Relativistic Einstein-Podolsky-Rosen correlations and localization

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We calculate correlation functions for a relativistic Einstein-Podolsky-Rosen–type experiment with massive Dirac particles. We take into account the influence of the Newton-Wigner localization and perform the calculations for a couple of physically interesting states. In particular, we show that localization inside detectors does not significantly affect correlation functions in the case when localization regions are macroscopic or in the case when Einstein-Podolsky-Rosen particles are in a definite momentum state (irrespectively of the size of localization regions).

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I. INTRODUCTION

Relativistic Einstein-Podolsky-Rosen–type (EPR–type) correlations have attracted a lot of attention in recent years (see, e.g., Refs. [1–13] and references therein). However, previous discussions did not take into account the localization of EPR particles inside detectors, in spite of the fact that such a localization usually accompanies the spin projection measurement. In the following, we will distinguish the localization region and detector region (compare, e.g., bubble chamber). The importance of localization of relativistic qubits in the context of curved space-time has been also stressed in a recent paper [14].

The main purpose of our paper is to include the localization in the discussion of EPR–type spin correlations. However, there are some issues we have to overcome. First of all, various spin operators have been used in the discussion of relativistic correlations (see, e.g., Refs. [15–19]). Second, the notion of localization is not well defined in relativistic quantum mechanics [20].

The definition of a proper position operator is a long-standing problem of relativistic quantum mechanics. Such a position operator is expected to fulfill natural demands: it should have commuting, Hermitian components, it should transform like a three-vector under rotations, its components should fill canonical commutation rules with momentum operators, and it should transform covariantly under Lorentz boosts. Unfortunately, an operator satisfying all of the above conditions does not exist; only some of these conditions can be satisfied simultaneously. In our opinion, the simplest choice is to give up Lorentz covariance. The operator satisfying all of the conditions but covariance was introduced by Newton and Wigner [21] and is called the Newton-Wigner (NW) position operator. Of course, other position operators are also used, for example, the center-of-mass position operator introduced by Pryce [22,23] (this operator has noncommuting components).

Notice that the definition of a relativistic spin operator is connected with the localization problem because spin can be defined as a difference between total and orbital angular momentum [Eq. (10)], where the orbital angular momentum is defined with the help of the position operator (see also [24]). This relationship is especially important when we consider localization and spin measurement simultaneously.

In our recent paper [25], we have shown that for Dirac particles the most appropriate spin operator is an operator related to the NW position operator [21]. Therefore, in this paper, we use the NW localization and the corresponding spin operator. Notice that the same spin operator is used in the quantum field theory formalism. In relativistic quantum information theory, other spin operators are also in use. For instance, Czachor [15] uses the spin operator corresponding to the Pryce localization.

We consider EPR correlations in a singlet state of two spin-$\frac{1}{2}$ particles assuming that spin projection measurements take place in finite-volume regions (detectors). We derive a general formula for the correlation function and then consider some special cases. In particular, we show that localization inside detectors does not significantly affect the correlation function in the case when detectors are macroscopic and in the case when EPR particles are in definite momentum state (irrespectively of the size of detectors).

II. TWO-PARTICLE SINGLET IN THE RELATIVISTIC SETTING

The carrier space $\mathcal{H}$ of the unitary representation of the Poincaré group for spin $\frac{1}{2}$ is spanned by the eigenvectors of four-momentum $\hat{P}^\mu$. These states are denoted as $|p,\sigma\rangle$, $\sigma = \pm\frac{1}{2}$, and are normalized as follows:

$$\langle p,\sigma|k,\lambda\rangle = 2p^0\delta^3(\mathbf{p}-\mathbf{k})\delta_{\sigma\lambda}. \tag{1}$$

Under the action of the Lorentz group, the basis states transform as

$$U(\Lambda)|p,\sigma\rangle = \mathcal{D}^{1/2}_{\lambda\sigma}(R(\Lambda,p))|\Lambda p,\lambda\rangle, \tag{2}$$

where $\mathcal{D}^{1/2}$ is the spin-$\frac{1}{2}$ representation of the rotation group and $R(\Lambda,p)$ is a Wigner rotation. The explicit form of a Wigner rotation is the following: $R(\Lambda,p) = L^{-1}_{\Lambda p}\Lambda L_p$, where L_p is a standard boost transforming the rest-frame four-momentum $(m,\mathbf{0})$ into four-momentum p:

$$L_p = \begin{pmatrix} \frac{p^0}{m} & \frac{\mathbf{p}^T}{m} \\ \frac{\mathbf{p}}{m} & I + \frac{\mathbf{p}\otimes\mathbf{p}^T}{m(m+p^0)} \end{pmatrix}. \tag{3}$$

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The Lorentz-covariant singlet two-particle state has the form (see, e.g., [1])

$$|\varphi\rangle = \int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\varphi(k,p)\mathcal{M}_{\sigma\lambda}(k,p)|k,\sigma\rangle \otimes |p,\lambda\rangle, \quad (4)$$

where $\varphi(k,p)$ is a scalar function, and for the particles with the same mass m the matrix $\mathcal{M}$ reads as

$$\mathcal{M}(k,p)_{\sigma\lambda} = -i(2m\sqrt{(m+p^0)(m+k^0)})^{-1} \times ([(m+k^0)(m+p^0) - \mathbf{k}\cdot\mathbf{p} - i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{p})]\sigma_2)_{\sigma\lambda}. \quad (5)$$

In the above equation, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ and σ_i are the standard Pauli matrices.

Lorentz covariance of a state means that it has a well-defined manifestly covariant transformation rule under Lorentz group action. For the state defined in Eq. (4), with the help of Eqs. (2) and (5), we get the following transformation rule:

$$U(\Lambda)\otimes U(\Lambda)|\phi\rangle = \int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\varphi(\Lambda^{-1}k,\Lambda^{-1}p)\mathcal{M}_{\sigma\lambda}(k,p)|k,\sigma\rangle \otimes |p,\lambda\rangle, \quad (6)$$

where Λ is a Lorentz transformation. For further details on the definition and properties of Lorentz-covariant states, we refer the reader to [1].

III. SPIN AND LOCALIZATION

It is well known that the spin square operator can be uniquely defined in terms of the generators of the Poincaré group as

$$\hat{\mathbf{S}}^2 = -\frac{1}{m^2}\hat{W}^\mu\hat{W}_\mu, \quad (7)$$

where $\hat{W}^\mu$ is the Pauli-Lubanski four-vector: $\hat{W}^\mu = \frac{1}{2}\epsilon^{\nu\alpha\beta\mu}\hat{P}_\nu\hat{J}_{\alpha\beta}$, and $\hat{J}_{\alpha\beta}$ are the generators of the Lorentz group. In spite of that, the definition of the relativistic spin operator has been widely discussed in the literature (see, e.g., [15,16,25–27] and references therein). In the enveloping algebra of the Poincaré algebra, we can define a relativistic spin operator

$$\hat{\mathbf{S}} = \frac{1}{m}\left(\hat{\mathbf{W}} - \hat{W}^0\frac{\hat{\mathbf{P}}}{\hat{P}^0+m}\right). \quad (8)$$

This operator is linear in the components of $\hat{W}^\mu$, transforms like a vector under rotations and like a pseudovector under reflections, commutes with space-time observables, and fulfills the standard canonical commutation relations (for the details see, e.g., [27]). Moreover, it has been shown [28] that the spin operator (8) is equivalent to the action of the mean-spin operator introduced by Foldy and Wouthuysen [26] in the Dirac theory. An exhaustive discussion of this operator, including its transformation properties, is given in [25,28].

The action of the operator (8) on the basis vectors is of the form

$$\hat{\mathbf{S}}|p,\sigma\rangle = \frac{\boldsymbol{\sigma}_{\sigma'\sigma}}{2}|p,\sigma'\rangle. \quad (9)$$

On the other hand, spin operator (8) can be defined as a difference between the total and orbital angular momentum

$$\hat{\mathbf{S}} = \hat{\mathbf{J}} - \hat{\mathbf{X}}\times\hat{\mathbf{P}}, \quad (10)$$

where $\hat{J}^i = \epsilon^{ijk}\hat{J}^{jk}$ and $\hat{\mathbf{X}}$ is the Newton-Wigner position operator.

We now briefly remind some basic properties of the NW position operator. An arbitrary one-particle state can be written as

$$|\psi\rangle = \int \frac{d^3\mathbf{p}}{2p^0}\psi_\sigma(p)|p,\sigma\rangle. \quad (11)$$

The action of the NW position operator on wave function in the momentum representation has the well-known form

$$\hat{\mathbf{X}}\psi_\sigma(p) = \left(i\nabla_{\mathbf{p}} - \frac{1}{2}\frac{i\mathbf{p}}{\mathbf{p}^2+m^2}\right)\psi_\sigma(p). \quad (12)$$

The eigenstates of this operator are

$$|\mathbf{x},\sigma\rangle = (2\pi)^{-3/2}\int \frac{d^3\mathbf{p}}{2p^0}\sqrt{2p^0}e^{-i\mathbf{p}\cdot\mathbf{x}}|p,\sigma\rangle. \quad (13)$$

Notice that the states (13) are not covariant, i.e., the Lorentz-transformed state $U(\Lambda)|\mathbf{x},\sigma\rangle$ is no longer an eigenstate of the NW position operator. Now, we can introduce a projector on a region Ω:

$$\hat{\Pi}_\Omega = \sum_\sigma \int_\Omega d^3\mathbf{x}\,|\mathbf{x},\sigma\rangle\langle\mathbf{x},\sigma|. \quad (14)$$

Using Eq. (13), we get

$$\hat{\Pi}_\Omega = \int \frac{d^3\mathbf{p}'}{\sqrt{2p'^0}}\frac{d^3\mathbf{p}}{\sqrt{2p^0}}\Delta_\Omega(\mathbf{p}'-\mathbf{p})\sum_\sigma |p',\sigma\rangle\langle p,\sigma|, \quad (15)$$

where

$$\Delta_\Omega(\mathbf{p}'-\mathbf{p}) = \frac{1}{(2\pi)^3}\int_\Omega d^3\mathbf{x}\,e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}. \quad (16)$$

Notice that

$$\Delta_{\mathbb{R}^3}(\mathbf{p}'-\mathbf{p}) = \delta^3(\mathbf{p}'-\mathbf{p}). \quad (17)$$

The spin projection measurement in the direction $\mathbf{n}$ in the region Ω is described by the following observable:

$$\mathbf{n}\cdot\hat{\mathbf{S}}_\Omega = (\mathbf{n}\cdot\hat{\mathbf{S}})\hat{\Pi}_\Omega, \quad (18)$$

where $\hat{\Pi}_\Omega$ and $\hat{\mathbf{S}}$ are given by Eqs. (14) and (9), respectively.

IV. RELATIVISTIC EPR CORRELATIONS

Now, let us consider an EPR–type experiment. That is, we assume that two particles in the state (4) are sent to two distant observers, Alice and Bob. Alice (Bob) measures the spin projection in the direction $\mathbf{a}$ ($\mathbf{b}$) provided that her (his) particle is localized inside the region A (B). It means that Alice measures the observable $(\mathbf{a}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{A}}$ while Bob $(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{B}}$. The normalized correlation function reads as

$$\mathcal{C}_\varphi^{\mathrm{AB}}(\mathbf{a},\mathbf{b}) = 4\frac{\langle\varphi|\hat{\Pi}_{\mathrm{A}}(\mathbf{a}\cdot\hat{\mathbf{S}})\otimes(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{B}}|\varphi\rangle}{\langle\varphi|\hat{\Pi}_{\mathrm{A}}\otimes\hat{\Pi}_{\mathrm{B}}|\varphi\rangle}. \quad (19)$$

The form of the denominator in Eq. (19) corresponds to the fact that we take into account only the pairs that are actually found inside the regions A and B and the function has to be appropriately normalized. Using Eqs. (4), (9), (15), and (18), we get

$$\begin{aligned}&\langle\varphi|\hat{\Pi}_{\mathrm{A}}(\mathbf{a}\cdot\hat{\mathbf{S}})\otimes(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{B}}|\varphi\rangle\\&=\frac{1}{4}\int\frac{d^3\mathbf{k}'\,d^3\mathbf{k}\,d^3\mathbf{p}'\,d^3\mathbf{p}}{\sqrt{2k'^0 2k^0 2p'^0 2p^0}}\varphi^*(k',p')\varphi(k,p)\\&\quad\times\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(k,p)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(k',p')\}\\&\quad\times\Delta_{\mathrm{A}}(\mathbf{k}'-\mathbf{k})\Delta_{\mathrm{B}}(\mathbf{p}'-\mathbf{p}),\end{aligned}\tag{20}$$

where Eq. (5) implies

$$\begin{aligned}&\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(k,p)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(k',p')\}\\&=-\left(2m^2\sqrt{(m+p^0)(m+k^0)(m+p'^0)(m+k'^0)}\right)^{-1}\\&\quad\times\{[\mathbf{a}\cdot(\mathbf{k}\times\mathbf{p})][\mathbf{b}\cdot(\mathbf{k}'\times\mathbf{p}')]+[\mathbf{a}\cdot(\mathbf{k}'\times\mathbf{p}')]\\&\quad\times[\mathbf{b}\cdot(\mathbf{k}\times\mathbf{p})]-(\mathbf{a}\cdot\mathbf{b})[(\mathbf{k}\times\mathbf{p})\cdot(\mathbf{k}'\times\mathbf{p}')]\\&\quad+(\mathbf{a}\cdot\mathbf{b})[(m+k^0)(m+p^0)-\mathbf{k}\cdot\mathbf{p}]\\&\quad\times[(m+k'^0)(m+p'^0)-\mathbf{k}'\cdot\mathbf{p}']\\&\quad-(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{k}'\times\mathbf{p}')[(m+k^0)(m+p^0)-\mathbf{k}\cdot\mathbf{p}]\\&\quad+(\mathbf{k}\times\mathbf{p})[(m+k'^0)(m+p'^0)-\mathbf{k}'\cdot\mathbf{p}']]\}.\end{aligned}\tag{21}$$

The denominator of the right-hand side of Eq. (19) takes the form

$$\begin{aligned}\langle\varphi|\hat{\Pi}_{\mathrm{A}}\otimes\hat{\Pi}_{\mathrm{B}}|\varphi\rangle&=\int\frac{d^3\mathbf{k}'\,d^3\mathbf{k}\,d^3\mathbf{p}'\,d^3\mathbf{p}}{\sqrt{2k'^0 2k^0 2p'^0 2p^0}}\varphi^*(k',p')\varphi(k,p)\\&\quad\times\mathrm{Tr}\{\mathcal{M}(k,p)\mathcal{M}(k',p')^\dagger\}\\&\quad\times\Delta_{\mathrm{A}}(\mathbf{k}'-\mathbf{k})\Delta_{\mathrm{B}}(\mathbf{p}'-\mathbf{p}),\end{aligned}\tag{22}$$

where

$$\begin{aligned}&\mathrm{Tr}\{\mathcal{M}(k,p)\mathcal{M}^\dagger(k',p')\}\\&=\left(2m^2\sqrt{(m+p^0)(m+k^0)(m+p'^0)(m+k'^0)}\right)^{-1}\\&\quad\times\{[(m+k^0)(m+p^0)-\mathbf{k}\cdot\mathbf{p}]\\&\quad\times[(m+k'^0)(m+p'^0)-\mathbf{k}'\cdot\mathbf{p}']\\&\quad+(\mathbf{k}\times\mathbf{p})\cdot(\mathbf{k}'\times\mathbf{p}')\}.\end{aligned}\tag{23}$$

As a reference point, let us recall here the correlation function in the scalar state without localization [1]:

$$\begin{aligned}&\mathcal{C}_\varphi(\mathbf{a},\mathbf{b})\\&=\left\{4\int\frac{d^3\mathbf{k}\,d^3\mathbf{p}}{2k^0 2p^0}|\varphi(k,p)|^2\mathrm{Tr}\{\mathcal{M}(k,p)\mathcal{M}^\dagger(k,p)\}\right\}^{-1}\\&\quad\times\int\frac{d^3\mathbf{k}\,d^3\mathbf{p}}{2k^0 2p^0}|\varphi(k,p)|^2\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(k,p)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\\&\quad\times\mathcal{M}^\dagger(k,p)\},\end{aligned}\tag{24}$$

where

$$\begin{aligned}&\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(k,p)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(k,p)\}\\&=-\frac{1}{m^2}\left\{(\mathbf{a}\cdot\mathbf{b})(m^2+kp)-(\mathbf{k}\times\mathbf{p})\left[(\mathbf{a}\times\mathbf{b})\right.\right.\\&\quad\left.\left.+\frac{(\mathbf{a}\cdot\mathbf{k})(\mathbf{b}\times\mathbf{p})-(\mathbf{b}\cdot\mathbf{p})(\mathbf{a}\times\mathbf{k})}{(m+k^0)(m+p^0)}\right]\right\}\end{aligned}\tag{25}$$

and

$$\mathrm{Tr}\{\mathcal{M}(k,p)\mathcal{M}^\dagger(k,p)\}=1+\frac{kp}{m^2}.\tag{26}$$

In this paper, we use natural units. However, for a moment let us consider a particle with mass m and write Eq. (16) including explicitly all constants:

$$\Delta_\Omega(\mathbf{p}'-\mathbf{p})=\frac{1}{(2\pi\hbar)^3}\int_\Omega d^3\mathbf{x}\,e^{-i\frac{(\mathbf{p}'-\mathbf{p})}{mc}\cdot\frac{\mathbf{x}}{\lambda}},\tag{27}$$

where $\lambda=\frac{\hbar}{mc}$ is a Compton wavelength of a particle. As an example, let us assume that the region Ω is a cube with the center located at $\mathbf{R}$. One can easily show that in this case

$$\Delta_\square(\mathbf{p}'-\mathbf{p})=\frac{1}{(mc)^3}e^{-i\frac{\mathbf{p}'-\mathbf{p}}{mc}\cdot\frac{\mathbf{R}}{\lambda}}\prod_{j=1}^{3}\frac{l}{2\pi\lambda}\frac{\sin\left(\frac{[R(\mathbf{p}'-\mathbf{p})]_j}{mc}\frac{l}{2\lambda}\right)}{\frac{[R(\mathbf{p}'-\mathbf{p})]_j}{mc}\frac{l}{2\lambda}},\tag{28}$$

where R is a rotation transforming the cube to the position with edges parallel to coordinate system axes. The derivation of Eq. (28) we give in Appendix A. Now, for elementary particles, the Compton wavelength is very small, for example, for electron [29]

$$\lambda_e=3.86\times10^{-13}\ \mathrm{m}.\tag{29}$$

For macroscopic regions, $\frac{l}{2\lambda}$ is very big and taking into account the formula [30]

$$\frac{\alpha}{\pi}\frac{\sin(\alpha x)}{\alpha x}\overset{\alpha\to\infty}{\longrightarrow}\delta(x),\tag{30}$$

we see that in this case

$$\Delta_\square(\mathbf{p}'-\mathbf{p})\to\delta^3(\mathbf{p}'-\mathbf{p}).\tag{31}$$

Generalizing this example, we see that for macroscopic regions the function $\Delta_\Omega(\mathbf{p}'-\mathbf{p})$ is practically equal to $\delta^3(\mathbf{p}'-\mathbf{p})$. Now, let us return to natural units.

Taking into account the above discussion, we get the following relations in the case of macroscopic regions of localization:

$$\begin{aligned}&4\langle\varphi|\hat{\Pi}_{\mathrm{A}}(\mathbf{a}\cdot\hat{\mathbf{S}})\otimes(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{B}}|\varphi\rangle\\&\quad\to\int\frac{d^3\mathbf{k}\,d^3\mathbf{p}}{2k^0 2p^0}|\varphi(k,p)|^2\\&\quad\quad\times\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(k,p)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(k,p)\}\end{aligned}\tag{32}$$

and

$$\begin{aligned}&\langle\varphi|\hat{\Pi}_{\mathrm{A}}\otimes\hat{\Pi}_{\mathrm{B}}|\varphi\rangle\\&\quad\to\int\frac{d^3\mathbf{k}\,d^3\mathbf{p}}{2k^0 2p^0}|\varphi(k,p)|^2\mathrm{Tr}\{\mathcal{M}(k,p)\mathcal{M}^\dagger(k,p)\}.\end{aligned}\tag{33}$$

Consequently, the correlation function has the same form as in the case when we do not take into account the localization [compare Eq. (24)]. Therefore, any nontrivial effects of localization on EPR correlation function can be expected only when volumes of localization regions are of order λ^3, where λ is a Compton wavelength of EPR particles.

V. SPECIAL CASES

In this section, we consider some special cases of the correlation function (19). We will try to find situations when the localization significantly affects the correlation function. From now on, we take the wave function $\varphi(k,p)$ to be of the following form:

$$\varphi(k,p) = \varphi(k)\varphi(p), \tag{34}$$

i.e., momentum profiles of the wave packets for both particles are chosen to be the same. Adopting (34) does not imply that our state is separable: entanglement is present in the structure of $\mathcal{M}(k,p)$, as given by Eq. (5). With this assumption, it is easily checked that (20) and (22) may be reduced to products of integrals of three types:

$$I_1^{\mathrm{A}}[\varphi] = \frac{1}{m}\int \frac{d^3\mathbf{k}'\, d^3\mathbf{k}\,(m+k'^0)(m+k^0)}{\sqrt{k'^0(m+k'^0)}\sqrt{k^0(m+k^0)}} \times \Delta_{\mathrm{A}}(\mathbf{k}'-\mathbf{k})\varphi^*(k')\varphi(k), \tag{35a}$$

$$I_2^{\mathrm{A}\,i}[\varphi] = \frac{1}{m}\int \frac{d^3\mathbf{k}'\, d^3\mathbf{k}\,(m+k'^0)k^i}{\sqrt{k'^0(m+k'^0)}\sqrt{k^0(m+k^0)}} \times \Delta_{\mathrm{A}}(\mathbf{k}'-\mathbf{k})\varphi^*(k')\varphi(k), \tag{35b}$$

$$I_3^{\mathrm{A}\,ij}[\varphi] = \frac{1}{m}\int \frac{d^3\mathbf{k}'\, d^3\mathbf{k}\, k^i k'^j\, \Delta_{\mathrm{A}}(\mathbf{k}'-\mathbf{k})\varphi^*(k')\varphi(k)}{\sqrt{k'^0(m+k'^0)}\sqrt{k^0(m+k^0)}} \tag{35c}$$

taken with appropriate coefficients. For Eq. (20), we have

$$\begin{aligned}
&\langle\varphi|\hat{\Pi}_{\mathrm{A}}(\mathbf{a}\cdot\hat{\mathbf{S}})\otimes(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_{\mathrm{B}}|\varphi\rangle\\
&= \frac{-1}{2^5}\{(\mathbf{a}\cdot\mathbf{b})\big[I_1^{\mathrm{A}}I_1^{\mathrm{B}} - (\mathbf{I}_2^{\mathrm{A}}\cdot\mathbf{I}_2^{\mathrm{B}}) - (\mathbf{I}_2^{\mathrm{A}}\cdot\mathbf{I}_2^{\mathrm{B}})^*\\
&\quad + \mathrm{Tr}\big[I_3^{\mathrm{A}}I_3^{\mathrm{B}*}\big] + \mathrm{Tr}\big[I_3^{\mathrm{A}}I_3^{\mathrm{B}}\big] - \mathrm{Tr}\big[I_3^{\mathrm{A}}\big]\mathrm{Tr}\big[I_3^{\mathrm{B}}\big]\big]\\
&\quad - (\mathbf{a}\times\mathbf{b})\cdot\big[(\mathbf{I}_2^{\mathrm{A}}\times\mathbf{I}_2^{\mathrm{B}}) + (\mathbf{I}_2^{\mathrm{A}}\times\mathbf{I}_2^{\mathrm{B}})^*\big]\\
&\quad + \mathbf{a}^T\big[I_3^{\mathrm{A}}(I_3^{\mathrm{B}})^* - I_3^{\mathrm{B}}(I_3^{\mathrm{A}})^* + (I_3^{\mathrm{A}})^*I_3^{\mathrm{B}} - (I_3^{\mathrm{B}})^*I_3^{\mathrm{A}}\big]\mathbf{b}\\
&\quad + \varepsilon^{ijk}\varepsilon^{qrs}a^i b^q\big[I_3^{\mathrm{A}\,jr}I_3^{\mathrm{B}\,ks} + (I_3^{\mathrm{A}\,jr}I_3^{\mathrm{B}\,ks})^*\big]\},
\end{aligned} \tag{36}$$

while for Eq. (22) we have

$$\begin{aligned}
&\langle\varphi|\hat{\Pi}_{\mathrm{A}}\otimes\hat{\Pi}_{\mathrm{B}}|\varphi\rangle\\
&= \frac{1}{2^3}\{I_1^{\mathrm{A}}I_1^{\mathrm{B}} - (\mathbf{I}_2^{\mathrm{A}}\cdot\mathbf{I}_2^{\mathrm{B}}) - (\mathbf{I}_2^{\mathrm{A}}\cdot\mathbf{I}_2^{\mathrm{B}})^* + \mathrm{Tr}\big[I_3^{\mathrm{A}}I_3^{\mathrm{B}*}\big]\\
&\quad - \mathrm{Tr}\big[I_3^{\mathrm{A}}I_3^{\mathrm{B}}\big] + \mathrm{Tr}\big[I_3^{\mathrm{A}}\big]\mathrm{Tr}\big[I_3^{\mathrm{B}}\big]\}.
\end{aligned} \tag{37}$$

A. Definite momentum state

Now, let us consider the simplest case of two particles with definite momenta. Therefore, we assume that

$$\varphi(k,p) \to 2q_a^0\delta^3(\mathbf{k}-\mathbf{q}_a)\,2q_b^0\delta^3(\mathbf{p}-\mathbf{q}_b)m^2, \tag{38}$$

where q_a and q_b designate the fixed four-momenta of particles a and b, respectively. Our state takes the form

$$|\varphi\rangle \to |\varphi_{q_aq_b}\rangle = \mathcal{M}(q_a,q_b)_{\sigma\lambda}|q_a,\sigma\rangle\otimes|q_b,\lambda\rangle. \tag{39}$$

Using explicit formulas given in Appendix B [Eqs. (B1), (B2), and (B3)] and inserting the explicit form of the matrix $\mathcal{M}(q_a,q_b)$ [Eq. (5)], we finally get in this case

$$\mathcal{C}^{\mathrm{AB}}_{\varphi_{q_aq_b}}(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b} + \frac{(\mathbf{q}_a\times\mathbf{q}_b)}{m^2+q_aq_b}\cdot\Bigg[(\mathbf{a}\times\mathbf{b}) + \frac{(\mathbf{a}\cdot\mathbf{q}_a)(\mathbf{b}\times\mathbf{q}_b)-(\mathbf{b}\cdot\mathbf{q}_b)(\mathbf{a}\times\mathbf{q}_a)}{(q_a^0+m)(q_b^0+m)}\Bigg]. \tag{40}$$

Comparing the correlation function (40) with the previous results for the correlation function without localization [1,17], we see that for definite momentum states localization does not affect the correlation function. This result is independent of the size of localization regions.

For further use, let us notice that Eq. (40) can be written in the following form:

$$\begin{aligned}
\mathcal{C}^{\mathrm{AB}}_{\varphi_{q_aq_b}}(\mathbf{a},\mathbf{b}) &= -\mathbf{a}\cdot\mathbf{b} + \{(\mathbf{n}\times\mathbf{m})\cdot(\mathbf{a}\times\mathbf{b})|\mathbf{q}_a||\mathbf{q}_b|\\
&\quad\times(m+q_a^0)(m+q_b^0) + [(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})\\
&\quad + (\mathbf{a}\cdot\mathbf{m})(\mathbf{b}\cdot\mathbf{m}) - 2(\mathbf{n}\cdot\mathbf{m})(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{m})]\mathbf{q}_a^2\mathbf{q}_b^2\}\\
&\quad\times\{\tfrac{1}{2}(m+q_a^0)^2(m+q_b^0)^2 + \tfrac{1}{2}\mathbf{q}_a^2\mathbf{q}_b^2\\
&\quad - (\mathbf{n}\cdot\mathbf{m})|\mathbf{q}_a||\mathbf{q}_b|(m+q_a^0)(m+q_b^0)\}^{-1},
\end{aligned} \tag{41}$$

where $\mathbf{n} = \mathbf{q}_a/|\mathbf{q}_a|$ and $\mathbf{m} = \mathbf{q}_b/|\mathbf{q}_b|$.

B. State with fixed particle momenta directions

Now, let us consider the more general situation when only the directions of particle momenta are fixed. Thus, let us denote directions of the momenta of the first and second particles by $\mathbf{n}$ and $\mathbf{m}$, respectively. We assume that the wave function has the following form:

$$\varphi(k,p) \to \frac{\sqrt{k^0(m+k^0)}}{\mathbf{k}^2}f(|\mathbf{k}|)\delta\left(\frac{\mathbf{k}}{|\mathbf{k}|}-\mathbf{n}\right) \times\frac{\sqrt{p^0(m+p^0)}}{\mathbf{p}^2}f(|\mathbf{p}|)\delta\left(\frac{\mathbf{p}}{|\mathbf{p}|}-\mathbf{m}\right), \tag{42}$$

where $\delta(\frac{\mathbf{k}}{|\mathbf{k}|}-\mathbf{n})$ is a Dirac delta projecting on a fixed direction, i.e.,

$$\int d\Omega(\alpha,\beta)\delta(\mathbf{n}(\alpha,\beta)-\mathbf{n})g(\mathbf{n}(\alpha,\beta)) = g(\mathbf{n}), \tag{43}$$

where $d\Omega(\alpha,\beta)$ is a differential solid angle.

In this case for the correlation function we find, with the help of Eqs. (36), (37), and (19), that

$$\begin{aligned}
&\mathcal{C}^{\mathrm{AB}}_{\mathbf{n},\mathbf{m}}(\mathbf{a},\mathbf{b})\\
&= -\mathbf{a}\cdot\mathbf{b} + \{(\mathbf{n}\times\mathbf{m})\cdot(\mathbf{a}\times\mathbf{b})\big[I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}}\\
&\quad + (I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}})^*\big] + 2\big[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n}) + (\mathbf{a}\cdot\mathbf{m})(\mathbf{b}\cdot\mathbf{m})\\
&\quad - 2(\mathbf{n}\cdot\mathbf{m})(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{m})\big]I_3^{\mathrm{A},\mathbf{n}}I_3^{\mathrm{B},\mathbf{m}}\}\{I_1^{\mathrm{A},\mathbf{n}}I_1^{\mathrm{B},\mathbf{m}}\\
&\quad + I_3^{\mathrm{A},\mathbf{n}}I_3^{\mathrm{B},\mathbf{m}} - (\mathbf{n}\cdot\mathbf{m})\big[I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}} + (I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}})^*\big]\}^{-1},
\end{aligned} \tag{44}$$

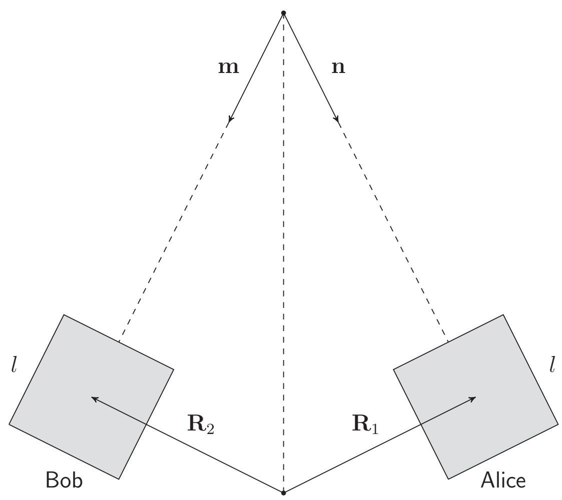

FIG. 1. The configuration of localization regions considered in Sec. V B. Both particles are localized inside cubes with sides l and with one face orthogonal to $\mathbf{n}$ (Alice) or $\mathbf{m}$ (Bob). The origin of the coordinate system we choose in such a way that $\mathbf{n}\cdot\mathbf{R}_1 = 0$ and $\mathbf{m}\cdot\mathbf{R}_2 = 0$.

where

$$I_1^{\mathrm{A},\mathbf{n}} = \int_0^\infty dt\,du\,(m+\sqrt{m^2+t^2})(m+\sqrt{m^2+u^2}) \times\Delta_{\mathrm{A}}((t-u)\mathbf{n})f^*(t)f(u), \quad (45a)$$

$$I_2^{\mathrm{A},\mathbf{n}} = \int_0^\infty dt\,du\,u(m+\sqrt{m^2+t^2})\Delta_{\mathrm{A}}((t-u)\mathbf{n})f^*(t)f(u), \quad (45b)$$

$$I_3^{\mathrm{A},\mathbf{n}} = \int_0^\infty dt\,du\,tu\Delta_{\mathrm{A}}((t-u)\mathbf{n})f^*(t)f(u). \quad (45c)$$

The relation of the above integrals with integrals (35) is given in Eq. (C1). It is worth to stress that for $\mathbf{m} = -\mathbf{n}$, the correlation function (44) is equal to

$$\mathcal{C}^{\mathrm{AB}}_{\mathbf{n},-\mathbf{n}}(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b}. \quad (46)$$

Thus, for particles propagating in opposite directions, localization does not change the correlation function.

Notice that Eq. (41) can be obtained from Eq. (44) under the following conditions:

$$\frac{I_1^{\mathrm{A},\mathbf{n}}I_1^{\mathrm{B},\mathbf{m}}}{I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}} + \left(I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}}\right)^*} \to \frac{\left(m+q_a^0\right)\left(m+q_b^0\right)}{2|\mathbf{q}_a||\mathbf{q}_b|}, \quad (47)$$

$$\frac{I_3^{\mathrm{A},\mathbf{n}}I_3^{\mathrm{B},\mathbf{m}}}{I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}} + \left(I_2^{\mathrm{A},\mathbf{n}}I_2^{\mathrm{B},\mathbf{m}}\right)^*} \to \frac{|\mathbf{q}_a||\mathbf{q}_b|}{2\left(m+q_a^0\right)\left(m+q_b^0\right)}. \quad (48)$$

Now, as an illustrative example, let us consider the configuration shown in Fig. 1. Thus, we assume that both particles are localized inside cubes with sides l and with one face orthogonal to the momentum of the corresponding particle. Without loss of generality, we can choose the coordinate system in such a way that $\mathbf{n}\cdot\mathbf{R}_1 = 0$ and $\mathbf{m}\cdot\mathbf{R}_2 = 0$. Therefore, in this case the correlation function does not depend on the distance between localization regions.

For such regions, the functions $\Delta_{\mathrm{A}}((t-u)\mathbf{n})$ and $\Delta_{\mathrm{B}}((t-u)\mathbf{m})$ [compare Eq. (16)] can be calculated explicitly. We get

$$\Delta_{\mathrm{A}}((t-u)\mathbf{n}) = \frac{2l^2}{(2\pi)^3}\frac{\sin\left[\frac{1}{2}l(t-u)\right]}{t-u} \quad (49)$$

and

$$\Delta_{\mathrm{B}}((t-u)\mathbf{m}) = \Delta_{\mathrm{A}}((t-u)\mathbf{n}). \quad (50)$$

To proceed further, let us assume also that the function f describing the profile of a wave packet [Eq. (42)] has the following Gauss-type form:

$$f(t) = \frac{1}{2\pi\sigma}(\sqrt{t^2+m^2}-m)^{1/2}e^{-\frac{(t-q)^2}{2\sigma^2}}. \quad (51)$$

The factor $\sqrt{\sqrt{t^2+m^2}-m}$ has been added to guarantee normalizability of the state $|\varphi\rangle$.

As we have noted before, nontrivial effects of localization should be looked for small values of l. In Fig. 2, we have depicted the correlation function in the considered state versus the average momentum of the particle divided by its mass

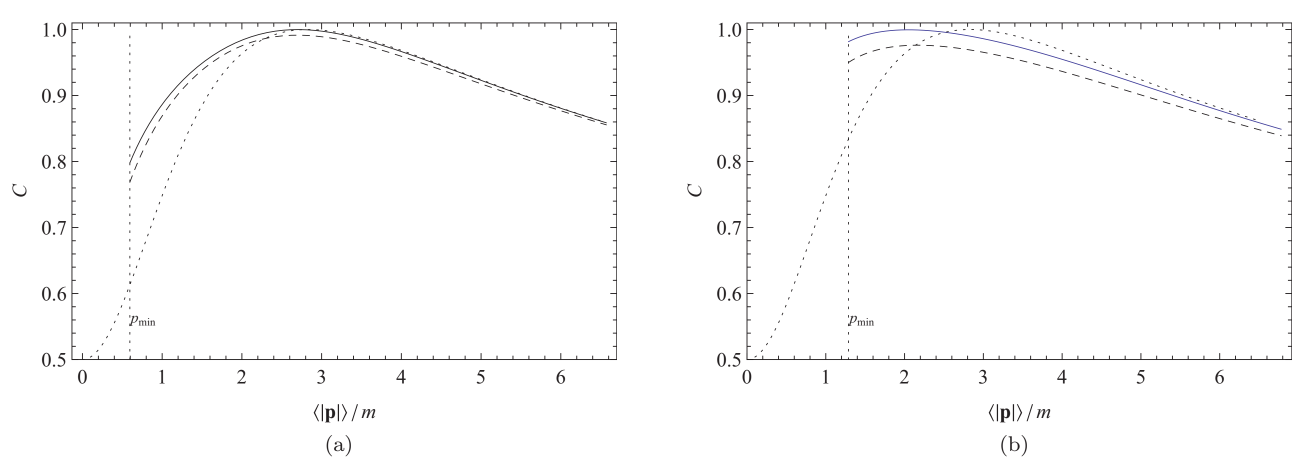

FIG. 2. Correlation function versus mean normalized particle momentum for fixed directions of particle momenta (Fig. 1) and for EPR particles with Gaussian-type momentum distribution profile [Eq. (51)]. Correlation function with localization inside detectors (solid line, $l = 0.25$) and without localization (dashed line) is compared to the case of particles with definite momenta (dotted line). Figures (a) and (b) correspond to narrower ($\sigma = 1$) and wider ($\sigma = 2$) profile of the wave packet, respectively. Momenta directions are $\mathbf{n} = \frac{1}{2}(1,-\sqrt{3},0)$ and $\mathbf{m} = \frac{1}{2}(-1,-\sqrt{3},0)$, while directions of spin projections are $\mathbf{a} = (0,1,0)$, $\mathbf{b} = \frac{1}{2}(\sqrt{3},-1,0)$.

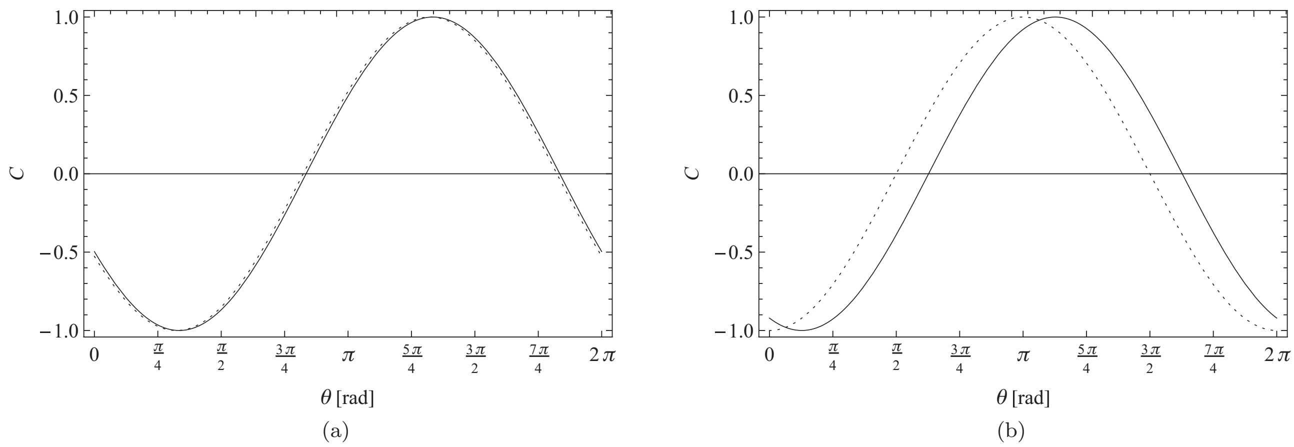

FIG. 3. Correlation function versus angle θ between directions of spin projections **a** and **b** for fixed directions of particle momenta (Fig. 1) and for EPR particles with Gaussian-type momentum distribution profile [Eq. (51)]. Correlation function with localization inside detectors (solid line) is compared to the case of particles with definite momenta (dotted line). Figure (a) corresponds to the mean value of momentum $\langle|\mathbf{p}|\rangle/m = 2.71659$ [maximum in Fig. (2a)], while (b) corresponds to the smallest possible values of mean value of momentum in Fig. (2a): $p_{\text{min}}/m = 0.594733$ (solid line) and 0 (dotted line). We take momenta directions $\mathbf{n} = \frac{1}{2}(1,-\sqrt{3},0)$ and $\mathbf{m} = \frac{1}{2}(-1,-\sqrt{3},0)$, spin projection directions as $\mathbf{a} = (0,1,0)$, $\mathbf{b} = (-\sin\theta,\cos\theta,0)$, momentum profile with $\sigma = 1$, and detector size $l = 0.25$. Notice that the entire curve is shifted toward right with increasing momentum.

for $\sigma = 1$ and 2. For the comparison, we have also added the curves corresponding to the correlation function without localization (but in the same state) and to the correlation function in the state with definite momenta [Eq. (41)]. Notice that particles in the state (51) can not have arbitrary mean momentum. That is why p_{min} appears in Fig. 2.

As expected, when $\sigma \to 0$, the correlation function tends to the correlation function without localization. The same conclusion holds for increasing size of localization regions l. In Fig. 3, we have plotted the dependence of the correlation function given in Eq. (44) in the configuration shown in Fig. 1 and for the state defined by the function (51) on an angle between vectors **a** and **b**.

C. Arbitrary momenta

Now, let us illustrate the behavior of the correlation function in the general case when particles momenta can take arbitrary values. We consider a simple example of wave function for which integrals (35) can be calculated explicitly. In this section, we again include explicitly all constants to controlling better upcoming approximations. We assume [compare Eq. (34)]

$$\varphi(k) = \frac{\sqrt{k^0(k^0+mc)}}{(mc)^2} e^{-\kappa\frac{k^0}{mc}}. \tag{52}$$

The advantage of this choice is that we can calculate explicitly the Fourier transform of the function $e^{-\kappa\frac{k^0}{mc}}$ (see Appendix D).

In this case, the integrals (35) can be expressed as follows:

$$I_1^{\Omega} = \int_{\Omega} \frac{d^3\mathbf{x}}{\lambda^3}[\Phi^0(|\mathbf{x}|,m,\kappa) + \Phi(|\mathbf{x}|,m,\kappa)]^2, \tag{53a}$$

$$I_2^{\Omega i} = \int_{\Omega} \frac{d^3\mathbf{x}}{\lambda^3}[\Phi^0(|\mathbf{x}|,m,\kappa) + \Phi(|\mathbf{x}|,m,\kappa)]\Phi^i(\mathbf{x},m,\kappa), \tag{53b}$$

$$I_3^{\Omega ij} = -\int_{\Omega} \frac{d^3\mathbf{x}}{\lambda^3}\Phi^i(\mathbf{x},m,\kappa)\Phi^j(\mathbf{x},m,\kappa), \tag{53c}$$

where the functions used in the above equations are defined in Eq. (D4). Their explicit form reads as

$$\Phi(\mathbf{x},m,\kappa) = \frac{4\pi\kappa}{(2\pi)^{3/2}} \frac{K_2\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)}{\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)^2}, \tag{54}$$

$$\Phi^0(\mathbf{x},m,\kappa) = \frac{4\pi}{(2\pi)^{3/2}}\left[\kappa^2\frac{K_3\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)}{\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)^3} - \frac{K_2\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)}{\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)^2}\right], \tag{55}$$

and

$$\Phi^j(\mathbf{x},m,\kappa) = \frac{4\pi\kappa i}{(2\pi)^{3/2}}\frac{x^j}{\lambda}\frac{K_3\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)}{\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda^2}}\right)^3}, \tag{56}$$

where K_n are modified Bessel functions of the second kind.

Now, as an example we consider the case $\kappa = 1$. We also assume for simplicity that both particles are localized inside the regions of the shape shown in Fig. 4. In this case, we have

$$I_1^{\mathrm{A}} = 2\pi(1-\cos\alpha)C_1(r_1,r_2), \tag{57a}$$

$$I_2^{\mathrm{A}j} = i\pi\sin^2\alpha\; n_{\mathrm{A}}^j\, C_2(r_1,r_2), \tag{57b}$$

$$I_3^{\mathrm{A}ij} = -\left[\frac{\pi}{3}(1-\cos\alpha)^2(2+\cos\alpha)\delta^{ij} + \pi\cos\alpha\,\sin^2\alpha\; n_{\mathrm{A}}^i n_{\mathrm{A}}^j\right]C_3(r_1,r_2), \tag{57c}$$

where

$$C_k(r_1,r_2) = \frac{2}{\pi}\int_{r_1}^{r_2}\frac{d|\mathbf{x}|}{\lambda}\frac{|\mathbf{x}|^{k+1}}{\lambda^{k+1}}\frac{\left[K_3\left(\sqrt{1+\frac{\mathbf{x}^2}{\lambda^2}}\right)\right]^2}{\left(1+\frac{\mathbf{x}^2}{\lambda^2}\right)^3}, \tag{58}$$

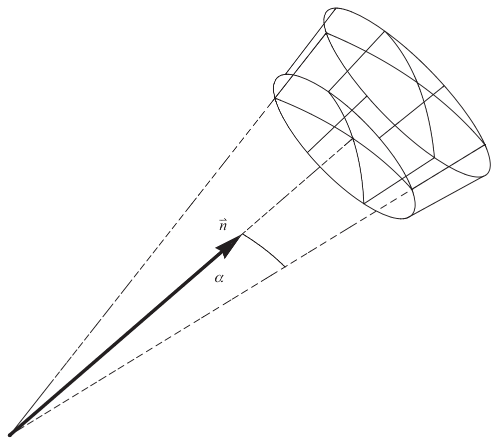

FIG. 4. Localization regions assumed to derive Eqs. (57). We assume that Alice and Bob localize particles inside regions of the same shape. Positions of localization regions are determined by the vectors $\mathbf{n}_A$ and $\mathbf{n}_B$, respectively. Both regions have the same angular width 2α.

for $k = 1,2,3$, and unit vector $\mathbf{n}_A = (\cos\varphi_A \sin\theta_A, \sin\varphi_A \sin\theta_A, \cos\theta_A)^T$ determines the center of the region A. Inserting integrals (57) into Eqs. (36) and (37) we get

$$\begin{aligned}
&\langle\varphi|\hat{\Pi}_A(\mathbf{a}\cdot\hat{\mathbf{S}})\otimes(\mathbf{b}\cdot\hat{\mathbf{S}})\hat{\Pi}_B|\varphi\rangle \\
&\quad= -\frac{\pi^2(1-\cos\alpha)^2}{16}\bigg\{(\mathbf{a}\cdot\mathbf{b})\bigg[2C_1^2 + (1+\cos\alpha)^2 \\
&\qquad\times(\mathbf{n}_A\cdot\mathbf{n}_B)C_2^2 + \frac{2}{9}(1+\cos\alpha+\cos^2\alpha)^2C_3^2\bigg] \\
&\qquad+(1+\cos\alpha)^2(\mathbf{a}\cdot\mathbf{n}_A)(\mathbf{b}\cdot\mathbf{n}_B) \\
&\qquad\times\left[C_2^2 + 2\cos^2\alpha\,(\mathbf{n}_A\cdot\mathbf{n}_B)C_3^2\right] \\
&\qquad-(1+\cos\alpha)^2(\mathbf{a}\cdot\mathbf{n}_B)(\mathbf{b}\cdot\mathbf{n}_A)C_2^2 \\
&\qquad-\frac{2}{3}\cos\alpha(1+\cos\alpha)(1+\cos\alpha+\cos^2\alpha) \\
&\qquad\times[(\mathbf{a}\cdot\mathbf{n}_A)(\mathbf{b}\cdot\mathbf{n}_A)+(\mathbf{a}\cdot\mathbf{n}_B)(\mathbf{b}\cdot\mathbf{n}_B)]C_3^2\bigg\}, \qquad (59)
\end{aligned}$$

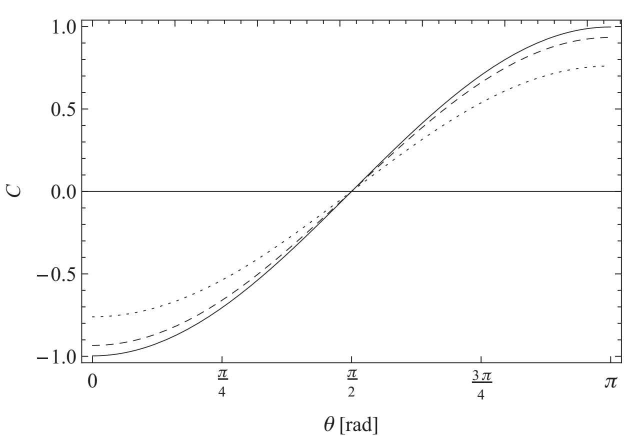

FIG. 5. Correlation function for arbitrary particle momenta [Eq. (62)] versus the angle θ between spin projection axes for different angular span of the detectors: $\alpha = \pi/60$ rad (solid line), $\alpha = \pi/12$ rad (dashed line), and $\alpha = \pi/6$ rad (dotted line). We take $\mathbf{n}_A = [\sin(\phi/2), \cos(\phi/2), 0]$, $\mathbf{n}_B = [\sin(\phi/2), -\cos(\phi/2), 0]$, $\mathbf{a} = (0,0,1)$, $\mathbf{b} = [\cos(\phi/2)\sin\theta, \sin(\phi/2)\sin\theta, \cos\theta]$, $\phi = 3\pi/4$.

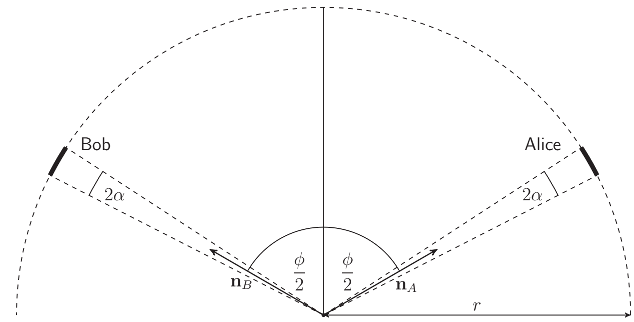

FIG. 6. Assumed configuration of EPR experiment. Vectors $\mathbf{n}_A$ and $\mathbf{n}_B$ lay in the xy plane.

and

$$\begin{aligned}
&\langle\varphi|\hat{\Pi}_A\otimes\hat{\Pi}_B|\varphi\rangle \\
&\quad= \frac{\pi^2(1-\cos\alpha)^2}{4}\left\{2C_1^2 + (1+\cos\alpha)^2(\mathbf{n}_A\cdot\mathbf{n}_B)C_2^2 + 2C_3^2\right\}.
\end{aligned} \qquad (60)$$

Now, in a realistic EPR–type experiment, observers should be separated by a macroscopic distance. Therefore, assuming that $\lambda \ll r_2 - r_1 \ll r_1$, we get

$$C_1^2(r_1,r_2) \ll C_3^2(r_1,r_2), \quad C_2^2(r_1,r_2) \ll C_3^2(r_1,r_2). \qquad (61)$$

Therefore, in this asymptotics the correlation function takes the following form:

$$\begin{aligned}
C_\varphi^{AB}(\mathbf{a},\mathbf{b}) = &-\tfrac{1}{9}(1+\cos\alpha+\cos^2\alpha)^2(\mathbf{a}\cdot\mathbf{b}) \\
&-(1+\cos\alpha)^2\cos^2\alpha\,(\mathbf{n}_A\cdot\mathbf{n}_B)(\mathbf{a}\cdot\mathbf{n}_A)(\mathbf{b}\cdot\mathbf{n}_B) \\
&+\tfrac{1}{3}\cos\alpha(1+\cos\alpha)(1+\cos\alpha+\cos^2\alpha) \\
&\times[(\mathbf{a}\cdot\mathbf{n}_A)(\mathbf{b}\cdot\mathbf{n}_A)+(\mathbf{a}\cdot\mathbf{n}_B)(\mathbf{b}\cdot\mathbf{n}_B)]. \qquad (62)
\end{aligned}$$

In Fig. 5 we have depicted the correlation function (62) in the configuration shown in Fig. 6 for $\mathbf{a} \perp \mathbf{n}_A$, $\mathbf{b} \perp \mathbf{n}_B$. For comparison, we have also shown the correlation function calculated in the same state [defined in Eq. (52)] but without localization.

VI. CONCLUSIONS

In conclusion, we have derived the correlation function in an arbitrary scalar state of two fermions assuming that spin projection measurement is associated with the localization of the particles [Eqs. (19)–(23)]. The most significant finding of our paper is that in a wide range of physically interesting situations, the influence of localization on the correlation function can be neglected. In particular, when EPR particles are in a definite momentum state, the correlation function with localization is exactly the same as the correlation function without localization. Moreover, the influence of localization can be neglected in the case when localization regions are macroscopic. As far as we know, such a situation took place in all correlation experiments with relativistic massive particles we are aware of. Up to date, there have been performed only three correlation experiments with massive relativistic protons: the Lamehi-Rachti–Mittig experiment [31] (CEN-Saclay), the Hamieh *et al.* experiment [32] (Kernfysisch

Versneller Instituut, Holland), and the Sakai *et al.* experiment [33] (RIKEN Accelerator Research Facility, Japan). There is also under preparation the experiment with relativistic Møller electrons (QUEST collaboration [34,35]). The present detectors (pixel arrays) used in such experiments are able to localize particles in regions of linear size $\sim 10^7 \lambda\!\!\!^{-}_e$ (the electron Compton wavelength). We have also considered examples when the localization modifies the correlation function.

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APPENDIX A: DERIVATION OF EQ. (28)

We are to calculate the following integral:

$$\Delta_\Omega(\mathbf{q}) = \frac{1}{(2\pi\hbar)^3}\int_\Omega d^3\mathbf{x}\,\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{x}}{mc\lambda\!\!\!^{-}}\right), \tag{A1}$$

where Ω is a cube with the center located at $\mathbf{R}$ and we have denoted $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. We make the following change of variables:

$$\mathbf{x} = \mathbf{y} + \mathbf{R}, \tag{A2}$$

and we get

$$\Delta_\Omega(\mathbf{q}) = \frac{\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{R}}{mc\lambda\!\!\!^{-}}\right)}{(2\pi\hbar)^3}\int_{\Omega_0} d^3\mathbf{y}\,\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{y}}{mc\lambda\!\!\!^{-}}\right), \tag{A3}$$

where the integration region Ω_0 is a cube with the center at (0,0,0) (Ω_0 is the cube Ω translated by the vector $-\mathbf{R}$). Now, let us denote by R the rotation around the origin and transforming Ω_0 into the cube with edges parallel to the coordinate system axes $\bar{\Omega}_0$:

$$R(\Omega_0) = \bar{\Omega}_0. \tag{A4}$$

Thus, changing the variables

$$\mathbf{z} = R\mathbf{y}, \tag{A5}$$

we have

$$\mathbf{q}\cdot\mathbf{y} = (R\mathbf{q})\cdot\mathbf{z}. \tag{A6}$$

Therefore,

$$\Delta_\Omega(\mathbf{q}) = \frac{\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{R}}{mc\lambda\!\!\!^{-}}\right)}{(2\pi\hbar)^3}\int_{\bar{\Omega}_0} d^3\mathbf{z}\,\exp\left[-i\frac{(R\mathbf{q})\cdot\mathbf{z}}{mc\lambda\!\!\!^{-}}\right], \tag{A7}$$

and, since $\bar{\Omega}_0$ has edges parallel to coordinate axes and the center at (0,0,0),

$$\Delta_\Omega(\mathbf{q}) = \frac{\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{R}}{mc\lambda\!\!\!^{-}}\right)}{(2\pi\hbar)^3}\prod_{j=1}^{3}\int_{-l/2}^{l/2} dz_j\,\exp\left[-i\frac{(R\mathbf{q})^j z_j}{mc\lambda\!\!\!^{-}}\right]. \tag{A8}$$

Now, performing simple integration, we arrive at Eq. (28).

APPENDIX B: DEFINITE MOMENTUM STATE

For the state with definite momenta [Eq. (39)], the numerator of the correlation function (19) is given by

$$\begin{aligned}\langle\varphi_{q_aq_b}|(\mathbf{a}\cdot\hat{\mathbf{S}}_\mathrm{A})(\mathbf{b}\cdot\hat{\mathbf{S}}_\mathrm{B})|\varphi_{q_aq_b}\rangle &= \frac{m^2q_a^0q_b^0}{(2\pi)^6}\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(q_a,q_b)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(q_a,q_b)\}\\ &\quad\times \mathrm{Vol(A)Vol(B)},\end{aligned} \tag{B1}$$

while the denominator has the form

$$\langle\varphi_{q_aq_b}|\hat{\Pi}_\mathrm{A}\otimes\hat{\Pi}_\mathrm{B}|\varphi_{q_aq_b}\rangle = \frac{4m^2q_a^0q_b^0}{(2\pi)^6}\mathrm{Tr}\{\mathcal{M}(q_a,q_b)\mathcal{M}^\dagger(q_a,q_b)\}\mathrm{Vol(A)Vol(B)}. \tag{B2}$$

By dividing the above formulas, we obtain

$$\mathcal{C}^{\mathrm{AB}}_{\varphi_{q_aq_b}}(\mathbf{a},\mathbf{b}) = \frac{\mathrm{Tr}\{(\mathbf{a}\cdot\boldsymbol{\sigma})\mathcal{M}(q_a,q_b)(\mathbf{b}\cdot\boldsymbol{\sigma}^T)\mathcal{M}^\dagger(q_a,q_b)\}}{\mathrm{Tr}\{\mathcal{M}(q_a,q_b)\mathcal{M}^\dagger(q_a,q_b)\}}. \tag{B3}$$

APPENDIX C: STATE WITH FIXED PARTICLE MOMENTA DIRECTIONS

In this case, integrals (35) are equal to

$$I_1^\mathrm{A} = I_1^{\mathrm{A},\mathbf{n}},\quad I_1^\mathrm{B} = I_1^{\mathrm{B},\mathbf{m}}, \tag{C1a}$$

$$I_2^{\mathrm{A}\,i} = n^i I_2^{\mathrm{A},\mathbf{n}},\quad I_2^{\mathrm{B}\,i} = m^i I_2^{\mathrm{B},\mathbf{m}}, \tag{C1b}$$

$$I_3^{\mathrm{A},ij} = n^in^j I_3^{\mathrm{A},\mathbf{n}},\quad I_3^{\mathrm{B}\,ij} = m^im^j I_3^{\mathrm{B},\mathbf{m}} \tag{C1c}$$

[compare Eq. (45)].

APPENDIX D: ARBITRARY MOMENTA

Using the following relations [36]

$$\int_0^\infty dt\, e^{-\alpha\sqrt{t^2+\beta^2}}\cos(\gamma t) = \frac{\alpha\beta}{\sqrt{\alpha^2+\gamma^2}}K_1(\beta\sqrt{\alpha^2+\gamma^2}) \tag{D1}$$

($\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta > 0$) and

$$xK_{n+1}(x) = nK_n(x) - xK_n'(x), \tag{D2}$$

where K_n is a modified Bessel function of the second kind, we obtain

$$\int\frac{d^3\mathbf{k}}{(mc)^3}e^{i\frac{\mathbf{k}}{mc}\cdot\frac{\mathbf{x}}{\lambda\!\!\!^{-}}}e^{-\kappa\frac{k^0}{mc}} = 4\pi\kappa\frac{K_2\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda\!\!\!^{-2}}}\right)}{\left(\sqrt{\kappa^2+\frac{\mathbf{x}^2}{\lambda\!\!\!^{-2}}}\right)^2}. \tag{D3}$$

For further convenience, let us define the following functions:

$$\Phi(\mathbf{x},m,\kappa) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3\mathbf{k}}{(mc)^3}e^{i\frac{\mathbf{k}}{mc}\cdot\frac{\mathbf{x}}{\lambda\!\!\!^{-}}}e^{-\kappa\frac{k^0}{mc}}, \tag{D4a}$$

$$\Phi^0(\mathbf{x},m,\kappa) = -\frac{\partial}{\partial\kappa}\Phi(\mathbf{x},m,\kappa), \tag{D4b}$$

$$\Phi^j(\mathbf{x},m,\kappa) = -i\lambda\!\!\!^{-}\frac{\partial}{\partial x^j}\Phi(\mathbf{x},m,\kappa). \tag{D4c}$$

Notice that

$$\Phi(\mathbf{x},m,\kappa) = \Phi(|\mathbf{x}|,m,\kappa), \tag{D5}$$

$$\Phi^0(\mathbf{x},m,\kappa) = \Phi^0(|\mathbf{x}|,m,\kappa), \tag{D6}$$

and

$$\boldsymbol{\Phi}(\mathbf{x},m,\kappa) = \frac{i\mathbf{x}}{\lambda\kappa}\left[\Phi^0(|\mathbf{x}|,m,\kappa) + \frac{1}{\kappa}\Phi(|\mathbf{x}|,m,\kappa)\right]. \tag{D7}$$

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