Quantum theory realizes all joint measurability graphs

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Joint measurability of sharp quantum observables is determined pairwise, and so can be captured in a graph. We prove the converse: any graph whose vertices represent sharp observables and whose edges represent joint measurability is realized by quantum theory. This leads us to show that it is not always possible to use Neumark dilation to turn unsharp observables into sharp ones with the same joint measurability relations.

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I. INTRODUCTION

One of the characteristic features of quantum theory is that not every two observables can be measured jointly. This raises the question, what rules govern the relationship of joint measurability between quantum observables? In this article, we prove that we can label the vertices of any given graph with sharp quantum observables in such a way that two observables are jointly measurable precisely when their vertices are connected by an edge. This leads us to a shortcoming of the idea that any quantum operation can be regarded as unitary evolution of a larger, *dilated*, system, and in particular that any unsharp quantum observable can be regarded as a sharp one on a dilated system. The caveat is that dilation does not respect joint measurability.

The latter result is important to be aware of for quantum information theorists, whose bread and butter is dilation [1]; in particular, unsharp quantum observables are used in quantum state discrimination [2–4], photonic qubit measurement [5], quantum state tomography [6], quantum cryptography [7], and remote state preparation [8]. The former result is of foundational interest in its own right. Joint measurability plays a pivotal role in *contextuality*, the phenomenon that the result of measuring an observable depends on which other observables it is measured jointly with. It has given rise to Gleason's theorem [9], Bell's inequalities [10–12], the Kochen-Specker theorem [13], Hardy's paradox [14], GHZ impossibility results [15], and generalized probabilistic theories [16,17]. All of these are under active study; see, e.g., [18–20].[1] In particular, there are (non)contextuality inequalities that are violated by quantum mechanics and hence can be used to experimentally detect quantum effects [22], that come from graph theory [23,24].

[1] Abramsky and Brandenburger [20] derive abstract Kochen-Specker, Bell, Hardy, and GHZ results "without any presupposition of quantum mechanics." Our results could be interpreted as strengthening this approach by showing that it fully captures such "characteristic mathematical structures of quantum mechanics, such as complex numbers, Hilbert spaces, operator algebras, or projection lattices," after all. See especially Sec. 7.1, which discusses [21]. That paper has results similar to those of this article, but with orthogonality instead of joint measurability, requiring extra conditions.

II. REALIZATION AS YES-NO QUESTIONS

Let G be a graph. Write $v,w,x,y,\ldots \in G$ for its vertices, and $v \sim w$ when v and w are connected by an edge. By convention, we agree that $v \sim v$ for any vertex v. Think of the vertices as observables, that are jointly measurable precisely when they are connected by an edge.

We will be concerned with several kinds of observables: all will be particular types of structures on a Hilbert space, but what joint measurability means will vary. By a *realization* of G as observables on a Hilbert space H, we mean a function $x \mapsto O_x$ that sends vertices to observables in such a way that O_x and O_y are jointly measurable if and only if x and y are connected by an edge. As the basic step, we will first consider *yes-no questions*, that is, *projections*. A set of projections is defined to be jointly measurable when each pair in it commutes. We now prove that any graph is realizable as projections on some Hilbert space.

Theorem 1. Any graph has a realization as projections on some Hilbert space.

Proof. First, consider the special case of a graph $G_{v,w}$ where all pairs of vertices are connected by an edge, except for two fixed vertices v,w that are not connected. Fix two projections on $\mathbb{C}^2$ that do not commute, for example:

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |+\rangle\langle +| = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We can use these to build a realization of $G_{v,w}$ as projections on $\mathbb{C}^2$. Define $p_v = |0\rangle\langle 0|$, $p_w = |+\rangle\langle +|$, and $p_x = 0$ for all other vertices $x \neq v,w$. By construction, all pairs p_x and p_y for vertices $x,y \in G_{v,w}$ commute, except for p_v and p_w. Hence $x \mapsto p_x$ realizes $G_{v,w}$ as projections on $\mathbb{C}^2$. We will denote the dependency on v and w of this realization by writing $p_x^{v \not\sim w}$ for p_x.

Now that we know how to obstruct a single pair of vertices from being jointly measurable, we return to an arbitrary graph

G. Let the Hilbert space $H = \bigoplus_{v \not\sim w} \mathbb{C}^2$ be the direct sum of copies of $\mathbb{C}^2$, where the direct sum ranges over all pairs of vertices that are not connected by an edge. For any vertex $x \in G$, then $p_x = \bigoplus_{v \not\sim w} p_x^{v \not\sim w}$ gives a well-defined projection on H [25]. Now, if $x \sim y$, then all $p_x^{v \not\sim w}$ and $p_y^{v \not\sim w}$ commute by construction, and so p_x and p_y commute. Similarly, if $x \not\sim y$, then $p_x^{x \not\sim y}$ and $p_y^{x \not\sim y}$ do not commute, and so p_x and p_y do not commute. All in all, we have constructed a realization $x \mapsto p_x$ of G as projections on H. ■

If $f : G_1 \to G_2$ is an injective function between graphs satisfying $f(v) \not\sim_2 f(w)$ when $v \not\sim_1 w$, then the realizations are related by $p_x = V^\dagger p_{f(x)} V$ for an isometry V.

III. DIMENSION BOUNDS FOR YES-NO QUESTIONS

There is a well-defined *minimal* dimension in which a graph with V vertices can be realized as projections. The construction in the proof of Theorem 1 showed that this minimal dimension is at most $2N$, where N is the number of non-edges, i.e., pairs of vertices that are not connected by an edge. Notice that Theorem 1 makes sense for graphs of arbitrary size; if the graph is infinite, then the number N should be regarded as a cardinal number. In particular, the theorem implies that finite graphs can be realized as projections on a finite-dimensional Hilbert space, namely in dimension $2N$. Clearly $N \leqslant \frac{|G|(|G|-1)}{2}$, so that the minimal dimension is at most $|G|(|G|-1)$; this inequality is saturated for graphs without edges, for which $N = \frac{|G|(|G|-1)}{2}$.

We will now show that the minimal dimension that any graph can be realized in is at most $|G|$.

Theorem 2. Any graph has a realization as projections on a Hilbert space whose dimension is at most $|G|$, the number of vertices of G.

Proof. If $|G|$ is an infinite cardinal number, then $|G|(|G|-1) = |G|$, and the claim follows from the above considerations.

We may therefore assume that the graph is finite. Consider the Hilbert space of dimension $|G| + N$, with orthonormal basis vectors $|x\rangle$ for each vertex $x \in G$ and $|\{v,w\}\rangle$ for each nonedge $v \not\sim w$. For each vertex $x \in G$, define a vector $|\psi_x\rangle = |x\rangle + \sum_{x \not\sim v} |\{x,v\}\rangle$, where the sum ranges over all vertices v not adjacent to x. For distinct vertices x and y then

$$\langle \psi_x \mid \psi_y \rangle = \begin{cases} 0, & x \sim y, \\ \langle \{x,y\} \mid \{y,x\} \rangle = 1, & x \not\sim y. \end{cases}$$

Thus $|\psi_x\rangle$ and $|\psi_y\rangle$ are orthogonal when $x \sim y$, but not orthogonal or parallel when $x \not\sim y$ (because $\langle \psi_x \mid \psi_y \rangle^2 = 1 < 2 \cdot 2 \leqslant \langle \psi_x \mid \psi_x \rangle \langle \psi_y \mid \psi_y \rangle$).

Letting p_x be the projection onto $|\psi_x\rangle$ constructs a realization $x \mapsto p_x$ as projections. Finally, notice that each p_x has rank 1. So we may restrict the Hilbert space down to just the linear span of the $|G|$ vectors $|\psi_x\rangle$. This restricts the realization $x \mapsto p_x$ to a Hilbert space of dimension at most $|G|$. ■

The construction in the proof relied on the fact that projections onto single vectors commute precisely when the vectors are parallel or orthogonal. This is closely related to *orthogonal representations* of graphs, which have been studied in the literature [26 Sec. 9.3]. For example, if the complement of the graph is connected after removing any $V - d - 1$ vertices, then one can assign unit vectors in $\mathbb{R}^d$ to the vertices such that all these vectors are different, and two vectors are orthogonal if and only if they share an edge. In general, if we insist that the projections p_x have rank one, then the minimal dimension in which the complement of the "path" graph

can be realized is $|G| - 1$ [27,28]. In that sense, Theorem 2 is very close to being optimal. We leave open the question of whether allowing p_x to have higher rank can lead to more efficient realizations.

IV. REALIZATION AS SHARP OBSERVABLES

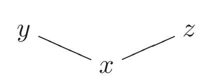
The above results easily extend from yes-no questions to *sharp observables*, that is, *projection valued measures* (PVMs). A PVM is a set P of mutually orthogonal projections that sum to 1. A family $P_1, P_2, \ldots$ of PVMs is jointly measurable when p and q commute for all $p \in P_i$ and $q \in P_j$ and all i,j [29]. Hence a specification of sharp quantum observables and which ones are jointly measurable is determined pairwise, and can also be captured in a graph.

Theorem 3. Any graph has a realization as PVMs on a Hilbert space whose dimension is at most the number of vertices.

Proof. Given a graph G with vertices $x, y, \ldots$, simply replace the projection p_x of Theorem 2 by the PVM $P_x = \{p_x, 1 - p_x\}$: the PVMs P_x and P_y are jointly measurable if and only if p_x and p_y commute. ■

In the joint measurability graph of all projections on a Hilbert space, a special role is played by *maximal cliques*: maximal sets of vertices, every two of which are connected by an edge. They correspond to PVMs P that are maximally fine-grained, in the sense that all $p \in P$ have rank one. More precisely, given such a PVM P, the set of all projections commuting with all $p \in P$ form a maximal clique. Conversely, a maximally fine-grained PVM can be recovered as the minimal projections in a maximal clique.[2]

It is not always possible to realize a graph as projections in a way that sends maximal cliques to PVMs. For a counterexample, consider the "fork" graph with three vertices and two edges,



Suppose there were a realization as projections with $p_x + p_y = 1 = p_x + p_z$. Then $p_y = 1 - p_x = p_z$, making p_x and p_y commute, contradicting the fact that $y \not\sim z$. We leave open the interesting question of characterizing which graphs can be realized as projections in a way that sends maximal cliques to PVMs.

We call a realization as projections $x \mapsto p_x$ *faithful* when distinct vertices $x \neq y$ give rise to distinct projections $p_x \neq p_y$.

[2] Given a maximal clique of projections in a Hilbert space, the C* algebra it generates is commutative. Therefore it has a commutative projection lattice. By maximality, this lattice coincides with the clique, which is therefore a Boolean sublattice of the full projection lattice.

The previous example might have given pause to the reader who intuitively expected a realization as projections of a graph to be faithful. The construction of Theorem 1 might not be faithful, because vertices $x \in G$ that are connected to all others end up being realized by the projection $p_x = 0$ commuting with anything. Any realization as projections can be made faithful as follows. Enlarge the Hilbert space to $H \oplus H'$, where H' has orthonormal basis $\{|x\rangle \mid x \in G\}$, and send $x \in G$ to $p_x \oplus |x\rangle\langle x|$. This is clearly faithful, and has the same commutativity properties as the original realization.

We can similarly extend to realizations as sharp quantum observables that are *not dichotomic*. If the vertices $x \in G$ are labeled with numbers $n_x \geqslant 2$, we can realize the graph as PVMs such that P_x has n_x elements. Enlarge the Hilbert space to $H \oplus \bigoplus_{x\in G} H_x$, where H_x has orthonormal basis $\{|3_x\rangle, \ldots, |n_x\rangle\}$, and send $x \in G$ to $P_x = \{p_x \oplus 0, (1-p_x) \oplus 0\} \cup \{0 \oplus |i\rangle\langle i| \mid i = 3, \ldots, n_x\}$. This has the same commutativity properties as the original realization.

In principle, one could imagine physical theories in which joint measurability of observables is not determined pairwise. (Indeed, we will see shortly that unsharp observables in quantum mechanics form a case in point.) To model joint measurability, we then have to generalize to *hypergraphs*, in which a hyperedge can connect any number of vertices [23,24]. Any graph induces a hypergraph, where a set of vertices forms a hyperedge when every two vertices in it are connected by an edge. Our definition of *realizability* easily carries over to hypergraphs: vertices still represent observables, and a set of vertices forms a hyperedge precisely when it is jointly measurable. Combining the above results with the well-known fact that sharp observables are jointly measurable when they commute [30,31], we obtain the following characterization: a hypergraph is realizable as sharp quantum observables if and only if it is induced by a graph.

Just as we have discussed dimension bounds for the realizations of graphs by projections, we can also ask what the minimal dimension is to realize a given graph by PVMs. As the proof of Theorem 3 shows, any realization as projections can be turned into a realization as PVMs, and hence the PVM minimal dimension is at most the projection minimal dimension. As witnessed by the multitude of proofs of the Kochen-Specker theorem in $\mathbb{C}^3$ and $\mathbb{C}^4$ [32], there is quite a lot of "room" already in these low dimensions, and one may wonder whether this is already enough to realize every graph as a PVM. This turns out not to be the case.

Theorem 4. There is no dimension d in which all graphs can be realized as PVMs.

Proof. For a given d, we construct a graph which cannot be realized in dimension d as follows. Let B_d be the number of partitions of $\{1, \ldots, d\}$; this is the dth Bell number. Now take a graph with $B_d + 1$ vertices designated as "action" vertices and $n := \lceil \log_2(B_d + 1) \rceil$ many "control" vertices. Enumerate the action vertices using bitstrings of length n. Then, action vertex v shares an edge with control vertex number k if and only if the kth bit in the bitstring associated to v is 1. Also, every two action vertices share an edge, while two control vertices may or may not share an edge.

This graph cannot be realized in dimension d: since every action vertex is connected to a different set of control vertices, no two action vertices can map to the same PVM. On the other hand, all these PVMs must be jointly measurable, and hence all their elements can be diagonalized in the same basis. In this fixed basis, every PVM therefore corresponds to a partition of $\{1, \ldots, d\}$. But since we have $B_d + 1$ many PVMs, which is higher than the number of partitions of $\{1, \ldots, d\}$, this is impossible. ■

V. UNSHARP OBSERVABLES AND NEUMARK DILATION

We now turn to the most general kind of *(unsharp) quantum observables*, namely *positive operator valued measures* (POVMs). These are defined as functions E from some *outcome space* I to operators on a Hilbert space that are bounded between 0 and 1 and form a resolution of the identity[3] $\sum_{i\in I} E(i) = 1$, and $0 \leqslant E(i) \leqslant 1$ for each $i \in I$. If $E(i)$ is a projection for each i, we actually have a PVM. Therefore we may also write $P(i)$ instead of p_i for PVMs $P = \{p_i \mid i \in I\}$. A family of POVMs $E_1, E_2, \ldots$ is defined to be jointly measurable when there exists a joint POVM E of which they are the marginals: if POVMs E_n have outcome space I_n, then E should have outcome space $\prod_n I_n$ and satisfy

$$E_1(i_1) = \sum_{i_2\in I_2, i_3\in I_3,\ldots} E(i_1,i_2,i_3,\ldots),$$

$$E_2(i_2) = \sum_{i_1\in I_1, i_3\in I_3,\ldots} E(i_1,i_2,i_3,\ldots),$$

and so on [29,34]. This reduces to the previously considered notions of joint measurability for yes-no questions and sharp quantum observables.

Neumark's famous *dilation* theorem says that any POVM can be dilated to a PVM on a larger Hilbert space, or in other words, that any POVM is the *compression* of a PVM on a larger Hilbert space: if E is a POVM on a Hilbert space H with outcome space I, then there exist a Hilbert space K, an isometry $V : H \to K$, and a PVM P on K with outcome space I, such that $E(i) = V^\dagger P(i) V$ [29,33]. This forms an important part of the philosophy that John Smolin called "the church of the larger Hilbert space," which holds that one need not care about unsharp observables as long as ancilla spaces are taken into account.

There is an extension of Neumark's dilation theorem for families of observables. We call a family $E_1, E_2, \ldots$ of POVMs, with outcome spaces $I_1, I_2, \ldots$, on a Hilbert space H *jointly dilatable* when there exist a Hilbert space K, an isometry $V : H \to K$, and a single PVM P with outcome space $\prod_n I_n$ such that $E_n(i) = \sum_j V^\dagger P(i,j) V$, where j ranges over $\prod_{m\neq n} I_m$, and we write (i,j) for the obvious element[4] of $\prod_m I_m$. It is now a matter of unfolding definitions to prove that

[3]While we only consider *discrete* POVMs here, all our results hold unabated for positive-operator valued measures on the Borel sets on a compact Hausdorff space, by reformulating them in the language of C* algebras and completely positive maps; see, e.g., Theorem 4.6 of [33].

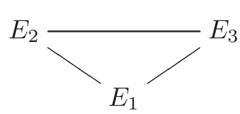
[4]That is, for $i \in I_n$ and $j \in \prod_{m\neq n} I_m$, the element $(i,j) \in \prod_n I_n$ has nth component i and other components given by the components of j.

a family of POVMs on a Hilbert space is jointly measurable if and only if it is jointly dilatable.

We now study how joint measurability behaves under Neumark dilation. Suppose POVMs E_1, E_2, and E_3 are compressions of PVMs P_1, P_2, and P_3 with respect to different isometries. If $\{E_1,E_2,E_3\}$ are jointly measurable there is a *single* isometry V_{123} that dilates the joint POVM E_{123} (to, say, P_{123}). But if the E_i are merely pairwise jointly measurable, then there exist PVMs P_{ij} and three isometries V_{ij} that dilate E_{ij} to P_{ij}. What we will show is that even if one has all three pairwise dilations P_{ij} via isometries V_{ij} at hand, it may be the case that there is no triplewise dilation P_{123} via any isometry V_{123}. Thus Neumark dilation cannot always turn unsharp observables into sharp ones with the same joint measurability relations. In this sense, Neumark dilation does not reflect joint measurability.

Theorem 5. There is a family $\{E_n\}$ of POVMs on a Hilbert space H that does not allow an isometry $V : H \to K$ and a family of PVMs $\{P_n\}$ (with the same outcome spaces as E_n) on K with $E_n(i) = V^\dagger P_n(i) V$ in such a way that a subset of $\{E_n\}$ is jointly measurable if and only if the corresponding subset of $\{P_n\}$ is jointly measurable.

Proof. Perhaps the simplest counterexample starts with a family $\{E_1,E_2,E_3\}$ of POVMs on the Hilbert space $H = \mathbb{C}^2$, every pair of which is jointly measurable, but which is not jointly measurable itself [12,18,30,31,34]. Its hypergraph is a "hollow triangle":



In other words, this (hyper)graph is realizable as POVMs.

In contrast, as noted above, joint measurability of PVMs is determined pairwise, which will lead us to a contradiction. Suppose PVMs $\{P_1,P_2,P_3\}$ as in the statement of the theorem existed. Then, by our assumptions, the pairwise joint measurability of the E_n would imply pairwise joint measurability of the P_n, so the P_n would necessarily be triplewise jointly measurable as well. In other words, then the $\{E_n\}$ would be (triplewise) jointly dilatable. But this contradicts the fact that the $\{E_n\}$ are not (triplewise) jointly measurable. In summary: joint measurability of the putative P_n would imply joint measurability of the E_n, since a joint POVM can be constructed as the compression of a joint PVM. ∎

We could interpret the previous theorem as a warning against an unreflected belief in "the church of the larger Hilbert space." If you care about (non-)joint measurability of observables, you cannot simply ignore unsharp quantum observables in favor of their dilated sharp observables, even if ancilla spaces are taken into account, and you have to take the unsharpness involved seriously.

This plays a role in quantum protocols that rely on unsharp observables that are not jointly measurable, in which case the usual analysis by dilation to sharp observables should not be used. For example, [5] explicitly constructs a PVM implementation of a POVM and mentions that this "faithfully represents the POVM." However, PVM implementations cannot always represent joint measurability relations within families of POVMs. We suspect that it may be possible to turn this apparent problem into a *feature* which can be exploited in new quantum information protocols. More concretely, we imagine situations in which a number of parties share some quantum information resource, but only certain subgroups of these parties are allowed joint access to it.

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[1] M. Keyl, Phys. Rep. **369**, 431 (2002).
[2] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
[3] D. Dieks, Phys. Lett. A **126**, 303 (1988).
[4] A. Peres, Phys. Lett. A **128**, 19 (1988).
[5] H. E. Brandt, Am. J. Phys. **67**, 434 (1999).
[6] A. J. Scott, J. Phys. A **39**, 13507 (2006).
[7] D. M. Appleby, H. B. Dang, and C. A. Fuchs, arXiv:0707.2071.
[8] N. Killoran, D. N. Biggerstaff, R. Kaltenbaek, K. J. Resch, and N. Lütkenhaus, Phys. Rev. A **81**, 012334 (2010).
[9] P. Busch, Phys. Rev. Lett. **91**, 120403 (2003).
[10] J. Bell, Physics **1**, 195 (1964).
[11] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[12] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
[13] S. Kochen and E. Specker, J. Math. Mech. **17**, 59 (1967).
[14] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
[15] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[16] J. Barrett, Phys. Rev. A **75**, 032304 (2007).
[17] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).
[18] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. **506**, 1 (2011).
[19] S. Abramsky and L. Hardy, Phys. Rev. A **85**, 062114 (2012).
[20] S. Abramsky and A. Brandenburger, New J. Phys. **13**, 113036 (2011).
[21] L. Lovász, M. Saks, and A. Schrijver, Linear Algebra and Its Applications **114**, 439 (1989).
[22] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. **101**, 020403 (2008).
[23] A. Cabello, S. Severini, and A. Winter, arXiv:1010.2163.
[24] T. Fritz, A. Leverrier, and A. B. Sainz, arXiv:1212.4084.
[25] C. Heunen, in *Computation, Logic, Games, and Quantum Foundations*, Lecture Notes in Computer Science, Vol. 7860, edited by B. Coecke, L. Ong, and P. Panangaden (Springer, Berlin, 2013), pp. 107–121.
[26] L. Lovász, "Geometric representations of graphs," 2009, http://www.cs.elte.hu/~lovasz/geomrep.pdf.

[27] L. Hogben, Lin. Alg. Appl. **428**, 2560 (2008).
[28] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, and W. Wang, SIAM J. Matrix Anal. Appl. **30**, 731 (2008).
[29] P. Busch, M. Grabowski, and P. Lahti, *Operational Quantum Physics* (Springer, Berlin, 1995).
[30] K. Kraus, *States, Effects, and Operations* (Springer, Berlin, 1983).
[31] G. Ludwig, *Foundations of Quantum Mechanics* (Springer, New York, 1983), Vol. 1.
[32] M. Pavičić, B. D. McKay, N. D. Megill, and K. Fresl, J. Math. Phys. **51**, 102103 (2010).
[33] V. Paulsen, *Completely Bounded Maps and Operator Algebras* (Cambridge University Press, Cambridge, 2002).
[34] T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. **38**, 1133 (2008).