Quantum Fisher information for states in exponential form

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We derive explicit expressions for the quantum Fisher information and the symmetric logarithmic derivative (SLD) of a quantum state in the exponential form $\rho = \exp(G)$; the SLD is expressed in terms of the generator G. Applications include quantum-metrology problems with Gaussian states and general thermal states. Specifically, we give the SLD for a Gaussian state in two forms, in terms of its generator and its moments; the Fisher information is also calculated for both forms. Special cases are discussed, including pure, degenerate, and very noisy Gaussian states.

I. INTRODUCTION

Quantum metrology studies the limit to the accuracy, set by quantum mechanics, with which physical quantities can be estimated by measurements. The basic idea is to determine an unknown parameter θ by probing a quantum state that depends on the parameter. Quantum metrology is important for various purposes, which include improving time and frequency standards [1,2], detecting gravitational waves [3,4], interferometry based on interacting systems [5,6], and magnetometry [7,8].

A standard scenario for quantum parameter estimation is to put a known initial state $\rho_{\rm in}$ through a quantum channel $\mathcal{E}_\theta$ that impresses θ on the system; the output state $\rho(\theta) = \mathcal{E}_\theta(\rho_{\rm in})$ is then subjected to a measurement. The goal is to find the optimal measurement strategy so that as much information as possible about θ is acquired. Although it is hard to solve the most general problem exactly, bounds on how accurately one can estimate a parameter can be obtained [9–12].

In classical parameter estimation theory, the Cramér-Rao bound (CRB) expresses a lower bound on the variance of an unbiased estimator $\theta_{\rm est}$,

$$\mathrm{var}(\theta_{\rm est}) \geqslant \frac{1}{\mathcal{I}_c(\theta)}, \tag{1.1}$$

where $\mathcal{I}_c(\theta)$ is the classical Fisher information [13]. Fisher's theory says that maximum likelihood estimation achieves the CRB asymptotically for a large number of trials [14,15]. For the quantum case, it was shown, in [16], that there exists an optimal quantum measurement whose classical Fisher information, obtained from the measurement outcomes, achieves the quantum Fisher information [16–19],

$$\mathcal{I}(\theta) = \mathrm{tr}[\rho(\theta) L^2(\theta)]. \tag{1.2}$$

Thus the inverse of the quantum Fisher information gives the quantum CRB on the variance of an estimator. The (Hermitian) operator $L(\theta)$, in Eq. (1.2), is the symmetric logarithmic derivative (SLD), defined implicitly by

$$\frac{d\rho(\theta)}{d\theta} = \frac{1}{2}\{L(\theta),\rho(\theta)\}, \tag{1.3}$$

where the curly braces denote the anticommutator. Knowing the SLD allows one to obtain not only the Fisher information but also the optimal measurement scheme.

Any full rank quantum state $\rho(\theta)$ can be written in exponential form,

$$\rho(\theta) = e^{G(\theta)}, \tag{1.4}$$

with the normalization absorbed into $G(\theta)$. The case that $\rho(\theta)$ is not invertible can be handled as a limit in which some eigenvalues of $G(\theta)$ go to minus infinity. The form (1.4) is useful when $G(\theta)$ takes a simple form, examples being Gaussian states and general thermal states. Gaussian states are important because of their appealing properties for quantum-metrology tasks [20–22] and their accessibility both to experimentalists and theorists. Thermal states are also useful for quantum-metrology tasks for at least two reasons: (i) The initial state is often a thermal state $\rho_{\rm in} = e^{-\beta H}/Z$, and the simple exponential form is preserved by a unitary channel U_θ. (ii) We can infer the temperature and the chemical potential by measuring the state $\rho(\theta) = e^{-\beta(H-\mu N)}/Z$, after the system is brought to thermodynamic equilibrium with a reservoir [23].

In Sec. II we consider the SLD for a quantum state in the exponential form (1.4). We show that the SLD can be expanded into a weighted sum of $dG/d\theta$ and its recursive, nested commutators with G. Simple expressions of the quantum Fisher information and the SLD are given in the basis where G is diagonalized. In Sec. III we apply the results of Sec. II to Gaussian states, and an explicit expression of the SLD in terms of the generator is derived. In Sec. IV, also for Gaussian states, the SLD and the quantum Fisher information are given in terms of the moments of position and momentum operators (or of creation and annihilation operators).

II. QUANTUM FISHER INFORMATION FOR STATES IN EXPONENTIAL FORM

A useful expression (see Eq. (2.1) of Ref. [24]) for density operators of the exponential form (1.4) is

$$\dot{\rho} = \int_0^1 e^{sG}\, \dot{G}\, e^{(1-s)G}\, ds, \tag{2.1}$$

where an overdot denotes a derivative with respect to θ. We now use the nested-commutator relation

$$e^G A e^{-G} = A + [G,A] + \frac{1}{2!}[G,[G,A]] + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{C}^n(A) = e^{\mathcal{C}}(A), \tag{2.2}$$

where $\mathcal{C}^n(A)$, a linear operation on A, denotes the nth-order nested commutator $[G, \dots, [G, A]]$, with $\mathcal{C}^0(A) = A$. Applying this relation to the expression (2.1), we get

$$\dot\rho\rho^{-1} = \dot{G} + \frac{1}{2!}[G, \dot{G}] + \frac{1}{3!}[G, [G, \dot{G}]] + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \mathcal{C}^n(\dot{G}) = h(\mathcal{C})(\dot{G}), \qquad (2.3)$$

where h is the generating function of the expansion coefficients in Eq. (2.3),

$$h(t) = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots = \frac{e^t - 1}{t}. \qquad (2.4)$$

Using the definitions (1.3) and (1.4), we also have

$$\dot\rho\rho^{-1} = \frac{1}{2}(L + e^G L e^{-G})$$

$$= \frac{1}{2}\left[L + \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{C}^n(L)\right] = r(\mathcal{C})(L), \qquad (2.5)$$

where the generating function is $r(t) = (e^t + 1)/2$. Suppose that the SLD adopts the form

$$L = \sum_{n=0}^{\infty} f_n \mathcal{C}^n(\dot{G}) = f(\mathcal{C})(\dot{G}), \qquad (2.6)$$

where the to be determined generating function f is specified by

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots. \qquad (2.7)$$

By putting Eq. (2.6) into Eq. (2.5) we have

$$\dot\rho\rho^{-1} = r(\mathcal{C})[f(\mathcal{C})(\dot{G})] = rf(\mathcal{C})(\dot{G}), \qquad (2.8)$$

where rf is the product of the two functions, and we use the identity $\mathcal{C}^n(\mathcal{C}^m(A)) = \mathcal{C}^{n+m}(A)$. Comparing Eq. (2.8) with Eq. (2.3) we have the relation among the generating functions,

$$f(t) = \frac{h(t)}{r(t)} = \frac{\tanh(t/2)}{t/2}$$

$$= \sum_{n=0}^{\infty} \frac{4(4^{n+1} - 1)B_{2n+2}}{(2n+2)!} t^{2n}, \qquad (2.9)$$

where B_{2n+2} is the $(2n+2)$th Bernoulli number. Comparing Eqs. (2.7) with (2.9) we have

$$f_n = \begin{cases} \frac{4(4^{n/2+1}-1)B_{n+2}}{(n+2)!}, & \text{for even } n, \\ 0, & \text{for odd } n. \end{cases} \qquad (2.10)$$

The vanishing of the odd-order f_n is a consequence of the Hermiticity of L, which makes $f(t)$ an even function.

The first four nonzero coefficients f_n are

$$f_0 = 1, \quad f_2 = -\frac{1}{12}, \quad f_4 = \frac{1}{120}, \quad f_6 = -\frac{34}{8!}. \qquad (2.11)$$

Although it appears that the f_n become negligible very fast, they revive at larger n, and the radius of convergence of the power series (2.7) is $t < \pi$. This limits the usefulness of the expansion (2.6); it is divergent when the difference between any two eigenvalues of G is greater than or equal to π. Fortunately, in many real problems, the recursive commutators

in Eq. (2.6) either terminate or repeat, enabling us to find an exact solution. In the latter case we can use analytic continuation to extend the result (2.6) beyond the domain of convergence.

Suppose that we work in the basis $|e_j\rangle$ where G is diagonal, i.e., $G|e_j\rangle = g_j|e_j\rangle$. This basis generally changes with θ, so we are considering here, as in the rest of this section, a particular value of θ. In this basis, Eq. (2.6) is equivalent to

$$L_{jk} = \langle e_j|L|e_k\rangle = f(g_j - g_k)\dot{G}_{jk}. \qquad (2.12)$$

The domain of Eq. (2.12) is not restricted to the radius of convergence $g_j - g_k < \pi$; it is well defined for any G, which is an example of analytic continuation. Using Eq. (2.3) we have

$$\dot\rho_{jk} = \langle e_j|\dot\rho|e_k\rangle = e^{g_k} h(g_j - g_k)\dot{G}_{jk}, \qquad (2.13)$$

and Eq. (2.12) can be converted to a formula familiar from Ref. [16],

$$L_{jk} = \frac{\dot\rho_{jk}}{e^{g_k} r(g_j - g_k)} = \frac{2\dot\rho_{jk}}{\rho_{jj} + \rho_{kk}}, \qquad (2.14)$$

where $\rho_{jj} = e^{g_j}$. This formula follows directly from the definition (1.3) of the SLD.

The Fisher information can now be calculated directly in this same basis,

$$\mathcal{I} = \sum_{j,k} e^{g_j}|L_{jk}|^2 = \sum_{j,k} e^{g_j} f^2(g_j - g_k)|\dot{G}_{jk}|^2. \qquad (2.15)$$

As a simple example we discuss the SLD and Fisher information for a qubit. Letting the Pauli matrices be denoted by σ_j, we can, without loss of generality, assume that the qubit state is diagonal in the eigenbasis of σ_3 and write the state as $\rho = \frac{1}{2}(\sigma_0 + \sigma_3 \tanh\gamma)$, where $\tanh\gamma$ is the expectation value of σ_3. This gives us

$$G = \gamma\sigma_3 - \ln(2\cosh\gamma)\sigma_0, \qquad (2.16)$$

$$\dot{G} = \dot\gamma(\sigma_3 - \sigma_0\tanh\gamma) + \tau_1\sigma_1 + \tau_2\sigma_2. \qquad (2.17)$$

Here $\dot\gamma$ accounts for the change in the eigenvalues of ρ as θ changes, and the real parameters τ_1 and τ_2 account for the change in eigenbasis of ρ as θ changes. Putting Eqs. (2.16) and (2.17) into Eq. (2.6) or (2.12) we have

$$L = \dot\gamma(\sigma_3 - \sigma_0\tanh\gamma) + \frac{\tanh\gamma}{\gamma}(\tau_1\sigma_1 + \tau_2\sigma_2). \qquad (2.18)$$

This expression can be verified by expanding the 2×2 density operator explicitly. The result for the Fisher information is

$$\mathcal{I} = \frac{\dot\gamma^2}{\cosh^2\gamma} + \frac{\tanh^2\gamma}{\gamma^2}(\tau_1^2 + \tau_2^2). \qquad (2.19)$$

When the eigenvalues of the density operator ρ are independent of θ, i.e., the change of ρ can be described by a unitary process, we have $\dot{G} = i[G, H] = i\mathcal{C}(H)$, where H is some Hermitian operator. Putting this expression into Eq. (2.6), we have the following formula for the SLD:

$$L = f(\mathcal{C})(\dot{G}) = if(\mathcal{C})\mathcal{C}(H) = 2i\tanh(\mathcal{C}/2)(H), \qquad (2.20)$$

which was first found by Knysh and Durkin (see Eq. (A3) of Ref. [25]).

III. GAUSSIAN STATES IN EXPONENTIAL FORM

In this section we apply the expansion (2.6) to Gaussian states, which naturally adopt the exponential form

$$\rho = e^G = \exp\left(-\tfrac{1}{2}\mathbf{r}^T \Omega \, \mathbf{r} + \mathbf{r}^T \boldsymbol{\eta} - \ln Z\right), \qquad (3.1)$$

where $\mathbf{r} = (x_1 \cdots x_n \ p_1 \cdots p_n)^T$ is the $2n$-dimensional vector of position and momentum operators, $\boldsymbol{\eta}$ is a real $2n$-dimensional vector, and $\Omega > 0$ is a $2n \times 2n$ real, symmetric matrix. The state (3.1) can be regarded as a thermal state, with $\beta = 1$, of the quadratic Hamiltonian

$$H = \tfrac{1}{2}\mathbf{r}^T \Omega \, \mathbf{r} - \mathbf{r}^T \boldsymbol{\eta}; \qquad (3.2)$$

notice that $Z = \mathrm{tr}(e^{-H})$.

The canonical commutation relations can be written as $[r_j, r_k] = i J_{jk}$, where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = -J^T = -J^{-1}, \qquad (3.3)$$

with $\mathbb{1}$ being the $n \times n$ identity matrix. For any Gaussian state, both G and $\dot{G}$ are degree-2 polynomials of the position and momentum operators, and thus so are all the recursive commutators in Eq. (2.6). Consequently, L is also a degree-2 polynomial of the position and momentum operators,

$$L = \mathbf{r}^T \Phi \, \mathbf{r} + \mathbf{r}^T \boldsymbol{\zeta} - \nu, \qquad (3.4)$$

where $\boldsymbol{\zeta}$ is a real $2n$-dimensional vector, Φ is a $2n \times 2n$ real, symmetric matrix, and ν can be determined by the trace-preserving condition

$$\nu = \mathrm{tr}(\rho \, \mathbf{r}^T \Phi \, \mathbf{r}). \qquad (3.5)$$

In order to use the expansion (2.6) efficiently, we write the quadratic Hamiltonian in the basis of creation and annihilation operators,

$$H = \tfrac{1}{2}\bar{\boldsymbol{a}} \, \Omega' \boldsymbol{a} - \bar{\boldsymbol{a}} \, \boldsymbol{\eta}', \qquad (3.6)$$

where $\boldsymbol{a}$ and $\bar{\boldsymbol{a}}$ are vectors of the creation and annihilation operators

$$\bar{\boldsymbol{a}} = (a_1^\dagger \cdots a_n^\dagger \ a_1 \cdots a_n), \qquad (3.7)$$

$$\boldsymbol{a} = (a_1 \cdots a_n \ a_1^\dagger \cdots a_n^\dagger)^T, \qquad (3.8)$$

with $a_j = (x_j + i p_j)/\sqrt{2}$; the matrix Ω' and the vector $\boldsymbol{\eta}'$ satisfy

$$\Omega' = V^\dagger \Omega V, \quad \boldsymbol{\eta}' = V^\dagger \boldsymbol{\eta}, \qquad (3.9)$$

where V is a unitary matrix linking the two bases, i.e., $V\boldsymbol{a} = \mathbf{r}$, or equivalently, $V^\dagger \mathbf{r} = \boldsymbol{a}$,

$$V^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}. \qquad (3.10)$$

Similarly, we can write the SLD as

$$L = \bar{\boldsymbol{a}} \, \Phi' \boldsymbol{a} + \bar{\boldsymbol{a}} \, \boldsymbol{\zeta}' - \nu, \qquad (3.11)$$

where $\Phi' = V^\dagger \Phi V$ and $\boldsymbol{\zeta}' = V^\dagger \boldsymbol{\zeta}$.

Without affecting the Fisher information, which is invariant under unitary transformations, we can displace the state (3.1)

so that $\boldsymbol{\eta} = 0$. Moreover, we now assume that the matrix Ω is in the diagonal form

$$\Omega = \begin{pmatrix} \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) & 0 \\ 0 & \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \end{pmatrix} = \Omega', \qquad (3.12)$$

which gives

$$G = -H - \ln Z = -\sum_{j=1}^n \epsilon_j \left(a_j^\dagger a_j + \frac{1}{2}\right) - \ln Z. \qquad (3.13)$$

This case is important, because any Gaussian state is equivalent to it up to a Gaussian unitary, i.e., a symplectic transformation of the creation and annihilation operators. The commutation relations between G and the creation and annihilation operators are straightforward:

$$[G, a_j] = \epsilon_j a_j, \quad [G, a_j^\dagger] = -\epsilon_j a_j^\dagger. \qquad (3.14)$$

Consequently, we have

$$f(\mathcal{C})(a_j) = f(\epsilon_j)a_j, \quad f(\mathcal{C})(a_j^\dagger) = f(\epsilon_j)a_j^\dagger, \qquad (3.15)$$

and for quadratic operators we have

$$f(\mathcal{C})(a_j^\dagger a_k) = f(\epsilon_k - \epsilon_j)a_j^\dagger a_k, \qquad (3.16)$$

$$f(\mathcal{C})(a_j a_k) = f(\epsilon_j + \epsilon_k)a_j a_k. \qquad (3.17)$$

Most generally, the derivative of G takes the form

$$\dot{G} = -\frac{1}{2}\bar{\boldsymbol{a}} \, \dot{\Omega}' \boldsymbol{a} + \bar{\boldsymbol{a}} \, \dot{\boldsymbol{\eta}}' - \frac{\dot{Z}}{Z}. \qquad (3.18)$$

Putting Eq. (3.18) into Eq. (2.6) and using the relations (3.15)–(3.17) we have

$$\nu = \dot{Z}/Z, \quad \boldsymbol{\zeta}'_j = f(\epsilon_j)\dot{\boldsymbol{\eta}}'_j, \qquad (3.19)$$

and

$$\Phi'_{jk} = \begin{cases} -\frac{1}{2}f(\epsilon_j - \epsilon_k)\dot{\Omega}'_{jk}, & \text{for } j,k \leqslant n \text{ or } j,k > n, \\ -\frac{1}{2}f(\epsilon_j + \epsilon_k)\dot{\Omega}'_{jk}, & \text{for all other cases,} \end{cases}$$

$$\qquad (3.20)$$

where $\epsilon_{j+n} = \epsilon_j$ for $j \leqslant n$. Equations (3.19) and (3.20) are explicit, and the only work required is to find the basis of the creation and annihilation operators, by a symplectic transformation, so that the Gaussian state is of the diagonal form (3.13).

Knowing the SLD allows one to calculate the Fisher information [see Eq. (4.29)],

$$\mathcal{I} = \tfrac{1}{2}\mathrm{tr}(\Gamma \Phi \Gamma \Phi) - \tfrac{1}{2}(J \Phi J^T \Phi) + \tfrac{1}{2}\boldsymbol{\zeta}^T \Gamma \boldsymbol{\zeta}, \qquad (3.21)$$

where Γ is the covariance matrix of the Gaussian state defined in Eq. (4.3). Going to the basis of creation and annihilation operators, we have

$$\mathcal{I} = \tfrac{1}{2}\mathrm{tr}(\Gamma' \Phi' \Gamma' \Phi') - \tfrac{1}{2}(J' \Phi' J' \Phi') + \tfrac{1}{2}\boldsymbol{\zeta}'^\dagger \Gamma' \boldsymbol{\zeta}', \qquad (3.22)$$

where

$$J' = J'^\dagger = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad (3.23)$$

and for Ω' taking the form (3.12), we have $\Gamma' = V^\dagger \Gamma V = \coth(\Omega'/2)$. Thus, the Fisher information can be calculated

explicitly,

$$\mathcal{I} = \sum_{j,k=1}^{n} (|\Phi'_{jk}|^2 + |\Phi'_{j,k+n}|^2) \coth \frac{\epsilon_j}{2} \coth \frac{\epsilon_k}{2}$$

$$+ |\Phi'_{j,k+n}|^2 - |\Phi'_{jk}|^2 + \sum_{j=1}^{n} |\zeta'_j|^2 \coth \frac{\epsilon_j}{2}. \quad (3.24)$$

IV. GAUSSIAN STATES BY MOMENTS

A number of authors have already discussed SLDs and quantum Fisher information for Gaussian states. Monras and Paris [26] investigated the problem of loss estimation with displaced squeezed thermal states. Pinel *et al.* [27,28] discussed parameter estimation with pure Gaussian states of arbitrarily many modes and general single-mode Gaussian states. Recently, Monras [29] found an equation—in terms of the moments—for the SLD of the most general Gaussian state. The Fisher information can be calculated once the SLD is known. Here we confirm Monras' results by using a different, somewhat simpler approach. Furthermore, we solve the resultant equation of the SLD with a symplectic transformation. Special cases are also discussed, which include pure, degenerate, and very noisy Gaussian states.

Most generally, the symmetrically ordered characteristic function of a Gaussian quantum state takes the form

$$\chi_S(\boldsymbol{\xi}) \equiv \mathrm{tr}(\rho\, e^{i \mathbf{r}^T \boldsymbol{\xi}}) = \exp\left(-\tfrac{1}{4} \boldsymbol{\xi}^T \Gamma \boldsymbol{\xi} + i \boldsymbol{\delta}^T \boldsymbol{\xi}\right), \quad (4.1)$$

where $\boldsymbol{\delta}$ is a real $2N$-dimensional vector, and $\Gamma > 0$ is a $2N \times 2N$ real, symmetric matrix. The vector $\boldsymbol{\delta}$ and the matrix Γ represent the means and the covariance matrix of the Gaussian state,

$$\delta_j = \mathrm{tr}(\rho r_j), \quad (4.2)$$

$$\Gamma_{jk} = \mathrm{tr}(\rho \{\Delta r_j, \Delta r_k\}), \quad (4.3)$$

where $\Delta r_j = r_j - \delta_j$. Without loss of generality, the mean $\boldsymbol{\delta}$ can be removed by a displacement,

$$\rho \rightarrow e^{-i\mathbf{r}^T J \boldsymbol{\delta}} \rho\, e^{i\mathbf{r}^T J \boldsymbol{\delta}}, \quad (4.4)$$

and we assume $\boldsymbol{\delta} = 0$ from now on.

A. Calculating the SLD

Taking a derivative with respect to θ on both sides of Eqs. (4.2) and (4.3) and using the definition (1.3) we have

$$\dot{\delta}_j = \tfrac{1}{2}\mathrm{tr}(\{\rho, L\}\, r_j) = \tfrac{1}{2}\mathrm{tr}(\rho\, \{L, r_j\}), \quad (4.5)$$

$$\dot{\Gamma}_{jk} = \tfrac{1}{2}\mathrm{tr}(\{\rho, L\}\{r_j, r_k\}) = \tfrac{1}{2}\mathrm{tr}(\rho\, \{L, \{r_j, r_k\}\}). \quad (4.6)$$

To calculate the traces in Eqs. (4.5) and (4.6) we introduce the following function, which we call the *partially symmetrically ordered characteristic function*:

$$\chi_P(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \equiv \tfrac{1}{2}\mathrm{tr}\left(\rho\, \{e^{i\mathbf{r}^T \boldsymbol{\xi}_1}, e^{i\mathbf{r}^T \boldsymbol{\xi}_2}\}\right)$$

$$= \chi_S(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) \cos\left(\tfrac{1}{2}\boldsymbol{\xi}_1^T J \boldsymbol{\xi}_2\right). \quad (4.7)$$

Denoting the partial derivative with respect to the jth element of $\boldsymbol{\xi}_{1,2}$ by $\partial_j^{(1,2)}$ we have

$$\dot{\delta}_j = -i\mathcal{L}^{(1)}\partial_j^{(2)} \chi_P\big|_{\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = 0}, \quad (4.8)$$

$$\dot{\Gamma}_{jk} = -2\mathcal{L}^{(1)}\partial_{jk}^{(2)} \chi_P\big|_{\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = 0}, \quad (4.9)$$

where $\partial_{jk} = \partial_j \partial_k$ and

$$\mathcal{L} = -\sum_{m,n} \Phi_{mn} \partial_{mn} - i\sum_l \zeta_l \partial_l - \nu. \quad (4.10)$$

Putting Eq. (4.10) into Eqs. (4.8) and (4.9) we have

$$\dot{\delta}_j = -\left(\sum_l \zeta_l \partial_l^{(1)}\right) \partial_j^{(2)} \chi_P\bigg|_{\boldsymbol{\xi}_{1,2}=0} = \tfrac{1}{2}\sum_l \Gamma_{jl}\, \zeta_l, \quad (4.11)$$

$$\dot{\Gamma}_{jk} = 2\left(\sum_{m,n} \Phi_{mn} \partial_{mn}^{(1)} + \nu\right) \partial_{jk}^{(2)} \chi_P\bigg|_{\boldsymbol{\xi}_{1,2}=0}$$

$$= \left[\tfrac{1}{2}\mathrm{tr}(\Gamma\Phi) - \nu\right]\Gamma_{jk} + (\Gamma\Phi\Gamma + J\Phi J)_{jk}, \quad (4.12)$$

where all the odd-order derivatives are neglected, because they vanish at $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = 0$ for $\boldsymbol{\delta} = 0$. By using the trace-preserving condition,

$$0 = \mathrm{tr}(L\rho) = \mathcal{L}\chi_S|_{\boldsymbol{\xi}=0} = \tfrac{1}{2}\mathrm{tr}(\Gamma\Phi) - \nu, \quad (4.13)$$

we have the following matrix forms:

$$\dot{\boldsymbol{\delta}} = \tfrac{1}{2}\Gamma\boldsymbol{\zeta}, \quad (4.14)$$

$$\dot{\Gamma} = \Gamma\Phi\Gamma - J\Phi J^T. \quad (4.15)$$

Equation (4.15) is an implicit matrix equation, which is generally hard to solve. A way to circumvent such difficulty is by using a symplectic transformation. Any covariance matrix Γ can be brought into the following standard (canonical) form by a symplectic transformation S satisfying $SJS^T = J$,

$$S\Gamma S^T = \Gamma_s = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad (4.16)$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \geqslant \mathbb{1}$ is a diagonal matrix (equality holds, i.e., $\lambda_j = 1$ for $j = 1, \ldots, n$, only for pure states). In the basis that Γ is standard, Eq. (4.15) reads

$$\dot{\Gamma}_s = \Gamma_s \Phi_s \Gamma_s - J\Phi_s J^T, \quad (4.17)$$

where $\dot{\Gamma}_s = S\dot{\Gamma}S^T$ and $J\Phi_s J^T = SJ\Phi J^T S^T$. Noticing that Γ_s and J commute, we have

$$\Gamma_s \dot{\Gamma}_s \Gamma_s + J\dot{\Gamma}_s J^T = \Gamma_s^2 \Phi_s \Gamma_s^2 - \Phi_s, \quad (4.18)$$

which can be solved explicitly since Γ_s is diagonal,

$$(\Phi_s)_{jk} = \frac{(\Gamma_s \dot{\Gamma}_s \Gamma_s + J\dot{\Gamma}_s J^T)_{jk}}{\lambda_j^2 \lambda_k^2 - 1}, \quad (4.19)$$

where $\lambda_{j+n} = \lambda_j$ for $j \leqslant n$. Once Φ_s is determined in terms of Γ_s and $\dot{\Gamma}_s$, an inverse symplectic transformation can transform it back to Φ. To end this subsection, we discuss some special cases where Eq. (4.19) can be simplified to forms which are manifestly symplectic covariant; this allows us to solve the SLD and the Fisher information without going to the standard basis.

For a very noisy Gaussian state where all $\lambda_j \gg 1$, we have the following relations:

$$\Phi_s \approx \Gamma_s^{-1} \dot{\Gamma}_s \Gamma_s^{-1}, \tag{4.20}$$

which is symplectic covariant and can be generalized to

$$\Phi \approx \Gamma^{-1} \dot{\Gamma} \Gamma^{-1}. \tag{4.21}$$

For the degenerate case where $\lambda_j = \lambda$ for all j, we have

$$\Phi_s = \frac{1}{\lambda^4 - 1} (\lambda^2 \dot{\Gamma}_s + J \dot{\Gamma}_s J^T), \tag{4.22}$$

which can be brought into the following symplectic covariant form:

$$\Phi = \frac{1}{\lambda^4 - 1} (\lambda^4 \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} + J \dot{\Gamma} J^T). \tag{4.23}$$

If the symplectic eigenvalues of Γ do not change, i.e., $\dot{\Gamma}$ is driven by some Gaussian unitary, we have

$$\dot{\Gamma} = \Gamma H J^T + J H \Gamma, \tag{4.24}$$

where $H = H^T$; this equation can be derived by considering the evolution of the covariance matrix (4.3) under the quadratic Hamiltonian $\mathbf{r}^T H \mathbf{r}/2$. With the condition $\lambda^2 \Gamma^{-1} = J \Gamma J^T$ for degenerate Gaussian states and Eq. (4.24) we have

$$\lambda^2 \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} = J \Gamma H + H \Gamma J^T = -J \dot{\Gamma} J^T, \tag{4.25}$$

and thus Eq. (4.23) can be simplified to

$$\Phi = \frac{\lambda^2}{\lambda^2 + 1} \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} = -\frac{1}{\lambda^2 + 1} J \dot{\Gamma} J^T. \tag{4.26}$$

For pure Gaussian states ($\lambda = 1$), we assume that the condition Eq. (4.25) is always satisfied; otherwise, Φ would diverge according to Eq. (4.23). By setting $\lambda = 1$ in Eq. (4.26), we have the following result for pure states:

$$\Phi = \tfrac{1}{2} \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} = -\tfrac{1}{2} J \dot{\Gamma} J^T. \tag{4.27}$$

Note that Eq. (4.27) is valid even if the pure Gaussian state actually goes through a nonunitary process which gives the same $\dot{\rho}$ as a unitary process for that pure state.

B. Quantum Fisher information

The Fisher information can be calculated by applying $\mathcal{L}$ on χ_P twice,

$$\mathcal{I} = \mathrm{tr}(\rho L^2) = \mathcal{L}^{(1)} \mathcal{L}^{(2)} \chi_P|_{\xi_{1,2}=0}. \tag{4.28}$$

Putting Eq. (4.10) into Eq. (4.28) and neglecting all derivatives of odd orders we have

$$
\begin{aligned}
\mathcal{I} &= \left(\sum_{j,k,l,m} \Phi_{jk} \Phi_{lm} \partial_{jk}^{(1)} \partial_{lm}^{(2)} + \sum_{j,k} 2\nu \Phi_{jk} \partial_{jk}^{(1)} - \sum_{j,k} \zeta_j \zeta_k \partial_j^{(1)} \partial_k^{(2)} + \nu^2 \right) \chi_P \Bigg|_{\xi_{1,2}=0} \\
&= \frac{1}{2} \mathrm{tr}(J \Phi J \Phi) + \frac{1}{2} \mathrm{tr}(\Gamma \Phi \Gamma \Phi) + \frac{1}{4} [\mathrm{tr}(\Gamma \Phi)]^2 - \nu \, \mathrm{tr}(\Gamma \Phi) + \frac{1}{2} \boldsymbol{\zeta}^T \Gamma \boldsymbol{\zeta} + \nu^2 \\
&= \frac{1}{2} \mathrm{tr}[(\Gamma \Phi \Gamma - J \Phi J^T) \Phi] + \frac{1}{2} \boldsymbol{\zeta}^T \Gamma \boldsymbol{\zeta} = \frac{1}{2} \mathrm{tr}(\dot{\Gamma} \Phi) + 2 \dot{\boldsymbol{\delta}}^T \Gamma^{-1} \dot{\boldsymbol{\delta}},
\end{aligned}
\tag{4.29}
$$

where the conditions (4.13), (4.14), and (4.15) are used to simplify the expressions; also note that the quantity $\mathrm{tr}(\dot{\Gamma} \Phi)$ is symplectically invariant, specifically,

$$\mathrm{tr}(\dot{\Gamma} \Phi) = \mathrm{tr}(\dot{\Gamma}_s \Phi_s). \tag{4.30}$$

For very noisy Gaussian states we have $\dot{\Gamma} \Phi = (\dot{\Gamma} \Gamma^{-1})^2$ by Eq. (4.21), and consequently, the quantum Fisher information reads

$$\mathcal{I}_{\mathrm{noisy}} \approx \tfrac{1}{2} \mathrm{tr}[(\dot{\Gamma} \Gamma^{-1})^2] + 2 \dot{\boldsymbol{\delta}}^T \Gamma^{-1} \dot{\boldsymbol{\delta}}. \tag{4.31}$$

For a degenerate Gaussian state, the quantum Fisher information can be derived by using Eq. (4.23),

$$\mathcal{I}_{\mathrm{degen}} = \frac{\mathrm{tr}[\lambda^4 (\dot{\Gamma} \Gamma^{-1})^2 - (\dot{\Gamma} J)^2]}{2(\lambda^4 - 1)} + 2 \dot{\boldsymbol{\delta}}^T \Gamma^{-1} \dot{\boldsymbol{\delta}}. \tag{4.32}$$

If the degenerate Gaussian state is driven by a Gaussian unitary, we have

$$\mathcal{I}_{\mathrm{degen}} = \frac{\lambda^2}{2(\lambda^2 + 1)} \mathrm{tr}[(\dot{\Gamma} \Gamma^{-1})^2] + 2 \dot{\boldsymbol{\delta}}^T \Gamma^{-1} \dot{\boldsymbol{\delta}}, \tag{4.33}$$

or equivalently,

$$\mathcal{I}_{\mathrm{degen}} = \frac{1}{2(\lambda^2 + 1)} \mathrm{tr}[(\dot{\Gamma} J)^2] + \frac{2}{\lambda^2} \dot{\boldsymbol{\delta}}^T J \Gamma J^T \dot{\boldsymbol{\delta}}, \tag{4.34}$$

where we use the identity $\Gamma^{-1} = J \Gamma J^T / \lambda^2$ for degenerate Gaussian states. In particular, Eqs. (4.33) and (4.34) work for all single-mode Gaussian states.

For pure Gaussian states we have

$$\mathcal{I}_{\mathrm{pure}} = \tfrac{1}{4} \mathrm{tr}[(\dot{\Gamma} \Gamma^{-1})^2] + 2 \dot{\boldsymbol{\delta}}^T \Gamma^{-1} \dot{\boldsymbol{\delta}}, \tag{4.35}$$

which coincides with Eq. (8) in [27], or equivalently,

$$\mathcal{I}_{\mathrm{pure}} = \tfrac{1}{4} \mathrm{tr}[(\dot{\Gamma} J)^2] + 2 \dot{\boldsymbol{\delta}}^T J \Gamma J^T \dot{\boldsymbol{\delta}}. \tag{4.36}$$

V. CONCLUSION

For a quantum state in exponential form we give expressions for the SLD, see Eqs. (2.6) and (2.12), and the quantum Fisher information, see Eq. (2.15). All these expressions are explicit and are useful for quantum-metrology problems with Gaussian or general thermal states (but are not restricted to these two kinds of states). We give the quantum Fisher information, see Eq. (3.24), for a Gaussian state in terms of its generator. Using a different approach we derive an equation for the SLD of an

arbitrary Gaussian state in terms of its moments, confirming a recent result by Monras [29]. We find that the resulting equation is symplectic covariant and can be solved exactly in the basis where the covariance matrix is in the standard form. Furthermore, the Fisher information in terms of the moments of a general Gaussian state is calculated; special cases are discussed, which include pure, degenerate, and very noisy Gaussian states.

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