \rho_{k\sigma,(1-k)\tau} |k; N - k, \sigma\rangle \langle 1 - k; N + k - 1, \tau|. \quad (26)$$

Clearly, only η can give a nonvanishing contribution to the expectation value $\text{Tr}[\rho A_1^o A_2^o]$, where $A_1^o A_2^o$ is a local operator whose components are both odd.¹¹ Therefore, the proposition is proven once we show that the state ρ_d can

be written in separable form. In order to prove this, let us make a change of basis in the second $M - 1$ partition passing from the Fock states to another set of separable states adapted to ρ , such that its components along this new basis satisfy $\rho_{k\sigma,k\tau} = \rho_{k\sigma,k\sigma} \delta_{\sigma\tau}$; notice that this is always possible through suitable local, unitary transformations diagonalizing the two matrices $\mathcal{M}_{\sigma\tau}^{(k)} \equiv [\rho_{k\sigma,k\tau}]$, $k = 0, 1$. In this new basis, ρ_d results a convex sum of projections on separable pure states and therefore it is itself separable. In conclusion, ρ is entangled if and only if its “off-diagonal” part η is nonvanishing. ■

Another very useful entanglement criteria involves the operation of partial transposition [37,38]: A state ρ for which the partially transposed density matrix $\tilde{\rho}$ is no longer positive is surely entangled. This lack of positivity can be quantified by the so-called negativity [60,61]:

$$\mathcal{N}(\rho) = \frac{1}{2}(\text{Tr}[\sqrt{\tilde{\rho}^\dagger \tilde{\rho}}] - \text{Tr}[\rho]), \quad (27)$$

which is nonvanishing only in the presence of a nonpositive $\tilde{\rho}$.

In the case of the $(1, M - 1)$ bipartition considered above, the partial transposition operation applied to the first partition gives results that are completely equivalent to the ones obtained in Proposition 2. Indeed, explicit computation shows that the negativity of the state in (23) is nonvanishing if and only if at least one of the off-diagonal components $\rho_{k\sigma,(1-k)\tau}$, $k = 0, 1$, are nonzero.

Although not exhaustive, the partial transposition criterion results more powerful than the one based on Corollary 1, allowing a complete characterization of the structure of entangled N -fermion states.

Proposition 3. A generic $(m, M - m)$ -mode bipartite state (22) is entangled if and only if it cannot be cast in the following block diagonal form:

$$\rho = \sum_{k=N_-}^{N_+} p_k \rho_k, \quad \sum_{k=N_-}^{N_+} p_k = 1, \quad \text{Tr}[\rho_k] = 1, \quad (28)$$

with

$$\rho_k = \sum_{\sigma,\sigma',\tau,\tau'} \rho_{k\sigma\sigma',k\tau\tau'} |k,\sigma; N - k,\sigma'\rangle \langle k,\tau; N - k,\tau'|, \quad (29)$$

$$\sum_{\sigma,\sigma'} \rho_{k\sigma\sigma',k\sigma\sigma'} = 1,$$

(i.e., at least one of its nondiagonal coefficients $\rho_{k\sigma\sigma',l\tau\tau'}$, $k \neq l$, is nonvanishing), or, if it can, at least one of its diagonal blocks ρ_k is nonseparable.¹²

Proof. Assume first that the state ρ cannot be written in block diagonal form; using techniques similar to the one adopted in dealing with boson systems [33], one can show that it is not left positive by the operation of partial transposition and therefore it is entangled. Next, take ρ in block diagonal form as in (28) and (29) above. If all its blocks ρ_k are separable, then clearly ρ itself results separable. Then, assume that at least one of the diagonal blocks is entangled. Mixing it with the remaining blocks as in (28) will not spoil its entanglement

¹¹Note that η is not a state, since in general it is not positive; it is the difference of two density matrices.

¹²For each block ρ_k , separability is understood with reference to the isomorphic structure $\mathbb{C}^{D_k} \otimes \mathbb{C}^{D_{N-k}}$ mentioned before (see Remark 3).

since all blocks ρ_k have support on orthogonal spaces; as a consequence, the state ρ results itself nonseparable. ■

Using this result, one can now study characteristic properties and geometry of the space \mathcal{F} of N -fermion states.

The first question that naturally arises concerns the strength of the entanglement content of a fermion state ρ against mixing with other states. This is measured by the so-called “robustness of entanglement” $R(\rho)$ [39,40,61,62]: It is the smallest, non-negative value the parameter t can take so that the (un-normalized) combination $\rho + t\rho_{\text{sep}}$ is separable, where ρ_{sep} span all separable fermion states.

Proposition 4. The robustness of entanglement of a generic $(m, M - m)$ -mode bipartite N -fermion state ρ is given by

$$R(\rho) = \sum_{k=0}^N p_k R(\rho_k), \quad (30)$$

for states that are in block diagonal form as in (28) and (29), while it is infinitely large otherwise.

The proof is very similar to the one given in [34] for bosonic states, so that it will not be repeated here. Nevertheless, it is worth stressing that, as in the case of bosons, fermion entanglement is in general much more robust than the one found in systems of distinguishable particles. Indeed, recalling the previous Proposition 3, one has that separable N -fermion states must necessarily be in block diagonal form. If the state ρ is not in this form, it can never be made block diagonal by mixing it with any separable one; therefore, in this case, the combination $\rho + t\rho_{\text{sep}}$ will never be separable, unless t is infinitely large, giving infinite robustness to the entangled state ρ .

A similar argument allows one to conclude that the structure of the space \mathcal{S} of separable fermion states is rather special: There always exist small perturbations of separable, necessarily block diagonal states that make them not block diagonal, hence entangled. This result should be compared to the one valid in the case of distinguishable particles, where instead almost all separable states remain separable under sufficiently small arbitrary perturbations [63,64].

Among the separable N -fermion states, the totally mixed one,

$$\rho_{\text{mix}} = \frac{1}{D} \sum_{k=N_-}^{N_+} \sum_{\sigma, \sigma'} |k, \sigma; N - k, \sigma'\rangle \langle k, \sigma; N - k, \sigma'|, \quad (31)$$

stands out because of its special properties. First of all, it lays on the border of the space \mathcal{S} of separable states, since in its vicinity one can always find nonseparable states of the form $\rho_{\text{mix}} + \epsilon \rho_{\text{ent}}$, $\epsilon > 0$, with ρ_{ent} any entangled state not in block-diagonal form.¹³ Furthermore, ρ_{mix} is the only state that remains separable for any choice of bipartition. Indeed, for any state $\rho_{\text{sep}} \neq \rho_{\text{mix}}$, separable in a given M -mode bipartition, it is possible to find a unitary Bogolubov transformation, defining a new M -mode bipartition, that maps it into an entangled one [34]. Only the state proportional to the unit matrix remains unchanged under any unitary transformation.

¹³On the contrary, recall that in the case of distinguishable particles, ρ_{mix} always lies in the interior of \mathcal{S} [60,63].

These results allows a rather precise description of the geometrical structure of the space \mathcal{F} of N -fermion states. As discussed above, by fixing a bipartition one selects the set \mathcal{S} of separable states, which forms a subspace of the convex space \mathcal{F} . Changing the bipartition through a Bogolubov transformation produces a new separable subspace, having in general only one point in common with the starting one, ρ_{mix} . Therefore, the global geometrical structure of the state space \mathcal{F} presents a sort of starlike shape formed by the various separable bipartition subspaces, all sharing just one point, the totally mixed state.

As a final comment, notice that all above results can be generalized to the case of systems where the total number of particles is not fixed, but commutes with all physical observables.¹⁴ In such a situation, a general density matrix ρ can be written as an incoherent mixture of states ρ_N with fixed number N of fermions:

$$\rho = \sum_N \lambda_N \rho_N, \quad \lambda_N \geq 0, \quad \sum_N \lambda_N = 1. \quad (32)$$

The state ρ is a convex combination of matrices ρ_N having support on orthogonal spaces, and therefore all previous arguments and results hold true for each component ρ_N .

IV. APPLICATIONS TO QUANTUM METROLOGY

One of the most promising developments in quantum technology, i.e., the application of quantum physics to practical technological realizations, is the possibility of achieving measurements of physical parameters with unprecedented accuracy.¹⁵ In a generic detection scheme, the parameter θ to be measured, typically a phase, is encoded into a state transformation occurring inside a measurement apparatus, generally an interferometric device. In the most common case of linear setups, this transformation can be modeled by a unitary mapping, $\rho \rightarrow \rho_\theta$, sending the initial state ρ into the final parameter-dependent outcome state:

$$\rho_\theta = e^{i\theta J} \rho e^{-i\theta J}, \quad (33)$$

where J is the devices-dependent, θ -independent operator generating the state transformation. The task of quantum metrology is to determine the ultimate bounds on the accuracy with which the parameter θ can be obtained through a measurement of ρ_θ and to study how these bounds scale with the available resources.

General quantum estimation theory allows a precise determination of the accuracy $\delta\theta$ with which the phase θ can be obtained in a measurement involving the operator J and the initial state ρ ; one finds that $\delta\theta$ is limited by the following

¹⁴In other terms, we are in the presence of a superselection rule [65]. For a similar reason, i.e., the conservation of the fermion “charge,” and in contrast with the boson case, fermion systems with a fluctuating number of particles result unphysical [66,67], so that density matrices that are coherent mixtures of states with different N are not admissible.

¹⁵The literature on the subject is vast; for a partial list, see [68–92] and references therein.

inequality [93–96]:

$$\delta\theta \geq \frac{1}{\sqrt{F[\rho, J]}}, \quad (34)$$

where the quantity $F[\rho, J]$ is the so-called “quantum Fisher information.” It can be defined through the symmetric logarithmic derivative $L, \partial_\theta \rho_\theta|_{\theta=0} = (\rho L + L \rho)/2 = -i [J, \rho]$, as

$$F[\rho, J] := \text{tr}[\rho L^2]. \quad (35)$$

Given a spectral decomposition of the input state, $\rho = \sum_j r_j |r_j\rangle\langle r_j|$, one explicitly finds

$$F[\rho, J] = 2 \sum_{i,j; r_i+r_j \neq 0} \frac{(r_i - r_j)^2}{r_i + r_j} |\langle r_i | J | r_j \rangle|^2, \quad (36)$$

which explicitly shows that $F[\rho, J]$ is independent from the parameter θ to be estimated. Furthermore, the quantum Fisher information is a continuous, convex function of the state ρ , and in general satisfies the inequality [97,98],

$$F[\rho, J] \leq 4 \Delta_\rho^2 J, \quad (37)$$

where $\Delta_\rho^2 J \equiv [\langle J^2 \rangle - \langle J \rangle^2]$ is the variance of the operator J in the state ρ , the equality holding only for pure initial states.

As a consequence of (34), better resolution in θ estimation corresponds to a larger quantum Fisher information. Therefore, once the measuring apparatus is given, i.e., the operator J is fixed, one can optimize the precision with which θ is determined by choosing an initial state ρ that maximizes $F[\rho, J]$.

In the case of devices using a system of N distinguishable particles, it has been shown that for any separable state ρ_{sep} the quantum Fisher information is bounded by N [88]:

$$F[\rho_{\text{sep}}, J] \leq N. \quad (38)$$

This means that by feeding the measuring apparatus with separable initial states, the best achievable precision in the determination of the phase shift θ is bounded in this case by the so-called shot-noise limit:

$$\delta\theta \geq \frac{1}{\sqrt{N}}. \quad (39)$$

This is also the best result attainable using classical (i.e., nonquantum) devices: The accuracy in the estimation of θ scales at most with the inverse square root of the number of available resources. Instead, quantum equipped metrology allows one to go below the shot-noise limit and in principle construct a new generation of sensors reaching unprecedented precision. And indeed, various detection protocols and input states ρ have been proposed, all able to yield sub-shot-noise sensitivities. Notice that, in view of the inequality (38), these input states need to be entangled.

This conclusion holds when the metrological devices used to estimate the physical parameter θ are based on systems of distinguishable particles. When dealing with identical particles, the above statement is not strictly correct and needs to be rephrased. Indeed, in the case of bosonic systems it has been explicitly shown that sub-shot-noise sensitivities may be obtained also via a nonlocal operation acting on separable input states [30]. In other terms, although some sort of nonlocality

is needed in order to go below the shot-noise limit, this can be provided by the measuring apparatus itself and not by the input state ρ , that indeed can be separable. This result has clearly direct experimental relevance, since the preparation of the suitable entangled input state may require in practice a large amount of resources.

When dealing with systems of N fermions, the situation may appear more problematic, since, due to the exclusion principle, a single mode can accommodate at most one fermion; therefore, the scaling with N of the sensitivity in the estimation of the parameter θ may worsen when compared to the boson case employing similar resources. In the case of bosons, a two-mode apparatus, e.g., a double-well interferometer, filled with N particles is sufficient to reach sub-shot-noise sensitivities. Instead, with fermions, a multimode interferometer [99–104] is needed in order to reach comparable sensitivities.

As an example, consider a system of N fermions in M modes, with M even, and let us fix the balanced bipartition $(M/2, M/2)$, in which each of the two parts contain $m = M/2$ modes, taking for simplicity $N \leq m$. As generator of the unitary transformation $\rho \rightarrow \rho_\theta$ inside the measuring apparatus let us take the following operator:

$$J_x^{(1)} = \frac{1}{2} \sum_{k=1}^m \omega_k (a_k^\dagger a_{m+k} + a_{m+k}^\dagger a_k), \quad (40)$$

where ω_k is a given spectral function, e.g., $\omega_k \simeq k^p$, with p integer. The apparatus implementing the above state transformation is clearly nonlocal with respect to the chosen bipartition: $e^{i\theta J_x^{(1)}}$ cannot be written as the product $A_1 A_2$ of two components made of operators referring only to the first and second partition, respectively. It represents a generalized, multimode beam splitter, and the whole measuring device behaves as a multimode interferometer.

Let us feed the interferometer with a pure initial state, $\rho = |\psi\rangle\langle\psi|$,

$$|\psi\rangle = |\underbrace{1, \dots, 1}_N, \underbrace{0, \dots, 0}_{m-N}, \underbrace{0, \dots, 0}_m\rangle = a_1^\dagger a_2^\dagger \dots a_N^\dagger |0\rangle, \quad (41)$$

where the fermions occupying the first N modes of the first partition; $|\psi\rangle$ is a Fock state and therefore it is separable, as discussed in the previous section. The quantum Fisher information can be easily computed since it is now proportional to the variance of $J_x^{(1)}$:

$$F[\rho, J_x^{(1)}] = 4 \Delta_\rho^2 J_x^{(1)} = \sum_{k=1}^N \omega_k^2. \quad (42)$$

Unless ω_k is k independent, $F[\rho, J_x^{(1)}]$ is larger than N and therefore the interferometric apparatus can beat the shot-noise limit in θ estimation, even starting with a separable state. Actually, for $\omega_k \simeq k^p$, one gets $F[\rho, J_x^{(1)}] \simeq O(N^{2p+1})$.

Remark 4. (i) Notice that, in this case, it is not the entanglement of the initial state that helps overcoming the shot-noise limit in the phase estimation accuracy; rather, it is the nonlocal character of the rotations operated by the apparatus on an initially separable state that allows $\delta\theta$ to be smaller than $1/\sqrt{N}$.

(ii) In the case of systems made of N *distinguishable* particles, for a collective operator $J = \sum_{i=1}^N J^{(i)}$, where $J^{(i)}$ are single-particle SU(2) rotation generators, and any state ρ , the following general bound on the quantum Fisher information holds:

$$F[\rho, J] \leq N^2, \quad (43)$$

providing an absolute limit for the accuracy on the parameter estimation: $\delta\theta \geq 1/N$. When the equality holds, one reaches the so-called Heisenberg limit, the ultimate sensitivity allowed by quantum metrology in this case. Instead, in the scenario described above, one can reach sub-Heisenberg sensitivities. This possibility has been discussed before, using, however, nonlinear metrology [105–112], i.e., for interferometric apparatus that cannot be described in terms of single-particle rotations. Furthermore, note that the result (42) and the ability to go beyond the Heisenberg limit is not a “geometrical” phenomena attributable to a phase accumulation even on empty modes [99]; rather, it is a genuine quantum effect, that scales as a function of the number of fermions, the resource available in the measure.

(iii) When dealing with boson systems, an interferometer based on standard beam splitters suffices in order to reach sub-shot-noise accuracy in parameter estimation. In such case the relevant operator is J_x , obtained from (40) by removing the spectral function ω_k ; it belongs to an $su(2)$ algebra. Similarly, also the generalized beam splitter operator (40) is part of a Lie algebra, although infinite dimensional. Indeed, let us define the three operators:

$$J_x^{(n)} = \frac{1}{2} \sum_{k=1}^m (\omega_k)^n (a_k^\dagger a_{m+k} + a_{m+k}^\dagger a_k), \quad (44)$$

$$J_y^{(n)} = \frac{1}{2i} \sum_{k=1}^m (\omega_k)^n (a_k^\dagger a_{m+k} - a_{m+k}^\dagger a_k), \quad (45)$$

$$J_z^{(n)} = \frac{1}{2} \sum_{k=1}^m (\omega_k)^n (a_k^\dagger a_k - a_{m+k}^\dagger a_{m+k}). \quad (46)$$

One easily checks that they satisfy the following commutation relations:

$$[J_i^{(n)}, J_j^{(m)}] = i \epsilon_{ijk} J_k^{(n+m)}, \quad i, j, k = x, y, z, \quad n, m \in \mathbb{Z}, \quad (47)$$

defining the $su(2)$ loop algebra (i.e., a centerless Kac-Moody algebra) [113, 114]. ■

In general, some sort of quantum nonlocality is nevertheless needed in order to attain sub-shot-noise accuracy in phase estimation. This can be most simply appreciated by changing the M -mode bipartition of our system of N fermions. Let us introduce new creation and annihilation operators b_k^\dagger, b_k through the following Bogolubov transformations:

$$b_k = \frac{a_k + a_{m+k}}{\sqrt{2}}, \quad b_{m+k} = \frac{a_k - a_{m+k}}{\sqrt{2}}, \quad k = 1, 2, \dots, m, \quad (48)$$

together with the Hermitian conjugate ones; the new M modes still obey canonical anticommutation relations: $\{b_k, b_j^\dagger\} = \delta_{kl}$. In this new representation, the operator $J_x^{(1)}$ in (40) gets

transformed into $J_z^{(1)}$,

$$J_x^{(1)} \rightarrow J_z^{(1)} = \frac{1}{2} \sum_{k=1}^m \omega_k (b_k^\dagger b_k - b_{m+k}^\dagger b_{m+k}). \quad (49)$$

Therefore, choosing again a balanced bipartition, $(M/2, M/2)$, in which half of the b_k modes are in the first component, while the remaining half are in the second one, the unitary transformation $\rho \rightarrow \rho_\theta$ inside the apparatus is now represented by a local operator,

$$e^{i\theta J_z^{(1)}} = e^{i\theta \sum_k \omega_k b_k^\dagger b_k / 2} e^{-i\theta \sum_k \omega_k b_{m+k}^\dagger b_{m+k} / 2}. \quad (50)$$

However, the initial state $|\psi\rangle$ is no longer separable in the new bipartition, since, in the new language, it results in a linear combination of 2^N different Fock states:

$$|\psi\rangle = \left(\frac{b_1^\dagger + b_{m+1}^\dagger}{\sqrt{2}} \right) \left(\frac{b_2^\dagger + b_{m+2}^\dagger}{\sqrt{2}} \right) \dots \left(\frac{b_N^\dagger + b_{N+1}^\dagger}{\sqrt{2}} \right) |0\rangle. \quad (51)$$

Despite these changes, the value of the quantum Fisher information for the given initial state and observable is unchanged and still expressed by (42), since it does not depend on the representation used to compute it. Therefore, if one is able to build an interferometric setup that can be described in terms of the modes b_k^\dagger, b_k instead of the original modes a_k^\dagger, a_k , then the accuracy $\delta\theta$ with which the phase θ may be determined can still be better than the shot-noise limit. In such a case, the improvement in sensitivity is due to the entanglement of the initial state and not to the nonlocality of the transformation that takes place inside the apparatus.

As a further remark, notice that in practical applications, instead of using a generalized rotation through operators of the form (44)–(46), it might be more convenient to implement parameter estimation via the dynamical state transformation generated by a Hamiltonian operator. A generic quadratic Hamiltonian for our N -fermion system can be written in the form,

$$H = \sum_{k=1}^M \Omega_k a_k^\dagger a_k, \quad (52)$$

with Ω_k a given energy dispersion relation. In most situations, the dependence on the parameter θ to be estimated arises as a proportionality coupling constant multiplying the Hamiltonian, $H_\theta \equiv \theta H$; in this case, the finite-time dynamical transformation occurring in the system is described by e^{-itH_θ} . This operator is clearly local in any $(m, M - m)$ bipartition, since it is the product of M transformations in the various modes: $e^{-itH_\theta} = \prod_k e^{-it\theta \Omega_k a_k^\dagger a_k}$. For an entangled initial state of the form,¹⁶

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|N; 0\rangle + |0; N\rangle), \quad (53)$$

where the two states $|N; 0\rangle, |0; N\rangle$ represents the situation in which the N fermions are all in the first, second component,

¹⁶This state is the multimode, N -fermion generalization of the state (6) considered in Sec. II.

respectively, of a generic $(m, M - m)$ bipartition, the quantum Fisher information can be readily computed:

$$F[|\Phi\rangle\langle\Phi|, H] = \left(\sum_{k=1}^N (\Omega_{m+k} - \Omega_k) \right)^2. \quad (54)$$

In the case of a linear dispersion relation, $\Omega_k \sim k$, it reduces to the simple form $F[|\Phi\rangle\langle\Phi|, H] = m^2 N^2$, providing a sub-Heisenberg-like sensitivity in the estimation of the quantity $t\theta$, hence of the parameter θ , once the evolution time t is fixed. Notice that in F the factor N^2 is a genuine quantum effect, while the dependence on the number of modes is a “geometrical” effect due to phase accumulation among all, even empty, modes [cf. Remark 4 (ii)].

V. OUTLOOK

One of the most important tasks in modern quantum physics is the characterization and quantification of nonclassical correlations, as they allow on one hand the implementation of classically unavailable protocols in information theory; on the other hand the realization of quantum devices and sensors outperforming the present available ones. In these developments, thanks to the recent advances in quantum optics and ultracold and superconducting physics, many-body systems composed by identical particles are playing a prominent role.

For such systems though, the usually adopted definitions of separability and quantum correlation are no longer viable since, due to the indistinguishability of the microscopic constituents, the natural particle Hilbert space decomposition on which these concepts are based is lost. One should then resort to a more general definition of locality, no longer given *a priori* once for all, rather, based on a choice of a bipartition (or more in general multipartition) of the operator algebra of “observables” of the system. In this framework, a state is declared separable if its expectation value on all local operators can be written in a product form, i.e., as a convex combination of products of local expectations. This new approach to quantum nonlocality is valid in all situations and, in particular, it reduces to the standard one when applied to systems of distinguishable particles.

The physical, characteristic properties of this new, generalized definition of separability, previously analyzed in a bosonic setting, has been studied here in the case of fermion systems. We have focused on many-body systems composed of a fixed number N of fermions that can occupy a given set of different states or modes. We stress that this model represents a very general paradigm, able to describe the behavior of various different situations in atomic and condensed matter physics, as those occurring in quantum phase transitions and matter interference phenomena.

The treatment of fermion systems requires more care than in the boson case because of the anticommutative character of the basic fermion algebra. As a result, in contrast to the bosonic case, the notion of locality for fermion systems is not directly related with that of commutativity; nevertheless, the intuition that entanglement should be connected to the presence of nonclassical correlations revealed through averages of local operators turns out to be correct also in this case. As a byproduct, a new entanglement criterion for fermion states is obtained. Using this criterion together with the partial transposition one, a complete classification of entangled N -fermion states has been explicitly given.

Similarly to what happens with N -boson states, the entanglement contained in N -fermion states turns out to be much more stable against mixing with other states than the one found in systems of distinguishable particles; this makes many-body systems made of identical constituents even more attractive for use in quantum technology applications.

In this respect, quantum metrology is the natural context in which systems of N fermions can be employed to construct quantum devices that outperform classical ones. Indeed, as discussed in the last section, multimode fermion quantum interferometers can be used to improve the accuracy in parameter estimation much beyond the so-called, classical, shot-noise limit, provided some sort of quantum nonlocality is present in the measuring process. However, this required nonlocality need not be encoded in the initial N -fermion state: It can be provided by the interferometric apparatus itself, which at this point can be fed with an initial separable state. As a result, no preliminary, resource-consuming, entanglement operation (like “squeezing”) on the state entering the apparatus is needed in order to get sub-shot-noise accuracies in parameter estimation. This fact clearly opens new perspectives in the realization of many-body-based quantum sensors capable of outperforming any available apparatus dedicated to the measurement of ultraweak effects.

Finally, let us briefly consider the case in which the fermionic system is describable in terms of a set of Majorana, Hermitian operators c_i , $i = 1, 2, \dots, 2M$, obeying the algebraic relations: $\{c_i, c_j\} = 2\delta_{ij}$. Clearly, also in this case the set of all polynomials in the operators c_i forms an algebra \mathcal{C} , to which the system observables belong. The adopted notions of algebraic bipartition, locality, and separability (see Definitions 1, 2, and 3) are very general and can be applied also to \mathcal{C} , so that most of the general results obtained in the case of complex fermion algebras hold also for the Hermitian ones. Nevertheless, the system Hilbert spaces differ in the two cases; in particular, the Majorana algebra does not admit a Fock representation. As a consequence, the detailed structure of entangled Majorana states differs from that reported in Sec. III, deserving a separate, expanded discussion that will be reported elsewhere.

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