Quantumness of correlations in indistinguishable particles

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We discuss a general notion of quantum correlations in fermionic or bosonic indistinguishable particles. Our approach is mainly based on the identification of the algebra of single-particle observables, which allows us to devise an activation protocol in which the *quantumness of correlations* in the system leads to a unavoidable creation of entanglement with the measurement apparatus. Using the distillable entanglement, or the relative entropy of entanglement, as entanglement measure, we show that our approach is equivalent to the notion of minimal disturbance in a single-particle von Neumann measurement, also leading to a geometrical approach for its quantification.

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I. INTRODUCTION

The notion of entanglement, first noted by Einstein, Podolsky, and Rosen [1], is considered one of the main features of quantum mechanics, and became a subject of great interest in the last few years due to its primordial role in quantum computation and quantum information [2–5]. However, entanglement is not the only kind of correlation presenting nonclassical features, and a great effort has recently been directed towards characterizing a more general notion of quantum correlations, the *quantumness* of correlations. The quantumness of correlations is revealed in different ways, and there are a wide variety of approaches, sometimes equivalent, in order to characterize and quantify it, e.g., through the "activation protocol," where the nonclassical character of correlations in the system is revealed by a unavoidable creation of entanglement between system and measurement apparatus in a local measurement [6,7]; or by the analysis of the minimum disturbance caused in the system by local measurements [8–10], which led to the seminal definition of *quantum discord* [8]; or even through geometrical approaches [11].

Despite being widely studied in systems of distinguishable particles, less attention has been given to the study of entanglement, or even a more general notion of quantum correlations, in the case of indistinguishable particles. In this case, the space of quantum states is restricted to symmetric $\mathcal{S}$ or antisymmetric $\mathcal{A}$ subspaces, depending on the bosonic or fermionic nature of the system, and the particles are no longer accessible individually, thus eliminating the usual notions of separability and local measurements, and making the analysis of correlations much subtler. In fact, there are a multitude of distinct approaches and an ongoing debate around the entanglement in these systems [12–22]. Nevertheless, despite the variety, the approaches consist essentially in the analysis of correlations under two different aspects: the correlations genuinely arising from the entanglement between the particles ("entanglement of particles") [14–18], and the correlations arising from the entanglement between the modes of the system ("entanglement of modes") [19–22]. These two notions of entanglement are complementary, and the use of one or the other depends on the particular situation under scrutiny. For example, the correlations in eigenstates of a many-body Hamiltonian could be more naturally described by particle entanglement, whereas certain quantum information protocols could prompt a description in terms of entanglement of modes. The modes notion associates a Fock space to the several distinguishable modes of a system of indistinguishable particles, which allows one to employ all the tools commonly used in distinguishable quantum systems. The entanglement of particles has different definitions which agree in some respects, and differ in others; but once one has opted for a certain definition, there are also several proposed methods to calculate it [23–27].

Note that the correlations between modes in a system of indistinguishable particles are subsumed in the usual analysis of correlations in systems of distinguishable ones. Thus we shall, in this work, characterize and quantify a general notion of quantum correlations (not necessarily entanglement) genuinely arising between indistinguishable particles. We shall call these correlations quantumness of correlations, to distinguish from entanglement, and it has an interpretation analogous to the quantumness of correlations in systems of distinguishable particles, as we shall see. One must however be careful with such phraseology, since systems of indistinguishable particles always have *exchange correlations* coming from the symmetric or antisymmetric nature of the wave function. The intrinsic exchange correlations are not included in the concept of the quantumness of correlations. We shall discuss these issues in more detail throughout the article.

The article is organized as follow. In Sec. II we briefly review the notion of quantumness of correlations in distinguishable subsystems, and their interplay with the measurement process via the activation protocol. In Sec. III we introduce the activation protocol for systems of indistinguishable particles, and in Sec. IV we characterize and quantify the quantumness of correlations in these systems. We conclude in Sec. V.

II. QUANTUMNESS OF CORRELATIONS

The concept of quantumness of correlations is related to the amount of inaccessible information of a composed system

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if we restrict to the application of local measurements on the subsystems [8,9,28,29]. Since quantumness of correlations can be created with local operations on the subsystems, it is also called the quantum properties of classical correlations [8,29]. A model for the description of a measurement process is given via decoherence [30], where in order to measure a quantum system we must interact it with a measurement apparatus, which is initially uncorrelated with the quantum system. This interaction, given by a unitary evolution, creates correlations between them, and thereby the measurement outcomes will be registered on the apparatus eigenbasis. A protocol that allows us to understand the interplay between a measurement process and the quantumness of correlations in a system is known as the *nonclassical correlations activation protocol*. This protocol shows that if and only if the system is strictly classically correlated, i.e., has no quantumness of correlations, there exists a local measurement on the subsystems that does not create entanglement between system and measurement apparatus [6,7,31]; or rather, if the system has quantumness of correlations, then it will inevitably create entanglement with the apparatus measurement in a local measurement process, hence the reference to "activation." A direct corollary of this protocol allows us to quantify the amount of quantumness of correlations by measuring the minimal amount of entanglement created between the system and the measurement apparatus during a local measurement process [32].

Given, for instance, a bipartite system S described by the state ρ_S, in order to apply a von Neumann measurement in this system we must interact it with a measurement apparatus $\mathcal{M}$, initially in an arbitrary state $|0\rangle\langle 0|_{\mathcal{M}}$. Suppose that we are able to apply global von Neumann measurements in such a system, e.g., a von Neumann measurement in the system eigenbasis $\{|i\rangle\}_{\mathcal{S}}$, $\rho_S = \sum_i \lambda_i |i\rangle\langle i|$. The system and the measurement apparatus must interact under the action of the following unitary transformation: $U_{S:\mathcal{M}}|i\rangle_S|0\rangle_{\mathcal{M}} = |i\rangle_S|i\rangle_{\mathcal{M}}$. We see that the interaction simply creates classical correlations between them: $\tilde{\rho}_{S:A} = U_{S:A}(\rho_S \otimes |0\rangle\langle 0_A|)U_{S:A}^\dagger = \sum_i \lambda_i |i\rangle\langle i| \otimes |i\rangle\langle i|$. If, however, we are restricted to apply local measurements, the measurement process will create entanglement between system and apparatus by their corresponding coupling unitary $U'_{S:A}$, unless the state is strictly classically correlated, as stated by the activation protocol. The minimal amount of entanglement $E(\tilde{\rho}_{S:A})$ which is created in a local measurement process is quantified by the quantumness of correlations $Q(\rho_S)$ of the system, i.e.,

$$Q(\rho_S) = \min_{U_{S:A}} E(\tilde{\rho}_{S:A}). \quad (1)$$

Different entanglement measures will lead, in principle, to different quantifiers for the quantumness of correlations. The only requirement is that the entanglement measure be monotone under local operations and classical communication (LOCC) maps [6,7,32]. Other measures of quantumness can be recovered with the activation protocol: the quantum discord [6], one-way work deficit [6], relative entropy of quantumness [7], and the geometrical measure of discord via trace norm [33], are some examples.

III. ACTIVATION PROTOCOL FOR INDISTINGUISHABLE PARTICLES

As aforesaid, quantum correlations between distinguishable particles can be interpreted via a unavoidable entanglement created with the measurement apparatus in a partial von Neumann measurement on the particles [6,7], i.e., in a measurement corresponding to a nondegenerate local observable. Note that although the approach is based on projective measurements, it is valid and well defined also for positive operator-valued measures (POVMs) : once the dimension and the partitioning of the ancilla can be arbitrarily chosen, general measurements can be realized through a direct application of Naimark's dilation theorem. In systems of indistinguishable particles the notion of "local measurement" will be implemented through the algebra of single-particle observables (see for example Ref. [14] for a detailed discussion), and based on this identification we shall set up an "activation protocol" for indistinguishable particles. The importance to study the correlations, particularly the entanglement, in terms of subalgebras of observables has been emphasized in [14,22,34–37], proving to be a useful approach for such analysis. The algebra of single-particle observables is generated by

$$\mathcal{O}_{sp} = M \otimes \mathcal{I} \otimes \cdots \otimes \mathcal{I} + \mathcal{I} \otimes M \otimes \cdots \otimes \mathcal{I} + \cdots + \mathcal{I} \otimes \cdots \otimes \mathcal{I} \otimes M, \quad (2)$$

where M is an observable in the Hilbert space of a single particle. We can express this algebra in terms of fermionic or bosonic creation $\{a_i^\dagger\}$ and annihilation $\{a_i\}$ operators, depending on the nature of the particles in the system. The algebra is generated by quadratic observables $\mathcal{O}_{sp} = \sum_{ij} M_{ij} a_i^\dagger a_j$ that can be diagonalized as $\mathcal{O}_{sp} = \sum_k \lambda_k \tilde{a}_k^\dagger \tilde{a}_k$, where $\tilde{a}_k^\dagger = \sum_j U_{kj} a_j^\dagger$ and U is the unitary matrix which diagonalizes M. Thus, since it is a nondegenerate algebra, the eigenvectors of their single-particle observables will be given by single Slater determinants, or permanents, for fermionic and bosonic particles respectively; more precisely, given by the set $\{\tilde{a}_{\vec{k}}^\dagger|\text{vac}\rangle\}$ where $\vec{k} = (k_1, \ldots, k_n)$, $k_i \in \{1, 2, \ldots, \text{dim}_{\text{single-particle}}\}$, represents the states of occupation of n particles, $\tilde{a}_{\vec{k}}^\dagger = \tilde{a}_{k_1}^\dagger \tilde{a}_{k_2}^\dagger \cdots \tilde{a}_{k_n}^\dagger |\text{vac}\rangle$, $\text{dim}_{\text{single-particle}}$ is the single-particle dimension, and $|\text{vac}\rangle$ is the vacuum state. The measurement of single-particle observables is therefore given by a von Neumann measurement, which we shall call hereafter as single-particle von Neumann measurement, according to the complete set of rank-1 projectors $\{\tilde{\Pi}_{\vec{k}} = \tilde{a}_{\vec{k}}^\dagger|\text{vac}\rangle\langle\text{vac}|\tilde{a}_{\vec{k}}\}$, $\sum_{\vec{k}} \tilde{\Pi}_{\vec{k}} = \mathcal{I}_{\mathcal{A}(\mathcal{S})}$, being $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{S}}$ the identity of the antisymmetric and symmetric subspaces, respectively.

Recall that a measurement can be described by coupling the system to a measurement apparatus, with the measurement outcomes being obtained by measuring the apparatus in its eigenbasis . Given a quantum state $\rho_{\mathcal{Q}}$, and a measurement apparatus $\mathcal{M}$ in a pure initial state $|0\rangle_{\mathcal{M}}$, such that $\rho_{\mathcal{Q},\mathcal{M}} = \rho_{\mathcal{Q}} \otimes |0\rangle\langle 0|_{\mathcal{M}}$, their coupling is given by applying a unitary U on the total state that will correlate system and apparatus, $\tilde{\rho}_{\mathcal{Q},\mathcal{M}} = U(\rho_{\mathcal{Q}} \otimes |0\rangle\langle 0|_{\mathcal{M}})U^\dagger$. Such unitary U realizes a single-particle von Neumann measurement $\{\Pi_{\vec{k}}\}$ if for any quantum state $\rho_{\mathcal{Q}}$ holds: $\text{Tr}_{\mathcal{M}}(U(\rho_{\mathcal{Q}} \otimes |0\rangle\langle 0|_{\mathcal{M}})U^\dagger) = \sum_{\vec{k}} \Pi_{\vec{k}} \rho_{\mathcal{Q}} \Pi_{\vec{k}}^\dagger$.

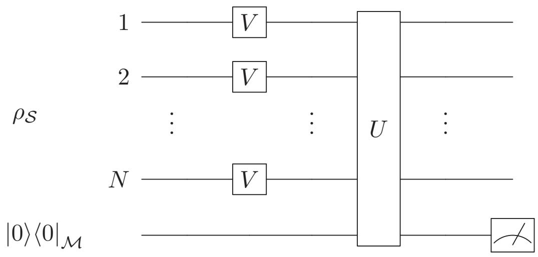

FIG. 1. Activation protocol for a system of indistinguishable particles, where ρ_S is the state of the system, $|0\rangle\langle 0|_\mathcal{M}$ represents the measurement apparatus, V are the single-particle unitary transformations and U is the unitary [as given by Eq. (3)] respective to a single-particle von Neumann measurement.

It is not hard to see how the unitary U must act in order to realize the $\{\Pi_{\vec{k}} = a^\dagger_{\vec{k}}|\text{vac}\rangle\langle\text{vac}|a_{\vec{k}}\}$, $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathcal{I}_{\mathcal{A}(\mathcal{S})}$ measurement. Let us first consider the notation $\{a^\dagger_{\vec{k}}|\text{vac}\rangle\} = \{|f(\vec{k})\rangle\}$, $f(\vec{k}) \in \{1,2,..,\dim_{\mathcal{A}(\mathcal{S})}\}$, with f being a bijective function of the sets $\{\vec{k}\}$ and $\{1,2,\ldots,\dim_{\mathcal{A}(\mathcal{S})}\}$, and $\dim_{\mathcal{A}(\mathcal{S})}$ is the dimension of the antisymmetric or symmetric subspaces. Given that the apparatus has at least the same dimension as the system, the unitary is given by

$$U|f(\vec{k})\rangle_\mathcal{Q} \otimes |j\rangle_\mathcal{M} = |f(\vec{k})\rangle_\mathcal{Q} \otimes |j \oplus f(\vec{k})\rangle_\mathcal{M}. \tag{3}$$

It is easy to show that such operator is indeed unitary; note that

$$U = \sum_{\vec{k},j} |f(\vec{k})\rangle|j \oplus f(\vec{k})\rangle\langle f(\vec{k})|\langle j|, \tag{4}$$

thus,

$$UU^\dagger = \sum_{\vec{k},j,\vec{k}',j'} \delta_{\vec{k},\vec{k}'}\delta_{j,j'}|f(\vec{k})\rangle|j \oplus f(\vec{k})\rangle\langle f(\vec{k}')|\langle j' \oplus f(\vec{k}')|, \tag{5}$$

and since $\{|f(\vec{k})\rangle_\mathcal{Q}\}_{\vec{k}}$ and $\{|j \oplus f(\vec{k})\rangle_\mathcal{M}\}_j$ form a complete set, we have that $UU^\dagger = \mathcal{I}_{\mathcal{A}(\mathcal{S})} \otimes \mathcal{I}_\mathcal{M}$.

Having defined the coupling unitary, we are now able to analyze the entanglement created between system and apparatus in a single-particle von Neumann measurement, $E_{\mathcal{Q},\mathcal{M}}$. Given a quantum state $\rho_\mathcal{Q}$, we intend to quantify the minimum of such entanglement over all single-particle von Neumann measurements, $\min_U E_{\mathcal{Q},\mathcal{M}}[U(\rho_\mathcal{Q} \otimes |0\rangle\langle 0|_\mathcal{M})U^\dagger]$. This quantity then corresponds to the quantumness of correlation in systems of indistinguishable particles. Note that such minimization is analogous to the activation protocol given in [7], but now for systems of indistinguishable particles, where the minimization is carried out on the single-particle unitary transformations $V^{\otimes n}$; see Fig. 1.

IV. RESULTS

Regardless of which entanglement measure is used, let us first see which set of states does not generate entanglement after the activation protocol, i.e., has no quantumness of correlations. We find that this set $\{\xi\}$ is given by states that possess a convex decomposition in orthonormal pure states described by single Slater determinants, or permanents,

$$\xi = \sum_{\vec{k}} p_{\vec{k}}\, \tilde{a}^\dagger_{\vec{k}}|\text{vac}\rangle\langle\text{vac}|\tilde{a}_{\vec{k}}, \quad \sum_{\vec{k}} p_{\vec{k}} = 1, \tag{6}$$

where $\tilde{a}^\dagger_{\vec{k}}|\text{vac}\rangle = V^{\otimes n} a^\dagger_{\vec{k}}|\text{vac}\rangle$, V is a unitary matrix, and $\{a^\dagger_{\vec{k}}\}$ an orthonormal set of creation operators.

Proof. We shall first show that states given by Eq. (6) do not generate entanglement, and then that they are the only ones. Let U be the coupling unitary corresponding to the $\{\Pi_{\vec{k}} = a^\dagger_{\vec{k}}|\text{vac}\rangle\langle\text{vac}|a_{\vec{k}} = |f(\vec{k})\rangle\langle f(\vec{k})|\}$, $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathcal{I}_{\mathcal{A}(\mathcal{S})}$ measurement. Applying the activation protocol on states given by Eq. (6), using $\bar{V} = V^\dagger$ as the single-particle unitary transformation, it follows that

$$\begin{aligned}\rho_{\mathcal{Q}:\mathcal{M}} &= U[(\bar{V}^{\otimes n}\xi\,\bar{V}^{\dagger^{\otimes n}})_\mathcal{Q} \otimes |0\rangle\langle 0|_\mathcal{M}]U^\dagger \\ &= \sum_{\vec{k}} p_{\vec{k}}\,|f(\vec{k})\rangle\langle f(\vec{k})|_\mathcal{Q} \otimes |f(\vec{k})\rangle\langle f(\vec{k})|_\mathcal{M},\end{aligned} \tag{7}$$

where $\rho_{\mathcal{Q}:\mathcal{M}} \in \text{Sep}(\mathcal{Q} \otimes \mathcal{M})$. The demonstration that such states correspond to the unique states that do not generate entanglement is given below. A separable state between system and measurement apparatus can be given by

$$\sigma = \sum_i p_i|\psi_i\rangle\langle\psi_i|_\mathcal{Q} \otimes |\phi_i\rangle\langle\phi_i|_\mathcal{M}, \tag{8}$$

noting that the sets $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ are not necessarily orthogonal. Since the activation protocol corresponds to a unitary operation, thus invertible, there must exist a set $\{|\eta_i\rangle\}$ of states for the system such that

$$U(V^{\otimes n})\,|\eta_i\rangle_\mathcal{Q} \otimes |0\rangle_\mathcal{M} = |\psi_i\rangle_\mathcal{Q} \otimes |\phi_i\rangle_\mathcal{M}, \tag{9}$$

and $\rho_\mathcal{Q} = \sum_i p_i\,|\eta_i\rangle\langle\eta_i|$. Expanding $\{|\eta_i\rangle\}$ on the basis $\{a^\dagger_{\vec{k}}|\text{vac}\rangle\}$ "transformed" by $V^{\dagger^{\otimes n}}$,

$$|\eta_i\rangle = \sum_{\vec{k}} c^{(i)}_{\vec{k}}\, V^{\dagger^{\otimes n}}\, a^\dagger_{\vec{k}}|\text{vac}\rangle, \tag{10}$$

we see from Eqs. (9) and (10) that

$$\begin{aligned}U(V^{\otimes n})\,|\eta_i\rangle \otimes |0\rangle &= \sum_{\vec{k}} c^{(i)}_{\vec{k}}\, a^\dagger_{\vec{k}}|\text{vac}\rangle \otimes |f(\vec{k})\rangle \\ &= |\psi_i\rangle \otimes |\phi_i\rangle.\end{aligned} \tag{11}$$

The above factorization condition imposes the following restriction: $c^{(i)}_{\vec{k}} = \gamma_i\,\delta_{\{\vec{k},g(i)\}}$, $\|\gamma_i\| = 1$, $g : \{i\} \mapsto \{\vec{k}\}$. Therefore,

$$\begin{aligned}\rho_\mathcal{Q} &= \sum_i p_i\,|\eta_i\rangle\langle\eta_i|, \\ &= \sum_i p_i \left(\sum_{\vec{k}} \gamma_i\,\delta_{\{\vec{k},g(i)\}}\, a^\dagger_{\vec{k}}|\text{vac}\rangle\right) \\ &\quad \times\left(\sum_{\vec{k}'}\langle\text{vac}|a_{\vec{k}'}\,\gamma_i^*\,\delta_{\{\vec{k}',g(i)\}}\right), \\ &= \sum_i p_i\, \underbrace{\|\gamma_i\|}_{1}\, a^\dagger_{g(i)}|\text{vac}\rangle\langle\text{vac}|a_{g(i)},\end{aligned} \tag{12}$$

i.e, the states with no quantumness of correlations as given by Eq. (6). ■

Example. Let us show an example of the approach in order to clarify the formalism and the above analysis. An interesting case concerns to the controversial bosonic quantum state $|\psi_b\rangle = \frac{1}{2}(b_0^\dagger b_0^\dagger + b_1^\dagger b_1^\dagger)|\text{vac}\rangle \in \mathcal{S}(\mathcal{H}^2 \otimes \mathcal{H}^2)$, where $\{b_i^\dagger\}$ are the bosonic creation operators. Such a state is considered both entangled by some authors [12,17,25] and nonentangled by others [14,15,18]. Note that such a state can actually be described by a single Slater permanent $|\psi_b\rangle = b_+^\dagger b_-^\dagger|\text{vac}\rangle$, where $b_\pm^\dagger = \frac{1}{\sqrt{2}}(b_0^\dagger \pm b_1^\dagger)$. Defining the coupling unitary U corresponding to the $\{\Pi_{\vec{k}} = b_{\vec{k}}^\dagger|\text{vac}\rangle\langle\text{vac}|b_{\vec{k}}\}$, $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathcal{I}_\mathcal{S}$, $\{\vec{k}\} = \{(0,0),(0,1),(1,1)\}$ measurement, and using the notation

$$b_0^\dagger b_0^\dagger|\text{vac}\rangle = |0\rangle,\ \ b_0^\dagger b_1^\dagger|\text{vac}\rangle = |1\rangle,\ \ b_1^\dagger b_1^\dagger|\text{vac}\rangle = |2\rangle, \quad (13)$$

we have that the unitary acts as follows:

$$U|k\rangle_\mathcal{Q} \otimes |0\rangle_\mathcal{M} = |k\rangle_\mathcal{Q} \otimes |k\rangle_\mathcal{M}. \quad (14)$$

Applying this unitary on the bosonic state, we generate an entangled state between system and apparatus, $U(|\psi_b\rangle_\mathcal{Q} \otimes |0\rangle_\mathcal{M}) = \frac{1}{2}(b_0^\dagger b_0^\dagger|\text{vac}\rangle \otimes |0\rangle + b_1^\dagger b_1^\dagger|\text{vac}\rangle \otimes |2\rangle)$; but this is not an unavoidable entanglement in order to realize that measurement, since we could apply, before the unitary coupling, the single-particle unitary transformation $V : |+\rangle = |0\rangle + i|1\rangle \mapsto |0\rangle, |-\rangle = |0\rangle - i|1\rangle \mapsto |1\rangle$, i.e,

$$V \otimes V : \begin{cases} b_+^\dagger \mapsto b_0^\dagger, \\ b_-^\dagger \mapsto b_1^\dagger. \end{cases} \quad (15)$$

We see now that the coupling between system and apparatus does not generate entanglement between them, $U[(V \otimes V)|\psi_b\rangle_\mathcal{Q} \otimes |0\rangle_\mathcal{M}] = U(b_0^\dagger b_1^\dagger|\text{vac}\rangle_\mathcal{Q} \otimes |0\rangle_\mathcal{M}) = b_0^\dagger b_1^\dagger|\text{vac}\rangle_\mathcal{Q} \otimes |1\rangle_\mathcal{M} \in Sep(\mathcal{Q} \otimes \mathcal{M})$, and thus such a state has no quantumness of correlations.

An important result to be emphasized in this analysis via the activation protocol relates to the establishment of an equivalence between the quantumness of correlations with the *distinguishable* bipartite entanglement between system and apparatus, showing the usefulness of the correlations between indistinguishable particles. Note that the set $\{\xi\}$ is simply the antisymmetrization or symmetrization of the distinguishable classically correlated states (states with distinguishable particles with no quantumness of correlations), and all their correlations are due to the exchange correlations; the activation protocol then shows that any kind of correlations between indistinguishable particles beyond the mere exchange correlations can always be activated or mapped into distinguishable bipartite entanglement between $\mathcal{Q} : \mathcal{M}$.

The correlations between indistinguishable particles can thereby be characterized by different types: the entanglement, the quantumness of correlations as discussed in this article, the correlations generated merely by particle statistics (exchange correlation), and the classical correlations. In fact, there are quantum states whose particles are classically correlated, not even possessing exchange correlations, such as pure bosonic states with all their particles occupying the same degree of freedom, $|\psi_b\rangle = \frac{1}{\sqrt{n!}}(b_i^\dagger)^n|\text{vac}\rangle$, or mixed states described by an orthonormal convex decomposition of such pure states, $\chi_b = \sum_i \frac{1}{n}(b_i^\dagger)^n|\text{vac}\rangle\langle\text{vac}|(b_i)^n$. See Fig. 2 for a schematic picture of these different kinds of correlations. Interesting questions to raise are how the notion of entanglement of particles is related to the quantumness of correlations, and if they are equivalent for pure states. We can note from Eq. (6) that, for pure states, the set with no quantumness of correlations is described by states with a single Slater determinant, or permanent, which is equivalent to the set of unentangled pure states. Actually there is an ongoing debate regarding the correct definition of particle entanglement [14–18], but at the same time there are strong physical reasons to consider particle entanglement in pure states as the correlations beyond the mere exchange correlations [14–16,18]. Concerning mixed states, it becomes clear that the set given by Eq. (6) is a subset of the unentangled one, thereby quantumness of correlations is a more general notion of correlations than entanglement.

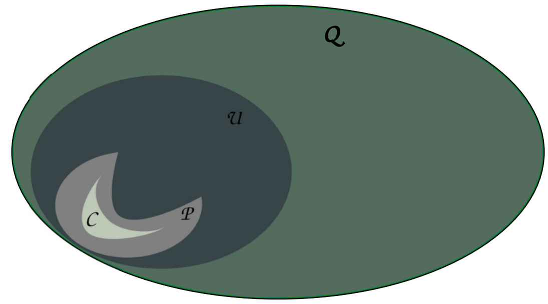

FIG. 2. (Color online) Schematic picture of the distinct types of correlations in systems of indistinguishable particles. The larger set ($\mathcal{Q}$) denotes the set of all fermionic, or bosonic, quantum states; the blue area ($\mathcal{U}$) represents the convex set of states with no entanglement; the gray area ($\mathcal{P}$) represents the non convex set of states with no quantumness of correlations, as defined in this article [Eq. (6)]; and the yellow area ($\mathcal{C}$) represents the nonconvex set of states with no exchange correlations due to the particle statistics, possessing only classical correlations. Note that for fermionic particles, the set $\mathcal{C}$ is a null set. The following hierarchy is identified: $\mathcal{C} \subset \mathcal{P} \subset \mathcal{U} \subset \mathcal{Q}$.

According to the activation protocol, different entanglement measures will lead, in principle, to different quantifiers for the quantumness of correlations. We can thus define the measure Q_E for quantumness of correlations, associated with the entanglement measure E, as follows:

$$Q_E(\rho_\mathcal{Q}) = \min_V E(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}), \quad (16)$$

where $\tilde{\rho}_{\mathcal{Q},\mathcal{M}} = U[(V^{\otimes n}\rho_\mathcal{Q} V^{\dagger^{\otimes n}}) \otimes |0\rangle\langle 0|_\mathcal{M}]U^\dagger$.

We shall consider two different entanglement measures for the bipartite entanglement: the physically motivated distillable entanglement E_D [38] and the relative entropy of entanglement E_r [39,40]. Note that the output states of the activation protocol have the so-called maximally correlated form [41] between system and measurement apparatus, $\tilde{\rho}_{\mathcal{Q},\mathcal{M}} = \sum_{\vec{l},\vec{l}'} \chi_{\vec{l},\vec{l}'}^V |f(\vec{l})\rangle\langle f(\vec{l}')|_\mathcal{Q} \otimes |f(\vec{l})\rangle\langle f(\vec{l}')|_\mathcal{M}$, with $\chi_{\vec{l},\vec{l}'}^V = (\Pi_{\vec{l}}^V)^\dagger \rho_\mathcal{Q} (\Pi_{\vec{l}'}^V)$, where $\Pi_{\vec{l}}^V = V^{\otimes n}\Pi_{\vec{l}}$ (see the Appendix). It is known that the entanglement for such states according to the distillable entanglement [42], as well as for the relative entropy of entanglement [41], is given by $E_{D(r)}(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}) = S(\tilde{\rho}_\mathcal{Q}) - S(\tilde{\rho}_{\mathcal{Q},\mathcal{M}})$, where $S(\rho) = -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy. The first term is given by $S(\tilde{\rho}_\mathcal{Q}) = S(\sum_{\vec{l}} (\Pi_{\vec{l}}^V)^\dagger \rho_\mathcal{Q} (\Pi_{\vec{l}}^V)|f(\vec{l})\rangle\langle f(\vec{l})|)$, i.e., the entropy of the projected state $\rho_\mathcal{Q}$ according to a single-particle von

Neumann measurement, and the second term is simply given by $S(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}) = S(\rho_{\mathcal{Q}})$, since it is invariant under unitary transformations. Thus we have that the quantumness of correlations measure is given by

$$Q_{E_{D(r)}}(\rho_{\mathcal{Q}}) = \min_{V}\left[S\left(\sum_{\vec{l}} \left(\Pi_{\vec{l}}^{V}\right)^{\dagger}\rho_{\mathcal{Q}}\left(\Pi_{\vec{l}}^{V}\right)|f(\vec{l})\rangle\langle f(\vec{l})|\right) - S(\rho_{\mathcal{Q}})\right], \tag{17}$$

which corresponds to the notion of minimum disturbance caused in the system by single-particle measurements. This result is in agreement with the analysis made in [43] for the particular case of two-fermion systems, and to the best of our knowledge is the only study attempting to characterize and quantify a more general notion of correlations between indistinguishable particles. Using arguments analogous to those in [11], it is possible to prove Eq. (17) is an equivalent expression to

$$Q_{E_{D(r)}}(\rho_{\mathcal{Q}}) = \min_{\xi} S(\rho_{\mathcal{Q}} \parallel \chi), \tag{18}$$

where $S(\rho \parallel \chi) = \mathrm{Tr}(\rho \ln \rho - \rho \ln \chi)$ is the relative entropy. The above equation introduces a geometrical approach to the particle correlation measure. Notably we see that, as well as for the quantumness of correlations in distinguishable subsystems, the quantumness of correlations between indistinguishable particles defined in this article has a variety of equivalent approaches in order to characterize and quantify it, as shown by the activation protocol [Eq. (16)], minimum disturbance [Eq. (17)] and geometrical approach [Eq. (18)].

V. CONCLUSION

In this work we discussed how to define a more general notion of correlation, called quantumness of correlations, in fermionic and bosonic indistinguishable particles, and presented equivalent ways to quantify it, addressing the notion of an activation protocol, the minimum disturbance in a single-particle von Neumann measurement, and a geometrical view for its quantification. An important result of our approach concerns to the equivalence of these correlations to the entanglement in distinguishable subsystems via the activation protocol, thus settling its usefulness for quantum information processing. It is interesting to note that the approach used in this work is essentially based on the definition of the algebra of single-particle observables, dealing here with the algebra of indistinguishable fermionic, or bosonic, single-particle observables, but we could apply the same idea for identical particles of general statistics, e.g., braid-group statistics, simply by defining the correct single-particle algebra of observables.

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APPENDIX: MAXIMALLY CORRELATED STATES

Let us show that the output states of the activation protocol for indistinguishable particles have the so called maximally correlated form between system and measurement apparatus. If $\{a_{\vec{k}}^{\dagger}|\mathrm{vac}\rangle\} = \{|f(\vec{k})\rangle\}$ is the system basis, U is the coupling unitary given by Eq. (3), and V is the unitary respective to the single-particle transformation, we have that

$$V^{\otimes n}\, a_{\vec{k}}^{\dagger}|\mathrm{vac}\rangle = \left(\sum_{l_1} v_{k_1 l_1} a_{l_1}^{\dagger}\right)\cdots\left(\sum_{l_n} v_{k_n l_n} a_{l_n}^{\dagger}\right)|\mathrm{vac}\rangle = \sum_{\vec{l}} v_{k_1 l_1}\cdots v_{k_n l_n}|f(\vec{l})\rangle, \tag{A1}$$

where $v_{k_i l_j}$ are the matrix elements of V. A general state for the system can be given as

$$\rho_{\mathcal{Q}} = \sum_{\vec{k},\vec{k}'} p_{\vec{k},\vec{k}'}|f(\vec{k})\rangle\langle f(\vec{k}')|; \tag{A2}$$

thereby,

$$V^{\otimes n}\rho_{\mathcal{Q}}V^{\dagger\otimes n} = \sum_{\vec{k},\vec{k}',\vec{l},\vec{l}'} p_{\vec{k},\vec{k}'}(v_{k_1 l_1}\cdots v_{k_n l_n}) \times (v_{k'_1 l'_1}\cdots v_{k'_n l'_n})^{\dagger}\,|f(\vec{l})\rangle\langle f(\vec{l}')|, = \sum_{\vec{l},\vec{l}'} \chi_{\vec{l},\vec{l}'}^{V}\,|f(\vec{l})\rangle\langle f(\vec{l}')|, \tag{A3}$$

where $\chi_{\vec{l},\vec{l}'}^{V} = \sum_{\vec{k},\vec{k}'} p_{\vec{k},\vec{k}'}(v_{k_1 l_1}\cdots v_{k_n l_n})(v_{\vec{k}'_1 l'_1}\cdots v_{\vec{k}'_n l'_n})^{\dagger}$. The output states of the activation protocol thus have the form

$$\rho_{\mathcal{Q}:\mathcal{M}} = U[(V^{\otimes n}\rho_{\mathcal{Q}}V^{\dagger\otimes n})\otimes|0\rangle\langle 0|_{\mathcal{M}}]U^{\dagger} = \sum_{\vec{l},\vec{l}'} \chi_{\vec{l},\vec{l}'}^{V}\,|f(\vec{l})\rangle\langle f(\vec{l}')|_{\mathcal{Q}}\otimes|f(\vec{l})\rangle\langle f(\vec{l}')|_{\mathcal{M}}, \tag{A4}$$

i.e., the maximally correlated form.

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