Topological phase structure of entangled qudits

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We discuss the appearance of fractional topological phases on cyclic evolutions of entangled qudits. The original result reported by Oxman and Khoury [Phys. Rev. Lett. **106**, 240503 (2011)] is detailed and extended to qudits of different dimensions. The topological nature of the phase evolution and its restriction to fractional values are related to both the structure of the projective space of states and entanglement. For maximally entangled states of qudits with the same Hilbert-space dimension, the fractional geometric phases are the only ones attainable under local SU(d) operations, an effect that can be experimentally observed through conditional interference.

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I. INTRODUCTION

Geometrical phases are a remarkable property of quantum phase evolutions, related to holonomies in the parameter space characterizing the quantum state vectors. The standard example is the phase acquired by a spin-$1/2$ particle undergoing a cyclic evolution described by a closed path in the Bloch sphere. The role of holonomies in the quantum phase evolution was pointed out by Berry [1] in connection with adiabatic transformations driven by a slowly varying time-dependent Hamiltonian. An analogous effect was studied in a seminal work by Pancharatnam [2] in a more elementary system, i.e., the polarization transformations in classical wave optics. A beautiful generalization of the Pancharatnam results to paraxial mode transformations was theoretically proposed in Refs. [3,4] and experimentally demonstrated in Ref. [5] where a Poincaré sphere representation was used for first-order paraxial modes. This representation was also used to discuss the geometric phase conjugation in an optical parametric oscillator [6]. More recently, the geometric representation of higher-order paraxial modes have been discussed in Refs. [7,8]. Another fundamental contribution to the theory of geometric phases was given by Mukunda and Simon in Refs. [9,10], where the kinematic aspects of the quantum state evolution were investigated and the geometric phase generalized to nonadiabatic evolutions. Since these seminal contributions, numerous works have been devoted to both fundamental and applied aspects of geometric phases.

In quantum information science, geometric phases were conceived as a robust means for implementing unitary gates that are useful for quantum computation [11,12]. The role of entanglement in the phase evolution of two-qubit systems was investigated in Refs. [13,14], and the topological nature of the corresponding geometric phases was investigated both theoretically [15–17] and experimentally in the context of spin-orbit transformations on a paraxial laser beam [18] and in nuclear magnetic resonance [19]. In a recent work, we investigated the crucial role played by the dimension of the Hilbert space on the topological phases acquired by entangled qudits [20,21]. The appearance of fractional phases is a remarkable property of two-qudit systems, also shared by multiple qubits [22,23]. Multidimensional entangled states can be realized on qudits encoded on the transverse position of quantum correlated photon pairs generated by spontaneous parametric down conversion [24–28]. Fractional phases were originally investigated in quantum Hall systems in connection with different homotopy classes in the configuration space of anyons. This topological structure has been conjectured to be a possible resource for fault-tolerant quantum computation [29]. These potential applications of geometric phases in quantum information science motivated a number of articles devoted to their implementation in quantum optical systems and their behavior under the influence of different kinds of reservoir [30–37]. Decoherence is recognized as the main difficulty for quantum information protocols in realistic physical systems. In this sense, quantum gates based on geometric phases are supposed to be a powerful tool. The extension of geometric phases to mixed quantum states in condensed-matter physics has been considered in connection with topological insulators and superconductors [38,39].

In the present work, we study the geometric phase acquired by entangled qudits under local unitary evolutions. The fractional phases predicted in Ref. [20] are developed in detail and generalized to qudits of different dimensions. A general expression is derived for the two-qudit geometric phase in terms of entanglement and the dimensions of their Hilbert spaces. We also discuss the holonomies of the phase evolution in terms of the parameters used to define the local SU(d) transformations applied to each qudit. The paper is organized as follows: in Sec. II, we discuss the role played by the purity of a single-qudit state in the geometric phase. Since, for pure states, two-qudit entanglement can be quantified by the purity of the partial density matrix of each qudit, the results of Sec. II are used in Sec. III to establish the role of entanglement in the geometric phase acquired by a two-qudit state under local unitary transformations. In Sec. IV, we present some numerical examples that illustrate the fractional phase values expected and the role played by entanglement. Finally, in Sec. V, we summarize our results and briefly discuss some future perspectives.

II. TOPOLOGICAL PHASES ON SINGLE QUDITS

Initially, we shall examine the properties of the geometric phases on unitary evolutions of single qudits and the role of the quantum state purity in the geometric phase. Our conclusions will be useful since two-qudit entanglement is frequently quantified through the purity of the partial density matrix describing one of the qudits. Therefore, we start by considering a single qudit initially prepared in a quantum state described by a general density matrix ρ_0. As this matrix must be Hermitian, it can be written in terms of the identity matrix and a basis $\{T_\alpha\}$ ($\alpha = 1,\ldots,d^2-1$) for the Hermitian traceless $d \times d$ matrices, that is,

$$\rho_0 = \frac{\mathbb{1}}{d} + q\sqrt{\frac{d-1}{d}}\,\hat{\mathbf{q}}\cdot\mathbf{T}. \tag{1}$$

The basis $\{T_\alpha\}$ is normalized according to $\mathrm{Tr}[T_\alpha T_\beta] = \delta_{\alpha\beta}$ and can serve as a set of generators of SU(d). We shall term $\mathbf{q} = q\,\hat{\mathbf{q}} \in \mathbb{R}^{d^2-1}$ as the *purity vector* (for qubits, it is the well-known Bloch vector), since its absolute value is related to the purity of the qudit state: $\mathrm{Tr}[\rho_0^2] = q^2 + (1-q^2)/d$, with $0 \leqslant q \leqslant 1$. From the kinematic approach by Mukunda and Simon [9,10], the geometric phase acquired by a time-evolving pure state $|\psi(t)\rangle$ is given by

$$\begin{aligned}\phi_g &= \arg\langle\psi(0)|\psi(t)\rangle + i\int dt\,\langle\psi(t)|\dot{\psi}(t)\rangle\\ &= \arg\{\mathrm{Tr}[\rho_0\,U(t)]\} + i\int dt\,\mathrm{Tr}[\rho_0\,U^\dagger\dot{U}],\end{aligned} \tag{2}$$

where $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$ is the density matrix of the initial state evolving under the action of the unitary operator $U(t)$. The second equality allows for a natural extension of the geometric phase for mixed states by taking the general density matrix (1). Also, it is important to rule out from the geometric phase any explicit phase evolution contained in $U(t)$. Let $U(t) = e^{i\phi(t)}\bar{U}(t)$, where $\bar{U}(t) \in \mathrm{SU}(d)$ for all t, with initial conditions $\phi(0) = 0$ and $\bar{U}(0) = \mathbb{1}$. It is straightforward to show that the explicit phase $\phi(t)$ does not contribute to the geometric phase, which is then given in terms of the SU(d) sector only:

$$\phi_g = \bar{\phi}_{tot} + i\int dt\,\mathrm{Tr}[\rho_0\bar{U}^\dagger\dot{\bar{U}}], \tag{3}$$

where we defined

$$\bar{\phi}_{tot} \equiv \arg\{\mathrm{Tr}[\rho_0\bar{U}(t)]\}. \tag{4}$$

Now it is useful to recall that for a general invertible matrix $\mathbb{A}$, we have [41]

$$\frac{d(\det\mathbb{A})}{dt} = (\det\mathbb{A})\mathrm{Tr}\left[\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\right]. \tag{5}$$

Since the evolution $\bar{U}(t)$ is closed in SU(d), we readily deduce that $\mathrm{Tr}[\bar{U}^\dagger\dot{\bar{U}}] = 0$. In addition, $d(\bar{U}^\dagger\bar{U})/dt = 0$ implies that $\bar{U}^\dagger\dot{\bar{U}}$ is anti-Hermitian so that it can be written as a combination of the T_α's with purely imaginary coefficients. Thus, we can introduce a useful *velocity* vector $\mathbf{u} \in \mathbb{R}^{d^2-1}$ such that $\bar{U}^\dagger\dot{\bar{U}} = i\mathbf{u}\cdot\mathbf{T}$. The geometric phase can be expressed in terms of the purity and velocity vectors as

$$\phi_g = \bar{\phi}_{tot} - q\sqrt{\frac{d-1}{d}}\int\hat{\mathbf{q}}\cdot\mathbf{dx}, \tag{6}$$

where $\mathbf{dx} \equiv \mathbf{u}dt$ is a connection.

For a pure state, an evolution over a time interval T is considered to be cyclic when it takes the system from a given initial state to a physically equivalent final state, i.e., when $\langle\psi(0)|\psi(T)\rangle = e^{i\phi_{tot}(T)}$. This condition can be generalized for mixed states as $\mathrm{Tr}[\rho_0 U(T)] = e^{i\phi_{tot}(T)}$. Now, let us inspect carefully this condition over totally mixed states: $q = 0$. In this case, it reduces to $\mathrm{Tr}[U(T)] = de^{i\phi_{tot}(T)}$, which implies $U(T) = e^{i\phi_{tot}(T)}\mathbb{1}$, and hence $\bar{U}(T) = e^{i\bar{\phi}_{tot}(T)}\mathbb{1}$. Since $\bar{U} \in \mathrm{SU}(d)$, $\det\bar{U} = e^{id\bar{\phi}_{tot}(T)} = 1$, so that $\bar{\phi}_{tot}(T) = 2n\pi/d$ ($n \in \mathbb{Z}$). For qubits, this corresponds to the two possible values 0 or π.

Therefore, expression (6) for a completely mixed state reduces to

$$\phi_g = \bar{\phi}_{tot} = \frac{2n\pi}{d}\quad(n \in \mathbb{Z}). \tag{7}$$

In principle, this result is of little physical relevance, since no interference can be measured on completely incoherent states. However, we can anticipate its important role on entangled states. Indeed, the partial trace of a maximally entangled pure state of a bipartite system produces completely mixed density matrices. In this case, we shall see that for cyclic evolutions, driven by local unitary operations, only the fractional phases in Eq. (7) can arise. However, they can now be measured through conditional interference, as long as the overall bipartite state is coherent. We shall put these arguments on a more formal ground in Sec. III.

A. The Cartan sector

An interesting refinement of the geometric phase structure is obtained by separating the basis $\{T_\alpha\}$ into diagonal and nondiagonal elements. The diagonal elements form the so-called Cartan subalgebra of SU(d). This separation in the density matrix and the evolution operator will make it possible to isolate the nonholonomic contribution to the geometric phase.

The first $d-1$ generators $T_1,\ldots,T_{d-1}$ can be taken as the diagonal elements [the only element for SU(2) is σ_z], and they will also be named as $\{H_\beta\}$, $\beta = 1,\ldots,d-1$. Then, we write $\{T_\alpha\} = \{H_\beta\} \cup \{P_\gamma\}$, where $\{P_\gamma\}$, $\gamma = 1,\ldots,d^2-d$, represents the remaining d^2-d nondiagonal generators. As a convention, dot products involving $\mathbf{H}$ ($\mathbf{P}$) will be used to represent Lie algebra elements restricted to the diagonal (off-diagonal) sector; that is, dot products between the full set of generators $\mathbf{T}$ and vectors having the last d^2-d (the first $d-1$) components vanishing.

Choosing a Hilbert-space basis that renders the initial density matrix diagonal, we may write

$$\begin{aligned}\rho_0 &= \frac{\mathbb{1}}{d} + q\sqrt{\frac{d-1}{d}}\,\hat{\mathbf{q}}\cdot\mathbf{H}\\ &= \frac{\mathbb{1}}{d} + q\sqrt{\frac{d-1}{d}}\,\mathrm{diag}[x_0\ldots x_{d-1}],\end{aligned} \tag{8}$$

where we defined $x_n \equiv \langle n|\,\hat{\mathbf{q}}\cdot\mathbf{H}\,|n\rangle$, with the properties $\sum_n x_n = 0$ and $\sum_n x_n^2 = 1$. In addition, we can use the factorization

$$\bar{U} = \bar{V}\,\exp(i\mathbf{h}\cdot\mathbf{H}), \tag{9}$$

where the parameters $\mathbf{h}$ map an $\mathbb{R}^{d-1}$ subspace and the SU(d) matrix $\bar{V}$ is such that $[\bar{V},H_\beta] \neq 0$ for at least one value of β, unless $\bar{V} = \mathbb{1}$.

In more formal terms, this corresponds to a coset factorization of the SU(d) group [42], so that $\bar{V} \in \mathrm{SU}(d)/\mathrm{U}(1)^{d-1}$. The latter manifold can in turn be written as a tensor product of different coset spaces. For SU(2), there is only one factor, SU(2)/U(1), which is topologically equivalent to the two-sphere S^2. Equivalently, the $\bar{V}$ factor can be defined by the following requirement: If

$$\bar{V} H_\beta \bar{V}^{-1} = H_\beta, \tag{10}$$

for every diagonal generator H_β, then necessarily $\bar{V} = \mathbb{1}$.

Using Eq. (9), we have

$$\bar{U}^\dagger \dot{\bar{U}} = e^{-i\mathbf{h}\cdot\mathbf{H}}\, \bar{V}^\dagger \dot{\bar{V}}\, e^{i\mathbf{h}\cdot\mathbf{H}} + i\dot{\mathbf{h}}\cdot\mathbf{H}, \tag{11}$$

where the velocity vector associated with the $\bar{V}$ sector can be separated into two orthogonal terms $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$, related to the Cartan subalgebra and the nondiagonal generators, respectively. They are defined according to

$$\bar{V}^\dagger \dot{\bar{V}} = i\mathbf{v}_\perp \cdot \mathbf{P} + i\mathbf{v}_\parallel \cdot \mathbf{H}. \tag{12}$$

Now, using the Baker-Campbell-Hausdorff formula and the fact that $[H_\alpha, P_\beta] \propto P_\gamma$, it is easy to show that the transformation $e^{-i\mathbf{h}\cdot\mathbf{H}}\bar{V}^\dagger\dot{\bar{V}}e^{i\mathbf{h}\cdot\mathbf{H}}$ leaves $\mathbf{v}_\parallel$ unchanged and makes $\mathbf{v}_\perp \to \mathbf{v}'_\perp$, so that

$$e^{-i\mathbf{h}\cdot\mathbf{H}}\bar{V}^\dagger\dot{\bar{V}}e^{i\mathbf{h}\cdot\mathbf{H}} = i\mathbf{v}'_\perp \cdot \mathbf{P} + i\mathbf{v}_\parallel \cdot \mathbf{H}. \tag{13}$$

Moreover, the orthonormality condition for the generators leads to $\mathrm{Tr}[(\bar{V}^\dagger\dot{\bar{V}})^2] = |\mathbf{v}_\perp|^2 + |\mathbf{v}_\parallel|^2$ and $|\mathbf{v}'_\perp|^2 = |\mathbf{v}_\perp|^2$. Therefore, $\mathbf{v}'_\perp$ corresponds to a rotation of $\mathbf{v}_\perp$ in a subspace orthogonal to the Cartan subspace where both $\hat{\mathbf{q}}$ and $\mathbf{v}_\parallel$ lie ($\mathbf{v}'_\perp \cdot \mathbf{v}_\parallel = \mathbf{v}'_\perp \cdot \hat{\mathbf{q}} = 0$). Finally, from Eq. (13), we get

$$\bar{U}^\dagger\dot{\bar{U}} = i\mathbf{v}'_\perp \cdot \mathbf{P} + i(\mathbf{v}_\parallel + \dot{\mathbf{h}}) \cdot \mathbf{H}, \tag{14}$$

which corresponds to the following decomposition of the velocity vector:

$$\mathbf{u} = \mathbf{v}'_\perp + \mathbf{v}_\parallel + \dot{\mathbf{h}}. \tag{15}$$

Since a diagonal representation has been assumed for ρ_0, only $\mathbf{v}_\parallel$ and $\dot{\mathbf{h}}$ will contribute to the integral term in the geometric phase. Noting that

$$\mathrm{Tr}[\rho_0 \bar{U}^\dagger \dot{\bar{U}}] = iq\sqrt{\frac{d-1}{d}}\hat{\mathbf{q}}\cdot(\mathbf{v}_\parallel + \dot{\mathbf{h}}), \tag{16}$$

and replacing in Eq. (6), we get

$$\phi_g = \bar{\phi}_{tot} - q\sqrt{\frac{d-1}{d}}\left[\hat{\mathbf{q}}\cdot\mathbf{h}(t) + \int \hat{\mathbf{q}}\cdot \mathbf{dx}_\parallel\right], \tag{17}$$

where $\mathbf{dx}_\parallel \equiv \mathbf{v}_\parallel dt$ and $\mathbf{h}(0) = \mathbf{0}$. The integral term represents a path-dependent (nonholonomic) contribution, built along the path followed on $\mathrm{SU}(d)/\mathrm{U}(1)^{d-1}$. When cyclic evolutions are considered, this term generalizes to SU(d) the usual solid angle contribution for paths on the Bloch sphere SU(2)/U(1), obtained for SU(2). Then, we shall define

$$\Phi = \oint \hat{\mathbf{q}}\cdot\mathbf{dx}_\parallel. \tag{18}$$

Let us denote as *partially cyclic* those evolutions that, at a given time $\bar{t}$, close a path in the $\mathrm{SU}(d)/\mathrm{U}(1)^{d-1}$ sector. With regard to the total phase, as $\bar{V}(\bar{t}) \in \mathrm{SU}(d)$ and $\bar{V}(0) = \mathbb{1}$, this would mean that $\bar{V}(\bar{t})$ must be the identity matrix times the exponential of a fractional phase. However, such $\bar{V}(\bar{t})$ would satisfy the condition (10) and, as a consequence, it must necessarily be the identity matrix. Therefore, for a partially cyclic evolution,

$$\bar{U}(\bar{t}) = \exp[i\mathbf{h}(\bar{t})\cdot\mathbf{H}], \tag{19}$$

and the geometric phase is given by

$$\phi_g = \bar{\phi}_{tot} - q\sqrt{\frac{d-1}{d}}[\hat{\mathbf{q}}\cdot\mathbf{h}(\bar{t}) + \Phi], \tag{20}$$

$$\bar{\phi}_{tot} \equiv \arg\left\{\mathrm{Tr}\left[\left(\frac{\mathbb{1}}{d} + q\sqrt{\frac{d-1}{d}}\hat{\mathbf{q}}\cdot\mathbf{H}\right)e^{i\,\mathbf{h}(\bar{t})\cdot\mathbf{H}}\right]\right\}. \tag{21}$$

For qubits, the Cartan sector reduces to a single parameter. The identification of the Cartan sector will be particularly useful to demonstrate the fractional phases for dimensions $d > 2$, since the number of parameters in the nondiagonal sector scales as d^2, while in the Cartan sector it scales as d. We will next build a useful representation for the Cartan sector that simplifies its parametrization and will be particularly useful for experimental proposals.

Let us now study how the fractional phases, generated in cyclic evolutions, are built. To simplify the discussion, consider evolutions restricted to the Cartan sector,

$$\bar{U}(t) = e^{i\mathbf{h}(t)\cdot\mathbf{H}} = \mathrm{diag}[e^{i\chi_0}, \ldots, e^{i\chi_{d-1}}], \tag{22}$$

with $\chi_n(t) \equiv \langle n|\mathbf{h}\cdot\mathbf{H}|n\rangle$ and $\sum_n \chi_n = 0$. For the initial density matrix given by Eq. (8), the geometric phase can be easily computed,

$$\phi_g = \bar{\phi}_{tot} - q\sqrt{\frac{d-1}{d}}\sum_{n=0}^{d-1} x_n\chi_n, \tag{23}$$

where the nontrivial total phase is

$$\bar{\phi}_{tot} = \arg\left\{\sum_{n=0}^{d-1}\left(\frac{1}{d} + q\sqrt{\frac{d-1}{d}}x_n\right)e^{i\chi_n}\right\}. \tag{24}$$

Now, for completely mixed states ($q = 0$), a quite subtle feature of the Cartan sector comes into play. The diagonal elements in $\bar{U}(t)$ are phasors in the complex plane. The state evolution will be cyclic when these phasors line up, making $\bar{U}$ proportional to the identity matrix. This will happen when $\Delta\chi_n \equiv \chi_0 - \chi_n = 2l_n\pi$, with $l_n \in \mathbb{Z}$. However, this alignment can only occur at fractional phase values. In order to see this, let us sum up all phase differences and make $\sum_n \Delta\chi_n = 2\pi L$, with $L \equiv \sum_n l_n$. On the other hand,

$$\sum_{n=0}^{d-1}\Delta\chi_n = \sum_{n=0}^{d-1}\chi_0 - \sum_{n=0}^{d-1}\chi_n = d\chi_0, \tag{25}$$

which brings us to the fractional solutions $\chi_0 = 2\pi\, L/d$ and $\chi_n = 2\pi\, L/d - 2\pi\, l_n$, as expected. Then, the nontrivial total phase is

$$\bar{\phi}_{tot} = \frac{2\pi L}{d}, \tag{26}$$

and this is the only contribution to the geometric phase acquired by completely mixed states.

B. Qubits

As an example, let us apply the ideas above to the simplest case of a single qubit. The normalized SU(2) generators can be written in terms of the Pauli matrices; the nondiagonal sector is composed of $P_1 = \sigma_x/\sqrt{2}$ and $P_2 = \sigma_y/\sqrt{2}$, while the Cartan sector corresponds to $H_1 = \sigma_z/\sqrt{2}$. Let us use a basis such that the initial density matrix is diagonal,

$$\rho_0 = \begin{bmatrix} \frac{1+q}{2} & 0 \\ 0 & \frac{1-q}{2} \end{bmatrix} = \frac{\mathbb{1} + q\,\sigma_z}{2}, \tag{27}$$

where $\{|0\rangle,|1\rangle\}$ are the eigenvectors of σ_z with eigenvalues $\{+1,-1\}$, respectively, and $0 \leqslant q \leqslant 1$. This initial state corresponds to the purity vector

$$\mathbf{q} = (q,0,0). \tag{28}$$

Then, suppose this qubit evolves under the action of a general SU(2) matrix,

$$\bar{U}(\theta,\varphi,\chi) = \bar{V}(\theta,\varphi)e^{i\chi\sigma_z},$$
$$\bar{V}(\theta,\varphi) = \exp(i\theta\hat{\mathbf{p}}\cdot\mathbf{P}) = \begin{bmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2}e^{-i\varphi} \\ i\sin\frac{\theta}{2}e^{i\varphi} & \cos\frac{\theta}{2} \end{bmatrix}, \tag{29}$$

where $\hat{\mathbf{p}} = (0,\cos\varphi,\sin\varphi)$, $\mathbf{h} = \sqrt{2}(\chi,0,0)$, and $\varphi(t)$, $\theta(t)$, and $\chi(t)$ are time-dependent real parameters with initial conditions $\varphi(0) = \theta(0) = \chi(0) = 0$. Here, $\varphi(t)$ and $\theta(t)$ can be identified with the angular coordinates on the Bloch sphere representation of a pure state. In fact, they are precisely the coordinates of the evolving state when it is initially prepared in $|0\rangle$ ($q = 1$). Therefore, we identify the state evolution as an explicit phase evolution $\chi(t)$ [not to be confused with the explicit phase $\phi(t)$ discarded above, since $\bar{U}$ is already an SU(2) matrix] and a path $[\theta(t),\varphi(t)]$ on the Bloch sphere.

The velocity vector $\mathbf{u} \in \mathbb{R}^3$ can be computed from the decomposition of $\bar{U}^\dagger\dot{\bar{U}}$ in terms of the SU(2) generators (Pauli matrices). It is more elegant to do it in two steps. Initially, we note that

$$\bar{U}^\dagger\dot{\bar{U}} = e^{-i\chi\sigma_z}\bar{V}^\dagger\dot{\bar{V}}e^{i\chi\sigma_z} + i\dot{\chi}\sigma_z, \tag{30}$$

and write $\bar{V}^\dagger\dot{\bar{V}} = i\mathbf{v}\cdot\mathbf{T}$, where $\mathbf{v}$ is the velocity vector along the path followed on the Bloch sphere. From Eq. (29), we obtain $\mathbf{v} = (v_h, v_{p1}, v_{p2})$, where

$$v_h = \sqrt{2}\dot{\varphi}\sin^2\left(\frac{\theta}{2}\right),$$
$$v_{p1} = \frac{1}{\sqrt{2}}(\dot{\theta}\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi), \tag{31}$$
$$v_{p2} = \frac{1}{\sqrt{2}}(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi).$$

The first term in Eq. (30) amounts to a rotation of $\mathbf{v}$ by an angle 2χ, generated by σ_z, so that

$$\mathbf{u} = \mathbf{v}' + \sqrt{2}(\dot{\chi},0,0), \tag{32}$$

where $\mathbf{v}' = R_z(2\chi)\mathbf{v}$. This rotation leaves the Cartan component of $\mathbf{v}$ unchanged so that, in this SU(2) parametrization, the connection becomes

$$\hat{\mathbf{q}}\cdot\mathbf{dx} = \sqrt{2}[d\chi + \sin^2(\theta/2)d\varphi]. \tag{33}$$

The first term is the holonomic contribution, while the second one (nonholonomic) is built along the path followed on the Bloch sphere. For a closed path, it gives

$$\Phi = \sqrt{2}\oint \sin^2\frac{\theta}{2}d\varphi = \frac{1}{\sqrt{2}}\iint \sin\theta d\theta d\varphi = \frac{\Omega}{\sqrt{2}}, \tag{34}$$

where the second equality results from Green's theorem, giving the usual solid angle contribution Ω, enclosed on the Bloch sphere.

Now, let us inspect these contributions for partially cyclic evolutions, that is, when $[\theta(t),\varphi(t)]$ follows a closed path on the Bloch sphere over a time interval $\bar{t}$, but $\chi(t)$ does not complete a full cycle. In this case, $\bar{U}(\bar{t}) = e^{i\chi(\bar{t})\sigma_z}$ and

$$\bar{\phi}_{tot} = \arg\{\cos\chi + iq\sin\chi\}, \tag{35}$$

which gives

$$\phi_g = \arctan(q\tan\chi) - q\left(\chi + \frac{\Omega}{2}\right). \tag{36}$$

For an initial pure state $|0\rangle$ ($q = 1$), one obtains the usual solid angle expression $\phi_g = -\Omega/2$. For completely mixed states ($q = 0$), the integral terms vanish and the only possible geometric phases are 0 or π .

C. Qutrits

There are eight generators of SU(3), usually represented in the form of Gell-Mann matrices. The Cartan sector is restricted to two diagonal matrices and the other six elements of the algebra are nondiagonal. Therefore, the SU(3) transformations are determined by six parameters in the nondiagonal sector and two in the Cartan sector. In order to focus on the fractional phases and the role played by the state purity, we shall restrict our study to transformations restricted to the Cartan sector. The nondiagonal parameters only bring geometric complexity, without much additional insight into the fractional phase structure.

First, we assume the qutrit basis is set to render diagonal the initial density matrix ρ_0. In terms of the two diagonal Gell-Mann matrices (apart from a slightly different normalization), we can parametrize the unit purity vector as $\hat{\mathbf{q}} = (\cos\theta, \sin\theta, 0, \ldots, 0)$. In this parametrization, the density matrix for the initial state becomes

$$\rho_0 = \frac{\mathbb{1}}{3} + q\sqrt{\frac{2}{3}}(\cos\theta\, H_1 + \sin\theta\, H_2),$$
$$= \frac{\mathbb{1}}{3} + \frac{2q}{3}\begin{bmatrix} \cos\left(\theta + \frac{2\pi}{3}\right) & 0 & 0 \\ 0 & \cos\left(\theta + \frac{4\pi}{3}\right) & 0 \\ 0 & 0 & \cos\theta \end{bmatrix}, \tag{37}$$

where

$$H_1 = -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad H_2 = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{38}$$

Then, the diagonal parameters are

$$
\begin{aligned}
x_0 &= \sqrt{\frac{2}{3}}\cos\left(\theta + \frac{2\pi}{3}\right),\\
x_1 &= \sqrt{\frac{2}{3}}\cos\left(\theta + \frac{4\pi}{3}\right),\\
x_2 &= \sqrt{\frac{2}{3}}\cos\theta.
\end{aligned} \tag{39}
$$

The density matrix eigenvalues belong to the interval [0,1], which limits the possible values of θ . For pure states ($q = 1$), only the discrete values 0, $2\pi/3$, and $4\pi/3$ are allowed. For $q < 1/2$, any value of θ gives a meaningful density matrix. Moreover, cyclic permutations of the basis vectors amount to transformations $\theta \to \theta + 2n\pi/3$ ($n \in \mathbb{Z}$), and the noncyclic permutations can be achieved by the same transformations followed by $\theta \to -\theta$. Therefore, without loss of generality, we can restrict our analysis to the domain $-\pi/3 \leqslant \theta \leqslant \pi/3$. Other θ values simply amount to a permutation of the diagonal elements in ρ_0. Nevertheless, in order to ensure that all diagonal elements belong to the allowed interval [0,1], we need to impose the restriction $\cos(\theta + 2n\pi/3) \geqslant -1/2q$. Therefore, we arrive at $-\theta_0 \leqslant \theta \leqslant \theta_0$, where

$$
\theta_0(q) = \begin{cases} \cos^{-1}(-1/2q) - 2\pi/3, & q \geqslant 1/2 \\ \pi/3, & q \leqslant 1/2. \end{cases} \tag{40}
$$

For pure states ($q = 1$), we are left with $\theta = 0$.

We now assume a diagonal SU(3) transformation,

$$
\bar{U}(t) = \begin{bmatrix} e^{i\chi_0(t)} & 0 & 0 \\ 0 & e^{i\chi_1(t)} & 0 \\ 0 & 0 & e^{i\chi_2(t)} \end{bmatrix}, \tag{41}
$$

where $\chi_2 = -(\chi_0 + \chi_1)$. In terms of the parameters characterizing $\bar{U}$ and ρ_0, the geometric phase becomes

$$
\phi_g = \bar{\phi}_{tot} - \frac{2q}{3}\left[\chi_0\cos\left(\theta + \frac{2\pi}{3}\right) + \chi_1\cos\left(\theta + \frac{4\pi}{3}\right) + \chi_2\cos\theta\right], \tag{42}
$$

where

$$
\begin{aligned}
\bar{\phi}_{tot} = \arg\Big\{ & e^{i\chi_0}\left[\frac{1}{3} + \frac{2q}{3}\cos\left(\theta + \frac{2\pi}{3}\right)\right] \\
& + e^{i\chi_1}\left[\frac{1}{3} + \frac{2q}{3}\cos\left(\theta + \frac{4\pi}{3}\right)\right] \\
& + e^{i\chi_2}\left[\frac{1}{3} + \frac{2q}{3}\cos\theta\right]\Big\}.
\end{aligned} \tag{43}
$$

We save this expression for our numerical investigation of the fractional phases acquired by entangled qutrits.

III. FRACTIONAL TOPOLOGICAL PHASES ON ENTANGLED QUDITS

We now turn to the main subject of this article: the fractional phases acquired by entangled qudits when subjected to local unitary operations. We shall restrict our analysis to overall pure states. However, the results of the previous section will naturally extend to combined quantum systems, with the special role of entanglement, as measured by the purity of the partial density matrices.

A. Singular value decomposition

We consider a two-qudit system with dimensions d_A and d_B ($d_A \leqslant d_B$). Let

$$
|\psi\rangle = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\alpha_{ij}|ij\rangle \tag{44}
$$

be the most general two-qudit *pure* state. We shall represent it by the $d_A \times d_B$ rectangular matrix α whose elements are the coefficients α_{ij}. With this notation, the associated norm becomes $\langle\psi|\psi\rangle = \mathrm{Tr}(\alpha^\dagger\alpha) = 1$, and the scalar product between two states is $\langle\phi|\psi\rangle = \mathrm{Tr}(\beta^\dagger\alpha)$, where β is the $d_A \times d_B$ matrix representing $|\phi\rangle$ in the chosen basis. In order to characterize a general vector in the Hilbert space, we note that any invertible matrix admits a singular value decomposition $\alpha = e^{i\phi}S_A K S_B^T$, where $S_j \in \mathrm{SU}(d_j)$ ($j = A,B$), K is a diagonal $d_A \times d_B$ rectangular matrix with real positive entries,

$$
K = [Q\ \ \mathbf{0}_{AB}], \tag{45}
$$

and Q is a $d_A \times d_A$ Hermitian diagonal square matrix ($K_{\alpha\alpha} = Q_{\alpha\alpha} \in \mathbb{R}$, with $0 \leqslant \alpha \leqslant d_A - 1$). Here, $\mathbf{0}_{AB}$ is a matrix of order $d_A \times (d_B - d_A)$, with all entries equal to zero. The normalization condition implies $\mathrm{Tr}[Q^2] = 1$.

Under local unitary operations $U_A(t)$ and $U_B(t)$, the coefficient matrix is transformed according to $\alpha(t) = U_A(t)\alpha(0)U_B^T(t)$. It can be readily seen that this kind of transformation preserves the singular decomposition and can be represented separately in each sector of the coefficient matrix:

$$
\alpha(t) = e^{i\phi(t)}S_A(t)K S_B^T(t), \tag{46}
$$

where $S_j(t) = \bar{U}_j(t)S_j(0)$, $\bar{U}_j$ is the SU(d_j) part of the corresponding local unitary operation, $U_j(t) = e^{i\phi_j(t)}\bar{U}_j(t)$, and $\phi(t) = \phi_0 + \phi_A(t) + \phi_B(t)$. Therefore, we identify the transformation in three sectors of the matrix structure: an explicit phase transformation $\phi_0 \mapsto \phi(t)$ and two local evolutions $S_j(0) \mapsto S_j(t)$ ($j = A,B$) in SU(d_j). The K sector remains invariant under local unitary operations. At this point, we would like to note that the singular value decomposition is not unique, since different choices of S_j may result in the same α . However, this is not a problem, as long as one picks up any choice compatible with the initial coefficient matrix $\alpha(t = 0)$. Then, the time evolution will be uniquely determined by the local unitary operations applied to the qudits.

In order to make a connection with the results of the previous section, it is important to unveil the physical meaning of the elements that participate in the singular value decomposition. It is easy to show that the reduced density matrices for qudits A and B are, respectively,

$$
\rho_A = \alpha\,\alpha^\dagger = S_A K K^\dagger S_A^\dagger \tag{47}
$$

and

$$
\rho_B = (\alpha^\dagger\alpha)^T = S_B K^\dagger K S_B^\dagger. \tag{48}
$$

Since K is a $d_A \times d_B$ rectangular matrix with $d_A \leqslant d_B$, one immediately sees that

$$K\,K^\dagger = Q^2, \tag{49}$$

and

$$K^\dagger\,K = \begin{bmatrix} Q^2 & \mathbf{0}_{AB} \\ \mathbf{0}_{BA} & \mathbf{0}_{BB} \end{bmatrix}, \tag{50}$$

where $\mathbf{0}_{BA}$ and $\mathbf{0}_{BB}$ are matrices of order $(d_B - d_A) \times d_A$ and $(d_B - d_A) \times (d_B - d_A)$, respectively, with all entries equal to zero. When $d_A = d_B$, we obtain $K^\dagger K = K K^\dagger = Q^2$. It will be useful to parametrize our matrices in terms of the generators $\{T^j_\alpha\}$ of SU(d_j), normalized as $\mathrm{Tr}\,[T^j_\alpha T^j_\beta] = \delta_{\alpha\beta}$. They can be separated into the Cartan subalgebra generators H^j_α with $[H^j_\alpha, H^j_\beta] = 0$, and the nondiagonal generators P^j_α. Since Q is a diagonal matrix, we can write

$$K K^\dagger = Q^2 = \frac{\mathbb{1}}{d_A} + q_A\sqrt{\frac{d_A - 1}{d_A}}\,\hat{\mathbf{q}}_A \cdot \mathbf{H}^A, \tag{51}$$

and

$$K^\dagger K = \frac{\mathbb{1}}{d_B} + q_B\sqrt{\frac{d_B - 1}{d_B}}\,\hat{\mathbf{q}}_B \cdot \mathbf{H}^B. \tag{52}$$

Here, $\mathbf{q}_j = q_j\,\hat{\mathbf{q}}_j \in \mathbb{R}^{d_j^2 - 1}$ is the purity vector associated with the reduced density matrix of qudit j ($j = A, B$). As before, its absolute value is related to the state purity of qudit j through $\mathrm{Tr}[\rho_j^2(0)] = q_j^2 + (1 - q_j^2)/d_j$. From $\mathrm{Tr}[(K\,K^\dagger)^2] = \mathrm{Tr}[(K^\dagger K)^2] = \mathrm{Tr}[Q^4]$, one easily shows that the norms of the two purity vectors are related by

$$q_B^2\,\frac{d_B - 1}{d_B} = q_A^2\,\frac{d_A - 1}{d_A} + \frac{d_B - d_A}{d_A d_B}. \tag{53}$$

Moreover, the components of $\hat{\mathbf{q}}_B$ are not all independent because of the zeros on the diagonal of $K^\dagger K$ as given by Eq. (50). In fact, there will be only d_A independent elements in $\hat{\mathbf{q}}_B$. Of course, if $d_A = d_B$, then $K K^\dagger = K^\dagger K = Q^2$ and the same generators as well as the same purity vector can be used for both qudits.

It is now important to identify the following invariants under local unitary evolutions: $\mathrm{Tr}[\rho_j^p]$, $p = 1, \ldots, d$, where ρ_j is the reduced density matrix with respect to qudit j. In fact, the invariants are j independent since one easily shows that $\mathrm{Tr}[\rho_A^p] = \mathrm{Tr}[\rho_B^p] = \mathrm{Tr}[Q^{2p}]$. The first one ($p = 1$) is simply the norm of the state vector, as already stated. The second invariant is related to the I *concurrence* of a two-qudit pure quantum state [40],

$$C = \sqrt{2(1 - \mathrm{Tr}\rho_j^2)} = \sqrt{1 - q_A^2}\,C_m, \tag{54}$$

where

$$C_m \equiv \sqrt{2\left(\frac{d_A - 1}{d_A}\right)} \tag{55}$$

is the I concurrence for maximally entangled states. The invariance of C expresses the well-known fact that entanglement is not affected by local unitary operations. The $p = d_A$ invariant can be rewritten in terms of the former and $\mathcal{D} = |\det Q|$. In particular, for qubits, we have $C = 2\mathcal{D}$. In order to exploit the role played by these invariants in the geometric phase, we shall make them explicit in the expression of Q^2 by expressing the norm of the purity vector in terms of the I concurrence, giving

$$\begin{aligned} Q^2 &= \frac{\mathbb{1}}{d_A} + \sqrt{\frac{C_m^2 - C^2}{2}}\;\hat{\mathbf{q}}_A \cdot \mathbf{H}^A \\ &= \frac{\mathbb{1}}{d_A} + \sqrt{\frac{C_m^2 - C^2}{2}}\,\mathrm{diag}\,[x_0 \ldots x_{d-1}]\,, \end{aligned} \tag{56}$$

with $x_n = \langle n|\hat{\mathbf{q}}_A \cdot \mathbf{H}^A|n\rangle$, $\sum_n x_n = 0$, and $\sum_n x_n^2 = 1$.

B. Fractional phases

Following [9,10], we shall define as cyclic those evolutions for which the initial and final state vectors are related by a global phase factor, $\alpha' = e^{i\theta}\alpha$, thus defining a closed path in the projective space of states $\mathcal{P}$. In other words, the final state of a cyclic evolution is physically equivalent to the initial one. The geometric phase acquired by a time-evolving pure state $\alpha(t)$ is given by

$$\begin{aligned} \phi_g &= \arg\langle\psi(0)|\psi(t)\rangle + i\int dt\,\langle\psi(t)|\dot{\psi}(t)\rangle \\ &= \arg\{\mathrm{Tr}[\alpha^\dagger(0)\alpha(t)]\} + i\int dt\,\mathrm{Tr}[\alpha^\dagger(t)\dot{\alpha}(t)], \end{aligned} \tag{57}$$

which corresponds to the total phase

$$\phi_{tot} \equiv \arg\{\mathrm{Tr}[\alpha^\dagger(0)\alpha(t)]\}, \tag{58}$$

minus the dynamical phase. We now use the singular value decomposition to investigate the contribution originated from each sector of the coefficient matrix. First, we can write the total phase as

$$\phi_{tot} = \phi(t) - \phi(0) + \bar{\phi}_{tot}, \tag{59}$$

where

$$\bar{\phi}_{tot} \equiv \arg\left\{\mathrm{Tr}\left[\alpha^\dagger(0)\bar{U}_A(t)\,\alpha(0)\bar{U}_B^T(t)\right]\right\} \tag{60}$$

is the contribution brought by the SU(d_j) sectors. We can investigate the dynamical phase using the singular value decomposition; using Eq. (46), we obtain

$$\begin{aligned} \mathrm{Tr}[\alpha^\dagger\dot{\alpha}] &= i\dot{\phi} + \mathrm{Tr}[S_A^\dagger \dot{S}_A K K^\dagger + K^\dagger K S_B^\dagger \dot{S}_B] \\ &= i\dot{\phi} + \mathrm{Tr}[\rho_A(0)\bar{U}_A^\dagger\dot{\bar{U}}_A + \rho_B(0)\bar{U}_B^\dagger\dot{\bar{U}}_B]. \end{aligned} \tag{61}$$

Note that the trivial phase evolution $\phi(t)$ cancels out when Eqs. (59) and (61) are used in the geometric phase expression, so that we are left with

$$\phi_g = \bar{\phi}_{tot} - \int \mathrm{Tr}[\rho_A(0)\bar{U}_A^\dagger\dot{\bar{U}}_A + \rho_B(0)\bar{U}_B^\dagger\dot{\bar{U}}_B]dt. \tag{62}$$

The reduced density matrices at $t = 0$ can also be expanded in terms of the identity matrix and the generators of SU(d_j) as

$$\rho_j(0) = \frac{\mathbb{1}}{d_j} + q_j\sqrt{\frac{d_j - 1}{d_j}}\,\hat{\mathbf{q}}'_j \cdot \mathbf{T}^j. \tag{63}$$

Note that $\hat{\mathbf{q}}_j$ and $\hat{\mathbf{q}}'_j$ are connected by an initial rotation in $\mathbb{R}^{d_j^2 - 1}$, which is contained in the adjoint representation of SU(d). This is determined by $S_j(0)$ through

$S_j(0)(\hat{\mathbf{q}}_j \cdot \mathbf{H}^j)S_j^\dagger(0) = \hat{\mathbf{q}}'_j \cdot \mathbf{T}^j$. In particular, if the local bases are chosen in order to diagonalize the initial two-qudit density matrix, then we can make $S_j(0) = \mathbb{1}$ and $\hat{\mathbf{q}}'_j = \hat{\mathbf{q}}_j$.

Let us consider a cyclic evolution over the time interval T, $\alpha(T) = e^{i\theta}\alpha(0)$. By defining the local velocity vectors $\mathbf{u}_j$ according to $\bar{U}_j^\dagger \dot{\bar{U}}_j = i\mathbf{u}_j \cdot \mathbf{T}^j$ and using the orthogonality condition for the generators, we arrive at

$$\phi_g = \bar{\phi}_{tot} - \sqrt{\frac{C_m^2 - C^2}{2}} \oint \hat{\mathbf{q}}'_A \cdot \mathbf{dx}_A - \sqrt{\frac{C_m^2 - C^2}{2} + \frac{d_B - d_A}{d_A d_B}} \oint \hat{\mathbf{q}}'_B \cdot \mathbf{dx}_B, \quad (64)$$

where $\mathbf{dx}_j = \mathbf{u}_j dt$.

Now, let us analyze the total phase. Since $\alpha(T) = e^{i\phi(T)} S_A(T) K S_B^T(T)$, and the K sector is time independent, the global phase θ acquired by the two-qudit system is composed of a trivial phase evolution plus the contributions from the SU(d_j) sectors,

$$\theta = \phi(T) - \phi(0) + \theta_A + \theta_B, \quad (65)$$

where $S_j(T) = e^{i\theta_j} S_j(0)$ $(j = A,B)$. However, $\det S_j(T) = e^{i d_j \theta_j} \det S_j(0)$, and since these evolutions are closed in the space of SU(d_j) matrices, we arrive at

$$\theta_j = 2\pi \frac{n_j}{d_j}, \quad (66)$$

where $n_j \in \mathbb{Z}$. Therefore, the trivial phase is canceled by the integral term and only fractional phase values can arise from the SU(d_j) sectors,

$$\bar{\phi}_{tot} = 2\pi \left(\frac{n_A}{d_A} + \frac{n_B}{d_B} \right). \quad (67)$$

Then, the geometric phase acquired in the cyclic evolution becomes

$$\phi_g = 2\pi \left(\frac{n_A}{d_A} + \frac{n_B}{d_B} \right) - \sqrt{\frac{C_m^2 - C^2}{2}} \oint \hat{\mathbf{q}}'_A \cdot \mathbf{dx}_A - \sqrt{\frac{C_m^2 - C^2}{2} + \frac{d_B - d_A}{d_A d_B}} \oint \hat{\mathbf{q}}'_B \cdot \mathbf{dx}_B. \quad (68)$$

Equation (68) evidences the roles played by entanglement and the dimensions of the qudit Hilbert spaces. When $d_A = d_B$, it reduces to the result reported in Ref. [20] with $n = n_A + n_B$. In this case, as anticipated in Sec. II, for maximally entangled states, the partial traces give a completely mixed density matrix for each qudit, so that only fractional geometric phases are allowed. Here, since the complete two-qudit state considered is pure, these fractional phases can be evidenced through conditional interference [21] when the qudits are locally operated with SU(d) transformations.

C. The two-qudit Cartan sector

Similarly to Sec. II A, the local unitary evolutions can be decomposed into the Cartan $U(1)^{d_j-1}$ sector and the coset manifold $SU(d_j)/U(1)^{d_j-1}$. This decomposition gives rise to two separate integral terms, as in Eq. (17). Let us assume that the local basis is chosen so as to make the matrix $\alpha(0)$ diagonal. In this representation, $S_j(0) = \mathbb{1}$ and the reduced density matrices at $t = 0$ are simply

$$\rho_A(0) = \frac{\mathbb{1}}{d_A} + \sqrt{\frac{C_m^2 - C^2}{2}} \hat{\mathbf{q}}_A \cdot \mathbf{H}^A,$$
$$\rho_B(0) = \frac{\mathbb{1}}{d_B} + \sqrt{\frac{C_m^2 - C^2}{2} + \frac{d_B - d_A}{d_A d_B}} \hat{\mathbf{q}}_B \cdot \mathbf{H}^B. \quad (69)$$

Now, we can employ the decomposition in Eq. (9),

$$\bar{U}_j = \bar{V}_j \exp(i\mathbf{h}_j \cdot \mathbf{H}^j) \quad (70)$$

$(j = A,B)$ and separate the Cartan sectors for each qudit evolution. The velocity vectors $\mathbf{u}_A$ and $\mathbf{u}_\mathbf{B}$ can be decomposed as in Eq. (15), $\mathbf{u}_j = \mathbf{v}'_{j\perp} + \mathbf{v}_{j\parallel} + \dot{\mathbf{h}}_j$. Since the reduced density matrices are written in a diagonal representation, $\hat{\mathbf{q}}_j \cdot \mathbf{v}'_{j\perp} = 0$ so that only $\mathbf{v}_{j\parallel}$ and $\dot{\mathbf{h}}_j$ will contribute to the integral term in the geometric phase. The contribution from $\dot{\mathbf{h}}_j$ is path independent (holonomic). The path-dependent (nonholonomic) contribution from $\mathbf{v}_{j\parallel}$ captures the geometric nature of the evolution in $SU(d_j)/U(1)^{d_j-1}$.

Suppose that, at time $\bar{t}$, a partially cyclic evolution occurs. Then, we have

$$\phi_g = \bar{\phi}_{tot} - \sqrt{\frac{C_m^2 - C^2}{2}} [\hat{\mathbf{q}}_A \cdot \mathbf{h}_A(\bar{t}) + \Phi_A] - \sqrt{\frac{C_m^2 - C^2}{2} + \frac{d_B - d_A}{d_A d_B}} [\hat{\mathbf{q}}_B \cdot \mathbf{h}_B(\bar{t}) + \Phi_B], \quad (71)$$

where

$$\Phi_j = \oint \hat{\mathbf{q}}_j \cdot \mathbf{dx}_{j\parallel}, \quad (72)$$

and $\mathbf{dx}_{j\parallel} = \mathbf{v}_{j\parallel} dt$ $(j = A,B)$.

For partially cyclic evolutions, the same argument leading to Eq. (19), here implies

$$\bar{U}_j(\bar{t}) = \exp[i\mathbf{h}_j(\bar{t}) \cdot \mathbf{H}^j], \quad (73)$$

$$\bar{\phi}_{tot} \equiv \arg\{\mathrm{Tr}[\alpha^\dagger(0) e^{i\mathbf{h}_A(\bar{t})\cdot\mathbf{H}^A} \alpha(0) e^{i\mathbf{h}_B(\bar{t})\cdot\mathbf{H}^B}]\}. \quad (74)$$

If, in addition, the evolution is cyclic, then the condition

$$\bar{U}_j(T) = \exp[i\mathbf{h}_j(T) \cdot \mathbf{H}^j] = \exp i(2\pi n_j/d_j)\mathbb{1} \quad (75)$$

must be satisfied, and the geometric phase is given by Eq. (71), with $\bar{t} \to T$, and $\bar{\phi}_{tot}$ given by the fractional values in Eq. (67); for qudits with equal dimensions $d_A = d_B = d$,

$$\phi_g = \frac{2\pi}{d}(n_A + n_B) - \sqrt{\frac{C_m^2 - C^2}{2}} \hat{\mathbf{q}} \cdot [\mathbf{h}_A(T) + \mathbf{h}_B(T)] - \sqrt{\frac{C_m^2 - C^2}{2}} (\Phi_A + \Phi_B). \quad (76)$$

For given n_j values, there is a discrete set $\{\mathbf{h}_j(T)\}_{n_j}$ of solutions to Eq. (75), forming a lattice in $\mathbb{R}^{d_j-1}$, which must be attained by $\mathbf{h}_j(t)$ in order to produce closed paths in the projective space of states $\mathcal{P}$. Those cyclic evolutions $\{\mathbf{h}_j(t)\}_0$ characterized by $n_j = 0$ also describe closed paths in SU(d_j), so they are topologically trivial, as SU(d_j) is simply connected. On the other hand, take for example cyclic evolutions $\{\mathbf{h}_j(t)\}_1$, characterized by $n_j = 1$. They correspond to topologically *nontrivial* closed paths in $\mathcal{P}$ as (i) they are open in SU(d_j), so the triviality of closed paths in SU(d_j) does not apply in this case, (ii) the lattices $\{\mathbf{h}_j(T)\}_0$ and $\{\mathbf{h}_j(T)\}_1$ are different, and (iii) the general condition (75), to keep the paths closed in $\mathcal{P}$, lead to discrete possibilities, with no solutions continuously interpolating the $n_j = 0$ and $n_j = 1$ lattices.

Note that closed paths with a fixed base point, and open paths with fixed endpoints, are fundamental elements to characterize the topological structure of a manifold. As is well known, the consideration of equivalence classes of closed paths, and the natural product based on their composition, leads to the first homotopy group.

For example, consider an evolution that interchanges a pair of anyons. This would correspond to a closed path in the configuration space of indistinguishable particles on the plane, as well as a closed path in the projective space of two-anyon states. To generate the fractional statistics phases, this type of evolution should be controlled. A similar physical content is contained in the necessary condition (75) to generate closed paths in the projective space $\mathcal{P}$ for a qudit pair.

D. Diagonal evolutions

Consider two qudits with the same Hilbert-space dimension d that are locally operated by diagonal SU(d) matrices $\bar{U}_j = \mathrm{diag}[e^{i\chi_{j0}} \ldots e^{i\chi_{j(d-1)}}]$ ($j = A,B$) starting from the initial state

$$|\psi(0)\rangle = \sum_{n=0}^{d-1} \left(\frac{1}{d} + q\sqrt{\frac{d-1}{d}} x_n \right)^{1/2} |nn\rangle. \tag{77}$$

In this case, $\rho_A(0) = \rho_B(0) = Q^2$. The geometric phase reduces to

$$\phi_g = \bar{\phi}_{tot} - \sqrt{\frac{C_m^2 - C^2}{2}} \sum_{n=0}^{d-1} x_n \chi_{T\,n}, \tag{78}$$

where $\chi_{Tn} = \chi_{An} + \chi_{Bn}$. Since the coefficient matrix and the local operations are diagonal, the evolution is partially cyclic at any time t. Then, we can use Eq. (74) to obtain

$$\bar{\phi}_{tot} = \arg\left\{ \sum_{n=0}^{d-1} \left(\frac{1}{d} + \sqrt{\frac{C_m^2 - C^2}{2}} x_n \right) e^{i\chi_{Tn}} \right\}. \tag{79}$$

We note that Eqs. (78) and (79) are very similar to (23) and (24). However, for diagonal transformations of entangled states (with $d_A = d_B$), the overall cyclic transformation can be composed of local noncyclic operations, since the total and geometric phase only depend on $\mathbf{h}_A(t) + \mathbf{h}_B(t)$. This fact is crucial for experimental investigations of the fractional phases and the role played by entanglement.

IV. EXAMPLES

A. Qubits revisited

As an illustration of the methods used in the previous section, we now consider a two-qubit system ($d_A = d_B = 2$) initially prepared in the state

$$|\psi(0)\rangle = \frac{\sqrt{1+q}|00\rangle + \sqrt{1-q}|11\rangle}{\sqrt{2}}, \tag{80}$$

with $0 \leqslant q \leqslant 1$. Note that any two-qubit pure state can be cast in this form by a suitable local basis choice. In this case, a single purity vector $\mathbf{q}$ can be used for both qubits. For the state given by Eq. (80), the concurrence is $C = \sqrt{1-q^2}$, and

$$Q^2 = \begin{bmatrix} \frac{1+q}{2} & 0 \\ 0 & \frac{1-q}{2} \end{bmatrix} = \frac{\mathbb{1}}{2} + \frac{\sqrt{1-C^2}}{2}\sigma_z. \tag{81}$$

Also, we may choose $S_A(0) = S_B(0) = \mathbb{1}$. The associated purity vector simply is $\mathbf{q} = (\sqrt{1-C^2},0,0)$. Let us assume that these qubits evolve under local unitary operators $U_j(t) = e^{i\phi_j(t)}\bar{U}_j(t)$ ($j = A,B$), where $\bar{U}_j$ is a SU(2) matrix acting on qubit j and ϕ_j is the corresponding global phase introduced by U_j. As in Sec. II B, we can make

$$\bar{U}_j(\theta_j,\varphi_j,\chi_j) = \bar{V}_j(\theta_j,\varphi_j)e^{i\chi_j\sigma_z}, \tag{82}$$

and

$$\bar{V}_j(\theta_j,\varphi_j) = \exp(i\theta_j \hat{\mathbf{p}}_j \cdot \mathbf{P}_j) = \begin{bmatrix} \cos\frac{\theta_j}{2} & i\sin\frac{\theta_j}{2}e^{-i\varphi_j} \\ i\sin\frac{\theta_j}{2}e^{i\varphi_j} & \cos\frac{\theta_j}{2} \end{bmatrix}, \tag{83}$$

where $\hat{\mathbf{p}}_j = (0, \cos\varphi_j, \sin\varphi_j)$. Here, $\varphi_j(t)$, $\theta_j(t)$, and $\chi_j(t)$ are time-dependent real parameters with initial conditions $\varphi_j(0) = \theta_j(0) = \chi_j(0) = 0$. As before, $\varphi_j(t)$ and $\theta_j(t)$ can be identified with angular coordinates on two separate Bloch spheres, one for each qubit. Thus, the velocity vector corresponding to each evolution is given by $\mathbf{v}_j = (v_h^j, v_{p1}^j, v_{p2}^j)$, where

$$\begin{aligned} v_h^j &= \sqrt{2}\,\dot{\varphi}_j \sin^2\left(\frac{\theta_j}{2}\right), \\ v_{p1}^j &= \frac{1}{\sqrt{2}}(\dot{\theta}_j\cos\varphi_j - \dot{\varphi}_j\sin\theta_j\sin\varphi_j), \\ v_{p2}^j &= \frac{1}{\sqrt{2}}(\dot{\theta}_j\sin\varphi_j + \dot{\varphi}_j\sin\theta_j\cos\varphi_j), \end{aligned} \tag{84}$$

and the component of $\mathbf{u}$ along the Cartan direction is

$$v_h^j + \sqrt{2}\dot{\chi}_j. \tag{85}$$

Now, from Eqs. (33) and (34), the geometric phase for a pair of qubits following a cyclic evolution under local unitary operations reduces to

$$\phi_g = n\pi - \frac{\sqrt{1-C^2}}{2}(\Omega_A + \Omega_B), \tag{86}$$

where $n = n_A + n_B$. This is a quite intuitive result in which we identify the topological contribution first predicted in Ref. [16], and the sum of the usual solid angle contributions from both qubits weighted by entanglement. For maximally entangled states, only the two fractional values are left.

It will be particularly interesting to investigate the geometric phase acquired under *partially cyclic* evolutions. For these evolutions, $[\theta_j(t),\varphi_j(t)]$ follows a closed path on the Bloch sphere, but $\chi_j(t)$ does not necessarily make a full cycle. In this case, the geometric phase becomes

$$\phi_g = \arctan[\sqrt{1-C^2}\tan(\chi_A+\chi_B)] - \sqrt{1-C^2}\left(\chi_A+\chi_B+\frac{\Omega_A+\Omega_B}{2}\right). \quad (87)$$

For product states ($C=0$), the χ_j terms cancel out and give no contribution to the geometric phase, which is then determined by the individual solid angles enclosed in the separate Bloch spheres. As the concurrence increases, the solid angle contributions diminish and a net effect of the χ_j terms appears as a stepwise variation of the geometric phase as a function of $\chi_T=\chi_A+\chi_B$. For maximally entangled states ($C=1$), the solid angle contributions completely vanish and the stepwise evolution degenerates to a discontinuous jump from 0 to π, which are the allowed fractional phases for qubits. This simple example illustrates the role played by entanglement in the way the geometric phase is built during the evolution.

The result given by Eq. (87) is a generalization of Eq. (9) in Ref. [20] for the case where both qubits are operated. It is also very similar to Eq. (36) in Sec. II B, especially if we notice that $\sqrt{1-C^2}=q$. Of course, this similarity is not surprising once we realize that the partial density matrices of the entangled qubits [$\rho_A(0)=\rho_B(0)=Q^2$; see Eq. (81)] are identical to the single-qubit mixed state considered in Eq. (27). Therefore, the entanglement signature on the geometric phase evolution is directly related to the purity of the partial traces of the two-qubit density matrix. However, there are two important differences between the two cases. First, since the two-qubit entangled state considered here is pure, we can expect the fractional phases to be experimentally observable. Second, the geometric phase acquired by the entangled qubits depends on $\chi_T=\chi_A+\chi_B$, which means that the overall cyclic transformation can be split into local noncyclic operations applied to the entangled qubits separately.

B. Qutrits

For the two-qutrit case, we will restrict our analysis to local evolutions in the 3×3 Cartan sector of each qutrit $\bar{U}_A(t)\otimes\bar{U}_B(t)$, where

$$\bar{U}_j(t) = \begin{bmatrix} e^{i\chi_{j0}} & 0 & 0 \\ 0 & e^{i\chi_{j1}} & 0 \\ 0 & 0 & e^{i\chi_{j2}} \end{bmatrix}, \quad (88)$$

with $j=A,B$ and $\chi_{j0}+\chi_{j1}+\chi_{j2}=0$.

Let us suppose that the local basis is chosen so as to leave the initial two-qutrit pure state in the form

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}[\sqrt{1+2q\cos(\theta+2\pi/3)}|00\rangle + \sqrt{1+2q\cos(\theta+4\pi/3)}|11\rangle + \sqrt{1+2q\cos\theta}|22\rangle], \quad (89)$$

where $0\leqslant q\leqslant 1$, $-\theta_0(q)\leqslant\theta\leqslant\theta_0(q)$, and $\theta_0(q)$ is given by (40). The maximal concurrence for qutrits is $C_m=\sqrt{4/3}$ and the concurrence of state (89) is $C=C_m\sqrt{1-q^2}$. Since the coefficient matrix $\alpha(0)$ and the local unitary operations are all diagonal, the calculation of the nontrivial total phase $\bar{\phi}_{tot}$ is significantly simplified as

$$\bar{\phi}_{tot} = \arg\{\mathrm{Tr}[Q^2\bar{U}_A(t)\bar{U}_B(t)]\}, \quad (90)$$

where

$$Q^2 = \frac{\mathbb{1}}{3} + \frac{2q}{3}\begin{bmatrix} \cos\left(\theta+\frac{2\pi}{3}\right) & 0 & 0 \\ 0 & \cos\left(\theta+\frac{4\pi}{3}\right) & 0 \\ 0 & 0 & \cos\theta \end{bmatrix}. \quad (91)$$

Also, the partial density matrices for qutrits A and B at $t=0$ are equal to Q^2. Therefore, the geometric phase becomes

$$\phi_g = \bar{\phi}_{tot} - \frac{2q}{3}\left[\chi_{T0}\cos\left(\theta+\frac{2\pi}{3}\right) + \chi_{T1}\cos\left(\theta+\frac{4\pi}{3}\right) + \chi_{T2}\cos\theta\right], \quad (92)$$

where $\chi_{Tn}=\chi_{An}+\chi_{Bn}$, and the nontrivial total phase is

$$\bar{\phi}_{tot} = \arg\left\{e^{i\chi_{T0}}\left[\frac{1}{3}+\frac{2q}{3}\cos\left(\theta+\frac{2\pi}{3}\right)\right] + e^{i\chi_{T1}}\left[\frac{1}{3}+\frac{2q}{3}\cos\left(\theta+\frac{4\pi}{3}\right)\right] + e^{i\chi_{T2}}\left[\frac{1}{3}+\frac{2q}{3}\cos\theta\right]\right\}. \quad (93)$$

Equations (92) and (93) are very similar to the single-qutrit result given by Eqs. (42) and (43). However, the diagonal phase shifts χ_n are replaced by the total phase shifts χ_{Tn}, showing the nonlocal character of the geometric phase. For maximally entangled states ($q=0\Rightarrow C=C_m$), the integral

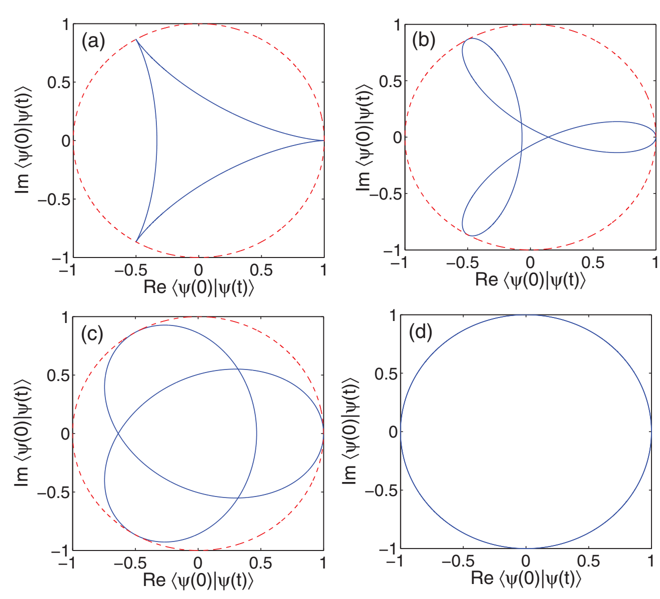

FIG. 1. (Color online) Parametric plot of the quantum state overlap for a two-qutrit evolution given by Eqs. (94). (a) $q=0$ ($C=\sqrt{4/3}$), (b) $q=0.2$, (c) $q=0.6$, (d) $q=1$ ($C=0$). The unit circle is depicted by a dashed red curve for reference.

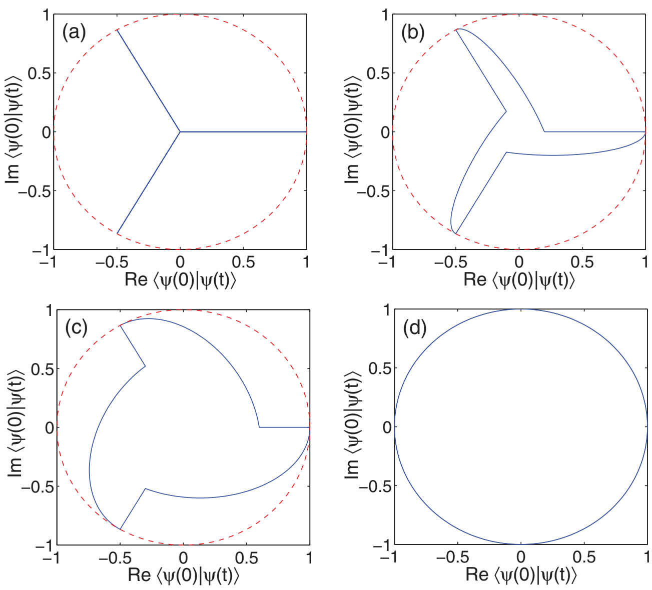

FIG. 2. (Color online) Parametric plot of the quantum state overlap for a two-qutrit evolution given by Eqs. (95). (a) $q = 0$ ($C = \sqrt{4/3}$), (b) $q = 0.2$, (c) $q = 0.6$, (d) $q = 1$ ($C = 0$). The unit circle is depicted by a dashed red curve for reference.

term vanishes, so that $\phi_g = \bar{\phi}_{tot}$. In Fig. 1, we show a parametric plot of the overlap $\langle\psi(0)|\psi(t)\rangle$ in the complex plane for $\theta = 0$ and different values of q. The diagonal phase shifts are evolved according to

$$\chi_{T0} = \chi_{T1} = t, \quad \chi_{T2} = -2t. \tag{94}$$

For the maximally entangled state ($q = 0$), the overlap presents sharp peaks, touching the unit circle at the fractional phases expected for cyclic evolutions of qutrits. As q is increased, the path followed in the complex plane degenerates to a circle for $q = 1$ (product state).

A second kind of evolution is considered in Fig. 2 in which the maximally entangled state follows sharp phase jumps between the fractional values. As entanglement is decreased, these jumps also degenerate to a continuous phase evolution (circle) for the product state. The diagonal phase shifts are evolved as

$$\begin{aligned}
\chi_{T0} &= -t,\\
\chi_{T1} &= \begin{cases} t & (0 \leqslant t \leqslant 2\pi/3)\\ 2\pi/3 & (2\pi/3 \leqslant t \leqslant 4\pi/3)\\ t - 2\pi/3 & (4\pi/3 \leqslant t \leqslant 2\pi)\\ 4\pi/3 & (2\pi \leqslant t \leqslant 8\pi/3)\\ t - 4\pi/3 & (8\pi/3 \leqslant t \leqslant 10\pi/3)\\ 2\pi & (10\pi/3 \leqslant t \leqslant 4\pi), \end{cases}\\
\chi_{T2} &= -(\chi_{T0} + \chi_{T1}).
\end{aligned} \tag{95}$$

It is interesting to inspect how the overlap path is affected when the diagonal phase shifts evolve at very different speeds. For example, consider a maximally entangled state evolving according to

$$\chi_{T0} = t, \quad \chi_{T1} = 30t, \quad \chi_{T2} = -31t. \tag{96}$$

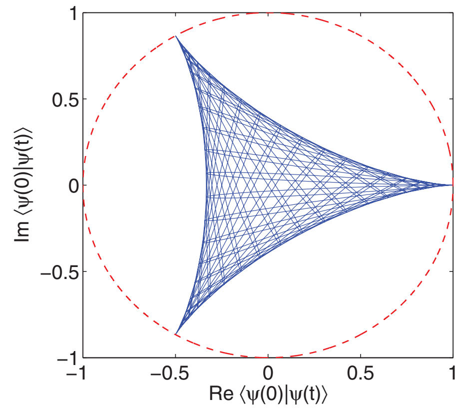

FIG. 3. (Color online) Parametric plot of the quantum state overlap for a pair of maximally entangled qutrits under the joint evolution given by Eqs. (96). The unit circle is depicted by a dashed red curve for reference.

The corresponding result is displayed in Fig. 3. A complicated trajectory appears within the limits of the perimeter defined by Fig. 1(a). This result raises the question of whether this fractional phase structure could still be observed under random local SU(3) transformations.

It is also interesting to compare the evolution of a maximally entangled state with partially entangled states having the same single-qutrit probability distribution. For example, let us take the initial state

$$\begin{aligned}
|\psi(0)\rangle = {}& \sqrt{\frac{q}{3}}(\,|00\rangle + |11\rangle + |22\rangle)\\
&+ \sqrt{\frac{1-q}{6}}(\,|01\rangle + |02\rangle + |12\rangle\\
&+ |20\rangle + |21\rangle + |10\rangle),
\end{aligned} \tag{97}$$

with q ranging between $1/3$ for the product state and 1 for the maximally entangled state. The probability distribution for qutrit A is

$$P_n^A = \sum_{m=0}^{2} |\langle nm|\psi(0)\rangle|^2 = \frac{1}{3}, \tag{98}$$

for $n = 0,1,2$, and similarly for P_n^B. To illustrate the role of entanglement, we can take the following parametric evolution:

$$\begin{aligned}
\chi_{A0} &= \chi_{A1} = t, \quad \chi_{A2} = -2t,\\
\chi_{B0} &= \chi_{B1} = 2t, \quad \chi_{B2} = -4t,
\end{aligned} \tag{99}$$

where the local phase shifts are asymmetrical. The parametric plot of the state overlap when the qutrits are subjected to the evolution given by Eqs. (99) is presented in Fig. 4. As we can see, only the maximally entangled state achieves maximum overlap at the fractional phases expected for qutrits. This comparison was used in Refs. [21] and [23] in the context of entangled photon pairs, where the quantum state overlap was associated to the visibility of two-photon interference fringes. Maximum visibility (overlap) can only occur with maximally entangled states at the allowed fractional phases.

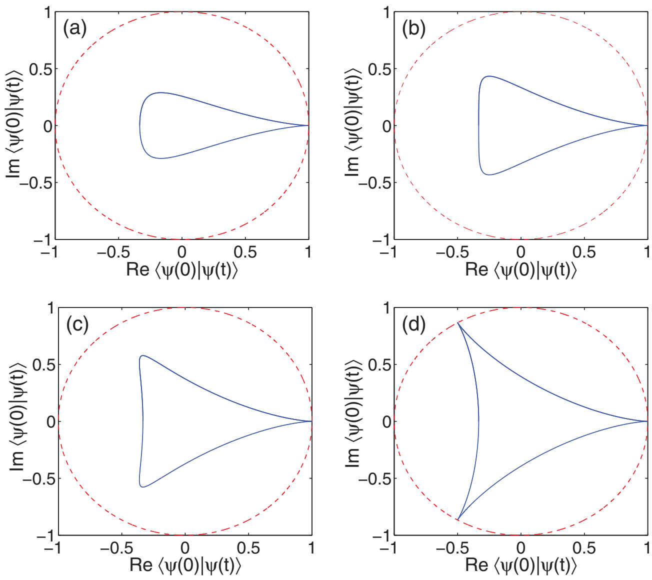

FIG. 4. (Color online) Parametric plot of the quantum state overlap for a two-qutrit evolution given by Eqs. (99). (a) $q = 1/3$ ($C = 0$), (b) $q = 1/2$, (c) $q = 2/3$, (d) $q = 1$ ($C = \sqrt{4/3}$). The unit circle is depicted by a dashed red curve for reference.

C. Qubit-qutrit

We now illustrate the simplest case with asymmetrical Hilbert spaces. Let us consider a qubit-qutrit system ($d_A = 2, d_B = 3$) initially prepared in the state

$$|\psi(0)\rangle = \frac{\sqrt{1+q}|00\rangle + \sqrt{1-q}|11\rangle}{\sqrt{2}}, \tag{100}$$

with $0 \leqslant q \leqslant 1$. The reduced density matrices of the qubit and the qutrit are

$$\rho_A(0) = Q^2, \quad \rho_B(0) = \begin{bmatrix} Q^2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{101}$$

where

$$Q^2 = \frac{\mathbb{1}}{2} + \frac{\sqrt{1-C^2}}{2}\sigma_z, \tag{102}$$

and $C = \sqrt{1-q^2}$ is the qubit-qutrit concurrence. First, let us suppose that both the qubit and the qutrit evolve under local diagonal operations such that

$$\bar{U}_A(t) = \begin{bmatrix} e^{i\chi_A} & 0 \\ 0 & e^{-i\chi_A} \end{bmatrix},$$

$$\bar{U}_B(t) = \begin{bmatrix} e^{i\chi_{B0}} & 0 & 0 \\ 0 & e^{i\chi_{B1}} & 0 \\ 0 & 0 & e^{i\chi_{B2}} \end{bmatrix}, \tag{103}$$

with $\chi_{B0} + \chi_{B1} + \chi_{B2} = 0$. Then, the geometric phase acquired is

$$\phi_g = \arctan\left[\sqrt{1-C^2}\tan\left(\chi_A + \frac{\chi_{B0}-\chi_{B1}}{2}\right)\right] - \sqrt{1-C^2}\left(\chi_A + \frac{\chi_{B0}-\chi_{B1}}{2}\right). \tag{104}$$

This result is identical to the two-qubit geometric phase if we make the identification $\chi_B \equiv (\chi_{B0} - \chi_{B1})/2$. Note that state (100) does not include the third component of the qutrit, which remains unaffected as long as only diagonal operations are performed. Thus, the qutrit behaves as an effective qubit and only two-qubit fractional phases can be observed.

In order to evidence the dual dimensional structure of the qubit-qutrit system, using only diagonal evolutions, we must consider an initial state with all qubit and qutrit components. We can build a simple numerical example with the following maximally entangled state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|12\rangle, \tag{105}$$

for which $Q^2 = \mathbb{1}_{2\times2}/2$, $S_A(0) = \mathbb{1}_{2\times2}$, and

$$S_B(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{106}$$

According to Eq. (68), the integral contribution for the geometric phase never vanishes for qudits with different dimensions, even for maximally entangled states. The dual dimension behavior, however, will still be present in the nontrivial total phase $\bar{\phi}_{tot}$. Let us consider the local diagonal evolutions given by Eqs. (103). In this case, we obtain

$$\bar{\phi}_{tot} = \arg\left\{\cos\left(\chi_A - \chi_{B2} - \frac{\chi_{B1}}{2}\right)\frac{e^{-i\chi_{B1}/2}}{2} + \cos\left(\chi_A - \chi_{B1} - \frac{\chi_{B2}}{2}\right)\frac{e^{-i\chi_{B2}/2}}{2}\right\}, \tag{107}$$

and

$$\phi_g = \bar{\phi}_{tot} - \frac{\chi_{B0}}{4}. \tag{108}$$

The fractional phases expected for a qubit-qutrit system are

$$\bar{\phi}_{tot} = n\pi + \frac{2m\pi}{3}, \tag{109}$$

with $n,m \in \mathbb{Z}$. Therefore, the qubit-qutrit system can exhibit the qubit, the qutrit, or a combination of both topological phases when subjected to cyclic evolutions under SU(2) ⊗ SU(3) operations. In Fig. 5, different diagonal evolutions are considered. In all cases, the qutrit is operated by

$$\chi_{B0} = \chi_{B1} = t, \quad \chi_{B2} = -2t, \tag{110}$$

while different evolutions are considered for the qubit. The dual phase behavior can be observed in Fig. 5(a), while Figs. 5(b) and 5(c) display the qutrit and qubit phases, respectively. The dual phase behavior also becomes evident when we make the qubit evolution much faster than the qutrit, as in Fig. 5(d). In this case, a complicated path is drawn by the quantum state overlap in the complex plane, touching the unit circle only when the fractional phases allowed for the qubit-qutrit system are attained.

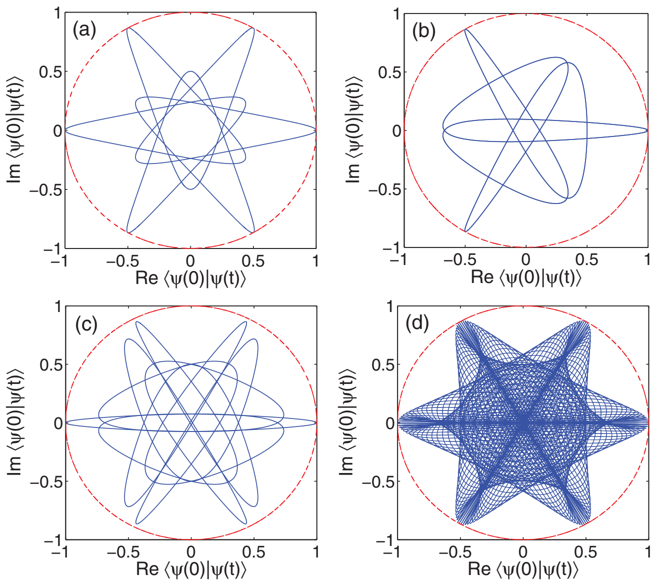

FIG. 5. (Color online) Parametric plot of the qubit-qutrit quantum state overlap for the initial condition given by Eq. (105). (a) $\chi_A = 1.5t$, (b) $\chi_A = 3t$, (c) $\chi_A = 3.5t$, (d) $\chi_A = 100t$. The unit circle is depicted by a dashed red curve for reference.

V. CONCLUSION

In this article, we presented a detailed description of the geometric phase acquired by entangled qudits operated by local unitary transformations. Our previous result [20] was detailed and extended to pairs of qudits with general dimensions d_A and d_B. This was achieved by utilizing the singular value decomposition of the coefficient matrix defined by the two-qudit quantum state. This decomposition involves a pair of matrices in SU(d_A) and SU(d_B), respectively, with the dimensions of the individual qudit Hilbert spaces. The fractional phase values naturally appear as the possible factors arising from cyclic evolutions of these local components. They completely encompass the geometric phase acquired by maximally entangled qudits with equal dimensions, subjected to cyclic evolutions. However, in a more general scenario where partially entangled states or different qudit dimensions are considered, the geometric phase can assume continuous values in addition to the fractional phase contribution.

To put in evidence the role played by entanglement and the SU(d) parameters of the local transformations, we used the geometric phase derived by Mukunda and Simon in Refs. [9,10], as well as the decomposition of the local transformations into the Cartan and coset SU(d)/U(1)$^{d-1}$ sectors. In particular, we showed that the geometric phase given by Mukunda and Simon gives rise to a holonomic contribution built in the Cartan sector and a nonholonomic one built in the coset sector. Our results regarding the fractional phases in higher dimensions were illustrated with numerical examples for two-qutrit and qubit-qutrit systems. This investigation could be applied to different experimental contexts, including entangled photon pairs created by spontaneous parametric down conversion, nuclear magnetic resonance, trapped ions, and other setups dealing with entangled states.

Qudit gates based on topological phases are a potentially robust means to implement quantum algorithms [43–45]. In order to demonstrate the usefulness of the fractional phases for quantum information protocols, it will be crucial to investigate the phase evolution under local random unitary transformations. It is well known that two-qubit entangled states are robust against certain kinds of noise [46], which motivated an alignment free quantum cryptography protocol [47,48]. We shall leave the investigation of the fractional phases under noisy evolutions to a future contribution.

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[1] M. V. Berry, Proc. R. Soc. London A **392**, 45 (1984).
[2] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247 (1956).
[3] S. J. van Enk, Opt. Comm. **102**, 59 (1993).
[4] M. J. Padgett and J. Courtial, Opt. Lett. **24**, 430 (1999).
[5] E. J. Galvez, P. R. Crawford, H. I. Sztul, M. J. Pysher, P. J. Haglin, and R. E. Williams, Phys. Rev. Lett. **90**, 203901 (2003).
[6] B. Coutinho dos Santos, C. E. R. Souza, K. Dechoum, and A. Z. Khoury, Phys. Rev. A **76**, 053821 (2007).
[7] G. Milione, H. I. Sztul, D. A. Nolan, and R. R. Alfano, Phys. Rev. Lett. **107**, 053601 (2011).
[8] G. Milione, S. Evans, D. A. Nolan, and R. R. Alfano, Phys. Rev. Lett. **108**, 190401 (2012).
[9] N. Mukunda and R. Simon, Ann. Phys. (NY) **228**, 205 (1993).
[10] N. Mukunda and R. Simon, Ann. Phys. (NY) **228**, 269 (1993).
[11] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) **403**, 869 (2000).
[12] L.-M. Duan, J. I. Cirac, and P. Zoller, Science **292**, 1695 (2001).
[13] E. Sjöqvist, Phys. Rev. A **62**, 022109 (2000).
[14] B. Hessmo and E. Sjöqvist, Phys. Rev. A **62**, 062301 (2000).
[15] R. Mosseri and R. Dandoloff, J. Phys. A **34**, 10243 (2001).
[16] P. Milman and R. Mosseri, Phys. Rev. Lett. **90**, 230403 (2003).
[17] P. Milman, Phys. Rev. A **73**, 062118 (2006).
[18] C. E. R. Souza, J. A. O. Huguenin, P. Milman, and A. Z. Khoury, Phys. Rev. Lett. **99**, 160401 (2007).
[19] Jiangfeng Du, Jing Zhu, Mingjun Shi, Xinhua Peng, and Dieter Suter, Phys. Rev. A **76**, 042121 (2007).
[20] L. E. Oxman and A. Z. Khoury, Phys. Rev. Lett. **106**, 240503 (2011).

[21] A. Z. Khoury, L. E. Oxman, B. Marques, A. Matoso, and S. Pádua, Phys. Rev. A **87**, 042113 (2013).
[22] Markus Johansson, Marie Ericsson, Kuldip Singh, Erik Sjöqvist, and Mark S. Williamson, Phys. Rev. A **85**, 032112 (2012).
[23] Markus Johansson, Antonio Z. Khoury, Kuldip Singh, and Erik Sjöqvist, Phys. Rev. A **87**, 042112 (2013).
[24] L. Neves, G. Lima, E. J. S. Fonseca, L. Davidovich, and S. Pádua, Phys. Rev. A **76**, 032314 (2007).
[25] I. Rodrigues, O. Cosme, and S. Pádua, J. Phys. B: At. Mol. Opt. Phys. **43**, 125505 (2010).
[26] S. P. Walborn, C. H. Monken, S. Pádua, and P. H. Souto Ribeiro, Phys. Rep. **495**, 87 (2010).
[27] M. França Santos, P. Milman, A. Z. Khoury, and P. H. Souto Ribeiro, Phys. Rev. A **64**, 023804 (2001).
[28] D. P. Caetano, P. H. Souto Ribeiro, J. T. C. Pardal, and A. Z. Khoury, Phys. Rev. A **68**, 023805 (2003).
[29] A. Kitaev, Ann. Phys. (NY) **303**, 2 (2003).
[30] K. M. F. Romero, A. C. A. Pinto, and M. T. Thomaz, Physica A **307**, 142 (2002).
[31] A. C. A. Pinto and M. T. Thomaz, J. Phys. A **36**, 7461 (2003).
[32] A. Carollo, M. F. Santos, and V. Vedral, Phys. Rev. A **67**, 063804 (2003).
[33] R. M. Serra, A. Carollo, M. F. Santos, and V. Vedral, Phys. Rev. A **70**, 044102 (2004).
[34] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **90**, 160402 (2003).
[35] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **92**, 020402 (2004).
[36] A. Carollo, G. Massimo Palma, A. Lozinski, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **96**, 150403 (2006).
[37] A. Carollo, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **96**, 020403 (2006).
[38] A. Rivas, O. Viyuela, and M. A. Martin-Delgado, Phys. Rev. B **88**, 155141 (2013).
[39] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, arXiv:1309.1174.
[40] P. Rungta, V. Buzek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).
[41] K. B. Petersen and M. S. Pedersen, http://matrixcookbook.com, version Nov. 15, 2012 (unpublished) [cf. Eq. (46), p. 8].
[42] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
[43] S. S. Bullock, D. P. O'Leary, and G. K. Brennen, Phys. Rev. Lett. **94**, 230502 (2005).
[44] A. Muthukrishnan and C. R. Stroud, , Phys. Rev. A **62**, 052309 (2000).
[45] R. Ionicioiu, T. P. Spiller, and W. J. Munro, Phys. Rev. A **80**, 012312 (2009).
[46] L. Aolita and S. P. Walborn, Phys. Rev. Lett. **98**, 100501 (2007).
[47] C. E. R. Souza, C. V. S. Borges, A. Z. Khoury, J. A. O. Huguenin, L. Aolita, and S. P. Walborn, Phys. Rev. A **77**, 032345 (2008).
[48] V. D'Ambrosio, E. Nagali, S. P. Walborn, L. Aolita, S. Slussarenko, L. Marrucci, and F. Sciarrino, Nat. Comm. **3**, 961 (2012).