Optimized entropic uncertainty for successive projective measurements

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We focus here on the uncertainty of an observable Y caused by a precise measurement of X. We illustrate the effect by analyzing the general scenario of two successive measurements of spin components X and Y. We derive an optimized entropic uncertainty limit that quantifies the necessary amount of uncertainty observed in a subsequent measurement of Y. We compare this bound to recently derived error-disturbance relations and discuss how the bound quantifies the information of successive quantum measurements.

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I. INTRODUCTION

It is well known that the Heisenberg uncertainty principle [1] is at the very heart of quantum mechanics. Through extensive investigations, it has been known that various distinctive properties of quantum mechanics can be derived from the principle [2]. However, its precise underlying meaning has so far eluded many attempts to explain its diverse features [3–7].

Heisenberg proposed the uncertainty relation after postulating the kinematics of quantum canonical variables that do not commute [1]. It says that as one tries to specify the position of an electron precisely, its conjugate variable, e.g., its momentum, is dispersed within a given precision. The mathematical formulation of the uncertainty was made by Kennard [8] as

$$\epsilon(Q)\eta(P) \geqslant \frac{\hbar}{2}, \tag{1}$$

where $\epsilon(Q)$ is the mean error that occurs when an observer measures the position of an electron, $\eta(P)$ is the disturbance of the electron's momentum P caused by the position measurement Q, and $\hbar$ is the Planck constant. Relation (1) uses the statistical variances between the two measurements and was later extended to arbitrary pairs of observables by Robertson [9]. By considering generalized observables X and Y, the lower bound is given by the commutator of the observables

$$\delta(X)\delta(Y) \geqslant \tfrac{1}{2}|\langle\psi|[X,Y]|\psi\rangle|, \tag{2}$$

where $\delta(X)$ is the standard deviation defined as $\delta(X) = \sqrt{|\langle\psi|(\hat{X} - \langle\hat{X}\rangle)^2|\psi\rangle|}$ and $[\hat{X},\hat{Y}]$ represents the commutator $[X,Y] = XY - YX$. The above relation (2) claims that in an arbitrary state $|\psi\rangle$, a pair of noncommuting observables cannot be well localized simultaneously.

In fact, the underlying meaning of two closely related uncertainty relations is not equivalent. Their subtle differences will become clearer when we consider the following three statements of uncertainty relations presented by Busch *et al.* [10]. Possible interpretations of the uncertainty relation can be that (i) it is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized, (ii) it is impossible to measure a system's position and momentum simultaneously, and (iii) it is impossible to measure position without disturbing momentum. In these statements, position and momentum represent two conjugate variables in a quantum measurement. We can classify the above relations into three categories of physical situations. First, the Robertson relation (2) is equivalent to statement (i), which identifies a fundamental limitation on preparing states whose noncommuting parameters cannot be well localized simultaneously with arbitrary precision. This is a statement about the property of a given ensemble, not about the statistics of measured data. Second, it follows from statement (ii) that the uncertainty relations apply to the simultaneous measurement of two different variables whose measurement is impossible to implement with arbitrary precision in principle. This means that the uncertainty is the property of the statistical distributions from the measurement setup rather than the state itself [3]. Third, Heisenberg's relation (1) is equivalent to statement (iii) since it describes the situation where a measurement of a variable, e.g., position Q, cannot avoid the disturbance on its conjugate variable P, where Q and P are noncommuting observables.

Recent efforts to generalize Heisenberg's relation (1) take into account various operational circumstances by uniting statements (i)–(iii). A universally valid error-disturbance uncertainty relation was derived in [4] as

$$\epsilon(X)\eta(Y) + \epsilon(X)\delta(Y) + \delta(X)\,\eta(Y) \geqslant \tfrac{1}{2}|\langle\psi|[\hat{X},\hat{Y}]|\psi\rangle|, \tag{3}$$

where the mean error and the disturbance are defined by $\epsilon(X)^2 = \sum_m \|M_m(m - X)|\psi\rangle\|^2$ and $\eta(Y)^2 = \sum_m \|[M_m,Y]|\psi\rangle\|^2$, respectively, if the apparatus M has a family $\{\hat{M}_m\}$ of measurement operators and $\|\cdots\|$ denotes the norm of the state vector [11]. This means that the measuring apparatus M has possible outcomes m with probability $\text{Prob}(m) = \|\hat{M}_m|\psi\rangle\|^2$ and the state of the object S after the measurement with the outcome m becomes $\hat{M}_m|\psi\rangle/\|\hat{M}_m|\psi\rangle\|$. It was also proved experimentally that the Heisenberg relation (1) is violated in spin measurements, while the improved relationship (3) remains valid [12]. Later, the error-disturbance relation was improved in a stronger form [6,13]. The error-disturbance uncertainty relation reduces to the Robertson uncertainty relations (2) when there is no error in the first measurement $\epsilon(X) = 0$ and the disturbance is replaced by the statistical deviation of the measurement Y as $\eta(Y) = \delta(Y)$.

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Inspired by an information-theoretic interpretation of quantum uncertainty, the trade-off relation for position and momentum observables has been obtained in terms of the Shannon entropy [14,15]. The relationship was later generalized for measurements on arbitrary continuous variables [16]. This is called the entropic uncertainty relationship (EUR). A generalization of the EUR to the discrete observables was proposed by Deutsch [17] and the bound of EUR was improved by Uffink in the following form [18]. Considering observables X and Y with nondegenerate spectra given by $X = \sum_i x_i |x_i\rangle\langle x_i|$ and $\hat{Y} = \sum_j y_j |y_j\rangle\langle y_j|$ with the natural logarithm, the Shannon entropies $H(X)$ and $H(Y)$ are defined as $H_\rho(X) = -\sum_i \mathrm{Tr}[\rho|x_i\rangle\langle x_i|] \ln \mathrm{Tr}[\rho|x_i\rangle\langle x_i|]$ and $H_\rho(Y) = -\sum_j \mathrm{Tr}[\rho|y_j\rangle\langle y_j|] \ln \mathrm{Tr}[\rho|y_j\rangle\langle y_j|]$ for a state expressed by a density matrix ρ. Then the EUR becomes

$$H_\rho(X) + H_\rho(Y) \geqslant -2 \ln c, \tag{4}$$

where the lower bound constant $c = |\max_{i,j} \langle x_i|y_j\rangle|$ is independent of the initial state. Here $\{|x_i\rangle\}$ and $\{|y_j\rangle\}$ are the corresponding complete sets of normalized eigenvectors with respect to operators X and Y. In general, it can be said that the EUR has a more fundamental lower bound than the variance-based uncertainty relation in the sense that the bound is independent of the prepared initial state, unlike in (2) and (3). On the other hand, the EUR in (4) is only limited by the prepared state ρ, like the Robertson inequality in (2). This means that the EUR provides a fundamental constraint on the state preparation as in the case of the operational uncertainty interpretation (iii) in [10].

In this paper we derive the uncertainty relationship characterized by the entropy under the circumstance of simultaneous measurements. We consider the case when two different measurements are performed successively on a single quantum system and find a fundamental entropic constraint that constitutes an entropic uncertainty relationship. The relationship has a different operational meaning from the original EUR in (4) and is comparable to the error-disturbance versions of the uncertainty relations in (1) and (3). We organize our discussion as follows. In Sec. II we compare the quantitative difference between the EUR and the variance-based uncertainty relation. We find that they are optimized in different regimes. In Sec. III the entropic uncertainty relationship for subsequent measurements is derived and generalized. We compare each term in the relationship and discuss their optimal physical meanings. In Sec. IV the optimized entropic uncertainty and the error disturbance relationship are compared and analyzed in detail. Our results are summarized in Sec. V.

II. COMPARISON BETWEEN ROBERTSON'S UNCERTAINTY RELATION AND THE EUR

In this section we compare the Robertson uncertainty relation (2) and the EUR (4) quantitatively to identify which is the more informative condition for a given quantum state. The former is a relationship based upon the variance of a statistical distribution and the latter is a characterization of uncertainty using Shannon's entropy.

A distinction between the relations is that their lower bounds behave differently: The bound for Robertson's uncertainty relation depends upon the prepared state, whereas the EUR does not. It is notable that the necessity of an independent lower bound of a state has been addressed in [17] and it is argued that such a bound is important when there is a dynamical evolution that transforms quantum states at each instance. Due to the difference, a direct comparison of the relationship is not straightforward in general.

Let us consider two general spin observables that are the simplest nontrivial example of incompatible measurements. Without loss of generality, they can be

$$\hat{X}(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y, \tag{5}$$
$$\hat{Y}(\phi) = \sin(\phi)\sigma_x + \cos(\phi)\sigma_y, \tag{6}$$

where σ_x and σ_y denote the Pauli matrices and ϕ characterizes the measurement angle between $\hat{X}$ and $\hat{Y}$. Their commutator is expressed as $[\hat{X},\hat{Y}] = 2i \cos 2\phi \sigma_z$. In their two extrema, when $\phi = 0$, the measurements are orthogonal and when $\phi = \pi/4$ they become identical.

Once the measurement operators are specified, an analytic description of the uncertainty relations (2) and (4) becomes possible in general. A pure state is defined in a Bloch vector sphere as $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$ and is depicted in Fig 1. In that case, the probability of outcomes for the measurements X and Y become

$$p_\pm^X = \tfrac{1}{2}[1 \pm \sin\theta \cos(\phi + \varphi)], \tag{7}$$
$$p_\pm^Y = \tfrac{1}{2}[1 \pm \sin\theta \sin(\phi - \varphi)], \tag{8}$$

which can be used for the evaluation of the spin variances and entropies. For the X measurement, they are

$$\delta(X) = \sqrt{1 - \langle\hat{X}\rangle} = \sqrt{1 - \left(p_+^X - p_-^X\right)}, \tag{9}$$
$$H_{|\psi\rangle}(X) = -p_+^X \ln p_+^X - p_-^X \ln p_-^X \tag{10}$$

and similar relations can be found for the Y measurement. The entropy $H_{|\psi\rangle}(X)$ is $H_\rho(X)$ when a state ρ is a pure state given by $\rho = |\psi\rangle\langle\psi|$. With these formulas, a direct comparison of the uncertainty relationship (2) and (4) can be made as follows.

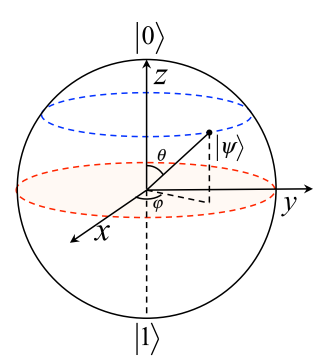

FIG. 1. (Color online) Pure state representation in a Bloch sphere. A prepared state is denoted by $|\psi\rangle = \cos\theta|0\rangle + \sin\theta e^{i\varphi}|1\rangle$, with polar angle $0 \leqslant \theta \leqslant \pi$ and azimuthal angle $0 \leqslant \varphi \leqslant 2\pi$. The north and south poles are chosen to correspond to eigenvectors of σ_z. Varying φ from 0 to $\pi/2$ with a fixed θ, we can consider state vectors on a circle (dashed blue). When $\theta = \pi/2$, it becomes a circle on the x-y plane (dashed red).

The uncertainty relations can be reformulated by the normalization, meaning that both sides of the relations are divided by their own lower bound. The normalized relations have the same bound 1 such that

$$\frac{H_{|\psi\rangle}(X)+H_{|\psi\rangle}(Y)}{-2\ln c}\geqslant 1,\quad \frac{\delta(X)\delta(Y)}{|\langle[X,Y]\rangle|/2}\geqslant 1, \tag{11}$$

where $c=\sqrt{(1+\sin 2\phi)/2}$ for $0<\phi<\pi/2$ and $|\langle[X,Y]\rangle|/2=|\cos 2\phi\cos\theta|$. The inequalities can be compared directly as they saturate to the same constant value.

Let us consider when the angle $\phi=0$, when two observables are orthogonal. In this case, the lower bound of (4) is given by a constant 1, whereas that of (2) is determined as a function of θ, $|\cos\theta|$. Then the left-hand sides (LHSs) of relations (11) are determined as functions of polar angle θ and azimuthal angle φ. In Fig. 2 these functions are plotted versus φ for fixed angles $\theta=0$, $3\pi/8$, $4\pi/9$, and $\theta\sim\pi/2$. This means that we take into account state vectors in a circle located halfway between the north pole and the equator, depicted by the dashed blue line in Fig. 1, and determined by θ.

This result is noteworthy. Figure 2 shows that the EUR (blue) tends to move into an optimized regime as the polar angle θ approaches $\pi/2$ from 0 [Figs. 2(a)–2(d)]. In contrast, Robertson's uncertainty relation (orange) diverges. Geometrically, it can be argued that when the state vector $|\psi\rangle$ is placed in the plane of two observables (the red plane in Fig. 1), the EUR in the first inequality (11) is optimized, whereas the Robertson uncertainty relation in the second inequality of (11) is optimized when $|\psi\rangle$ is aligned along the z axis. In particular, the state aligned along the z axis becomes a spin coherent state whose variances of the two measurements σ_x and σ_y are equivalent, as a constant equal to 1.

Similar behavior can be found in the case of two nonorthogonal observables too. For nonorthogonal observables, i.e., $\phi\neq n\pi/2$, where n is an integer, it can be found that the EUR has a minimal value for a state vector lying in the x-y plane, meaning $\theta=\pi/2$. At the same time, the uncertainty relation based on the standard deviation diverges at $\pi/2$ since its lower bound vanishes when $|\psi\rangle$ is given by an eigenvector of observables.

Consequently, it can be said that neither of the two relations is stronger in the case of discrete observables in general. Depending upon the state provided, the EUR and the Robertson uncertainty relation characterize the trade-off relationship differently. The EUR is the optimal relation when the state is located in the same plane as the two measurements, while the Robertson uncertainty relations is optimized when the state is in the plane orthogonal to both observables. For the case of continuous variable measurements, the situation is changed slightly in that the EUR for position and momentum observables is stronger than the relation based on standard deviation [16].

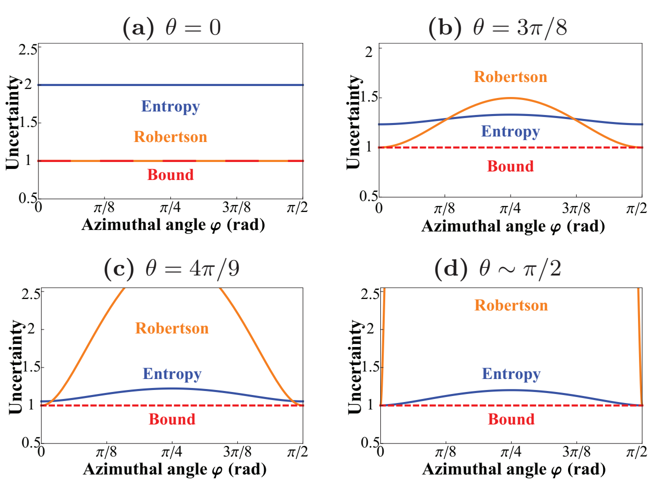

FIG. 2. (Color online) Graphs illustrating how EUR and Robertson's uncertainty relations behave with the probabilities of outcomes of observables $X(0)$ and $Y(0)$. To compare two relations, we plot the LHSs of the inequalities (11) against the azimuthal angle φ for chosen values of polar angle θ: (a) 0, (b) $3\pi/8$, (c) $4\pi/9$, and (d) $\pi/2$. The EUR is optimized as θ goes to $\pi/2$, whereas the Roberson uncertainty relation diverges. Relations (2) and (4) have a minimum value when $\theta=0$ and $\pi/2$ for fixed ϕ, respectively.

III. ENTROPIC UNCERTAINTY RELATION FOR SUCCESSIVE MEASUREMENTS

In this section we will show that it is possible to derive the entropic uncertainty relation for successive measurements and we consider the limit of our ability to measure two nondegenerate observables X and Y with arbitrary precision.

Following from Heisenberg's original insight, Srinivas derived the EUR for successive measurements as follows [19]. Consider observables X and Y with nondegenerate spectra

$$H_\rho(X)+H_{\mathcal{E}(\rho)}(Y)\geqslant -2\ln c, \tag{12}$$

where $\mathcal{E}(\rho)=\sum_i \hat{P}_i^X\rho\hat{P}_i^X$ and $\hat{P}_i^X\equiv|x_i\rangle\langle x_i|$. The second term $H_{\mathcal{E}(\rho)}(Y)$ is the Shannon entropy associated with the marginal of the joint probability $p(x_i,y_j)=\mathrm{Tr}[|y_j\rangle\langle y_j|\hat{P}_i^X\rho\hat{P}_i^X]$, defined as

$$H_{\mathcal{E}(\rho)}(Y)=-\sum_j \mathrm{Tr}[\mathcal{E}(\rho)|y_j\rangle\langle y_j|]\ln\{\mathrm{Tr}[\mathcal{E}(\rho)|y_j\rangle\langle y_j|]\}$$
$$=-\sum_j p(y_j)\ln p(y_j),$$

where $p(y_j)=\sum_i p(x_i,y_j)$. With the result he argued that this relation reflects statement (iii), which is the error disturbance of the uncertainty relation. However, it is not equivalent to Ozawa's universally valid error-disturbance relation because it does not include the effect of the measuring process. Here we propose an improved form of the EUR for successive measurements and highlight its differences from the error-disturbance uncertainty relation.

Assume that we perform a projective measurement described by nondegenerate projection operators $\{\hat{P}_i^X\}$. In the projection postulate, an input density matrix is changed to output states determined by corresponding outcomes. The probability of obtaining an outcome i is given by $p(x_i)=\mathrm{Tr}[\hat{P}_i^X\rho]$, where the input density matrix is ρ. In the case where we obtain the outcome i after the measurement, the output state ρ_i^X is given by [20]

$$\rho_i^X=\frac{\hat{P}_i^X\rho\hat{P}_i^X}{\mathrm{Tr}[\hat{P}_i^X\rho]}, \tag{13}$$

according to the projection postulate. In successive measurements, the probability of obtaining an eigenvalue a_i

of eigenstate $|x_i\rangle$ after the measurement of X is given by $p(x_i) = \mathrm{Tr}[\rho|x_i\rangle\langle x_i|]$. If we perform the measurement of Y on an output state obtained just after the first measurement of X, we obtain an eigenvalue b_j with a probability $p(y_j|x_i) = \mathrm{Tr}[\rho_i^X(|y_j\rangle\langle y_j|)]$. Then the joint probability $p(x_i,y_j)$ of outcomes x_i and y_j in successive measurements is given by $p(x_i)p(y_j|x_i)$. From the projection postulate the joint entropy of the probability distribution for the subsequent measurements is given by

$$H_\rho(X,Y) = -\sum_{i,j} p(x_i)p(y_j|x_i)\ln p(x_i)p(y_j|x_i).$$

The entropy $H(X,Y)$, defined in terms of the joint probability, means that an amount of uncertainty is present when a state is measured by successive measurements of X and Y. According to the subadditivity inequality, the joint entropy has a relation with the entropy of marginal distributions of the joint probability, i.e., the LHS of the entropic uncertainty relation (12), as

$$H_\rho(X) + H_{\mathcal{E}(\rho)}(Y) \geqslant H_\rho(X,Y). \tag{14}$$

Furthermore, the joint entropy satisfies the relation [19]

$$H_\rho(X,Y) \geqslant -2\ln c. \tag{15}$$

This relation implies a limitation of measuring observables X and Y that are not compatible with each other in successive measurements. The joint entropy $H(X,Y)$ can be decomposed into the entropy of X and the conditional entropy of Y given X such that

$$H_\rho(X,Y) = H_\rho(X) + H_\rho(Y|X) \geqslant -2\ln c, \tag{16}$$

where the conditional entropy of the observable Y given X for a density matrix ρ is defined as

$$H_\rho(Y|X) = \sum_i p(x_i)H_\rho(Y|X = x_i),$$

where $H_\rho(Y|x_i) = -\sum_j p(y_j|x_i)\ln p(y_j|x_i)$. It can be seen from relation (16) that the total uncertainty in successive measurements characterized by $H_\rho(X,Y)$ consists of the uncertainty of X and the averaged uncertainty of Y over outcomes x_i.

The bound of the relation (15) comes from the conditional entropy since the conditional entropy $H_\rho(Y|X)$ satisfies the relation

$$H_\rho(Y|X) \geqslant -2\ln c \tag{17}$$

for nondegenerate observables X and Y. It also follows that it is impossible to measure incompatible observables X and Y with certainty using successive projective measurements. Moreover, the joint entropy is composed of the entropy of X and the conditional entropy of Y given X. For entropy $H_\rho(X)$ it follows that the uncertainty characterizing a density matrix ρ and the conditional entropy $H_\rho(Y|X)$ leads to an averaged uncertainty in observable Y caused by the projective measurement of X. We will apply the above inequalities (12), (16), and (17) to make clear the relations among them.

Let us consider successive spin measurements assumed to satisfy the projection postulate. Measurements of $\hat{X}(\phi)$ and $\hat{Y}(\phi)$ in the relations (12)–(17) are designed to carry out the projective measurements of the Pauli matrices $\hat{\sigma}_\phi = \hat{\sigma}_x\cos\phi + \hat{\sigma}_y\sin\phi$ and $\hat{\sigma}_y$, respectively, as depicted in Fig. 3. The measurements are effectively the same to measure $\hat{X}(\phi)$ and $\hat{Y}(\phi)$ consecutively since they are just two observables that are separated by the angle ϕ.

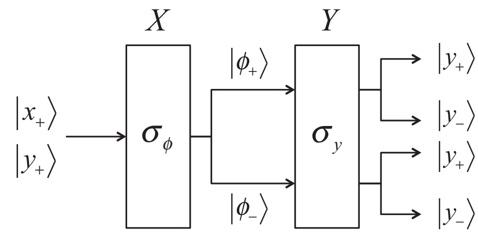

FIG. 3. Schematic of a successive measurement scheme for observables X and Y, used to clarify and compare EURs (12), (16), and (17). After preparing the input states $|x_+\rangle$ and $|y_+\rangle$, which are eigenstates of σ_x and σ_y, respectively, the successive measurement is assumed to measure observables $X = \sigma_\phi$ and $Y = \sigma_y$. It results in four possible outcomes.

Since each measurement has its own eigenvectors, it projects the input state onto a spin-up state $|+\rangle$ or a spin-down state $|-\rangle$ after the measurements as $\sigma_x|\pm\rangle = \pm|\pm\rangle$. In this way, its final result (a_i,b_j) emerges among four possible outcomes $\{(\pm,\pm),(\pm,\mp)\}$ as depicted in Fig. 3. Figure 4 shows the left-hand side of the EURs (12), (16), and (17) and the calculated lower bound as a function of ϕ. When we compare the graphs of three EURs in Fig. 4 our relation of the conditional entropy (17) is closest to the lower bound since the relation among them is such that

$$\begin{aligned} H_\rho(X) + H_{\mathcal{E}(\rho)}(Y) &\geqslant H_\rho(X,Y).\\ &= H_\rho(X) + H_\rho(Y|X)\\ &\geqslant H_\rho(Y|X) \geqslant -2\ln c, \end{aligned}$$

where $c = \max_{i,j}|\langle x_i|y_j\rangle|$. Three EURs have the same value when the input state is prepared in an eigenstate of the first measurement since an outcome is determined by the corresponding eigenvalue of the input state and the first measurement does not change the input state, i.e., $H_\rho(X) = 0$ and $H_{\epsilon(\rho)}(Y) = H_\rho(Y|X)$.

IV. COMPARISON BETWEEN THE EUR FOR SUCCESSIVE MEASUREMENTS AND THE ERROR-DISTURBANCE UNCERTAINTY RELATION

In his proposal for the EUR for successive measurements [19], Srinivas said that "to explore the influence of the measurement of one observable on the uncertainties in the outcomes of another, we have to formulate an uncertainty relation for successive measurements". However, the EUR for successive measurements does not reflect Heisenberg's microscope experiment, a thought experiment proposed in [1], since it does not consider the effect of the measuring process. On the other hand, in the error-disturbance relation (3) by Ozawa, the error is defined by the distance between a positive-operator-valued measure of an apparatus and an observable X and the disturbance due to information loss in the input state caused by the measuring process [4]. Thus,

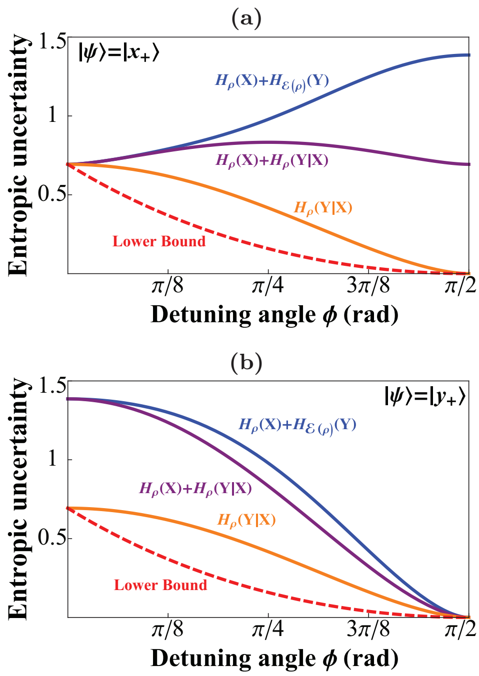

FIG. 4. (Color online) Graphs showing the LHSs of relations (12) (blue), (16) (purple), and (17) (orange) and the lower bound (dashed red) with respect to the detuning angle ϕ. Two input states $|x_+\rangle$ and $|y_+\rangle$ are considered. By inspection, we find that the LHS of the EUR for successive measurement (12) always has a larger value than the LHS of (16) due to subadditivity (14). The conditional entropy has a smaller value. Three relations have zero value only when two measurements are the same at $\phi = \pi/2$, but only the conditional entropy is saturated when two observables are mutually unbiased at $\phi = 0$. This means that the inner product of all pairs of each eigenstate is given by $1/\sqrt{d}$, where d is a dimension of the Hilbert space [21].

the error-disturbance relation is equivalent to statement (iii) and reflects Heisenberg's microscope experiment. This is in the sense that an effort to measure an observable X exactly increases the disturbance of another observable Y that is incompatible with X.

To formalize Heisenberg's intuition, the mean error and the disturbance are mathematically well defined using an indirect measurement with apparatus in [4]. This is because all quantum measurements can be described by the outcome from an indirect measurement. Under the assumption that the measuring apparatus M has a family of $\{M_m\}$ measurement operators, the error and disturbance are defined as [11]

$$\epsilon(X)^2 = \sum_m \|M_m(m - X)|\psi\rangle\|^2, \tag{18}$$

$$\eta(Y)^2 = \sum_m \|[M_m, Y]|\psi\rangle\|^2, \tag{19}$$

respectively, where $|\psi\rangle$ denotes an input state and $\|\cdots\|$ denotes the norm of the state vector. Theses quantities are characterized by a measuring process realized in apparatus M.

However, from the perspective of the error-disturbance relation, the EUR for successive measurements (16) is constructed under the assumption that a measuring apparatus designed for measuring an observable X precisely measures X, namely, there is no error in performing successive measurements. In this case, the error-disturbance uncertainty relation (3) reduces to

$$\delta(X)\eta(Y) \geqslant \tfrac{1}{2}|\langle [X,Y]\rangle| \tag{20}$$

since the error $\epsilon(X)$ vanishes from (3). Under the assumption of precise successive measurements, the error-disturbance uncertainty relation (3) and the EURs (16) and (17) restrict probabilities for the possible outcomes of measurements X and Y.

A natural question at this stage is which of these relations places more restrictions on the probabilities. We compare them by dividing them by their own lower bounds such that

$$\frac{H_{|\psi\rangle}(X) + H_{|\psi\rangle}(Y|X)}{-2\ln c} \geqslant 1, \tag{21}$$

$$\frac{H_{|\psi\rangle}(Y|X)}{-2\ln c} \geqslant 1, \tag{22}$$

$$\frac{\delta(X)\,\eta(Y)}{|\langle[\hat{X},\hat{Y}]\rangle|/2} \geqslant 1 \tag{23}$$

for strictly positive bounds. Using relations (21)–(23), we consider a successive measurement of observables $X(0) = \sigma_x$ and $Y(0) = \sigma_y$ with an input state vector $|\psi\rangle$. Then the probabilities of outcomes that are obtained by successive measurement of $X(0)$ and $Y(0)$ are restricted by the uncertainty relations. In Fig. 5 the LHSs of relations (23) and (21) are plotted together against the azimuthal angle φ for fixed polar angle θ. As a result, we can see in Fig. 5 that for all θ and ϕ, the LHS of (22) has the same value with the bound.

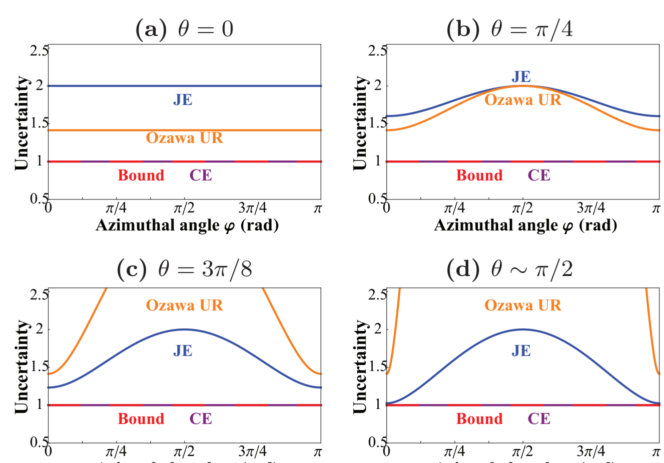

FIG. 5. (Color online) Graphs illustrating how the different EURs for successive measurements (16), (17), and the Ozawa's relation (3) impose restrictions on the probabilities of outcomes of observables $X(0)$ and $Y(0)$. The LHSs of relations (21) [joint entropy (JE) (blue)], (22) [conditional entropy (CE) (purple)], and (23) [Ozawa's relation (orange)] are plotted against the azimuthal angle φ for fixed values of the polar angle θ (0, $\pi/4$, $3\pi/8$, and $\pi/2$). By inspection, we find that the JE (21) decreases with respect to increasing θ, whereas the Ozawa relation (23) diverges. However, the conditional entropy of CE (22) is given by a constant 1 for all values of θ and ϕ.

This means that it imposes the highest restriction among the relations for successive measurements. In the case of nonorthogonal observables, the error-disturbance relationship divided by its lower bound has a minimum value 1 at $\varphi = \phi$ and maximum value $\sqrt{1+\tan^2\theta}$ at $\varphi = (\pi/2 + \phi)$. However, the EUR for successive measurements divided by its lower bound is independent of the input state $|\psi\rangle$, namely, it is only determined as function of ϕ and its value increases as ϕ goes to $(\pi/4 + n\pi/2)$.

V. CONCLUSION

In this work we derived the entropic uncertainty relation for subsequent measurements and compared it with the uncertainty relations based on the standard deviation using spin measurements. An alternative form of EUR for successive measurements was proposed in view of Heisenberg's statement [1] that "it is impossible to measure position without disturbing momentum".

Since Heisenberg, much debate and effort have been expended on formalizing its underlying meaning, while it is more recent that experiments have found different ways of demonstrating them [22–25]. A state-independent information-theoretic error-disturbance relation has also been proposed [26], which shows a trade-off relation between error and disturbance. However, the EUR for successive measurements does not coincide with the error-disturbance uncertainty relation. In our work we made clear the difference between them by plotting restrictions imposed on possible probabilities of outcomes of observables in successive projective measurements without error. From the results we can conclude that under the assumption of precise successive measurements, it is limited to obtaining outcomes with certainty. This limitation is clarified by the relation for conditional entropy.

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