Virtual monopoles in a bosonic atom–diatomic-molecule system

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We investigate the virtual monopoles in an ultracold bosonic atom–diatomic-molecule system with a three-level second-quantized model. In the quantum theory, we show that the monopole field of the ground state does not have a spherical symmetry. We calculate the monopole charge and find that it is an integer multiple of the elementary charge $g_0 = \frac{1}{2}$. This multiple exactly reflects the degeneracy properties of the ground state and strongly depends on the total particle number and the atom-number imbalance between two atomic species. In the mean-field limit, we illustrate that the system can create continuous monopole charges in the case of heteronuclear molecules. The underlying mechanism associated with the degeneracy properties and the application related to the quantum phase transition of the monopoles are briefly discussed as well.

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I. INTRODUCTION

Since 1984, the geometric phase [1,2] has become an exciting area of research due to its important applications in the fields of high-precision quantum measurement [3], quantum information processing [4], quantum computing [5,6], and quantum Hall effect [7], to name a few. In quantum mechanics, the geometric phase arises in a cyclic adiabatic evolution, which can be expressed as a circuit integral of the Mead-Berry connection or an integral of the Mead-Berry curvature [8] over a surface bounded by a closed path. The Mead-Berry connection as a vector-valued function is gauge dependent and hence the local Mead-Berry connection can never be physically observable. In contrast to the Mead-Berry connection, the Mead-Berry curvature is a gauge-invariant local manifestation of the geometric properties of the wave functions in the parameter space, and has proven to be an essential physical ingredient for understanding a variety of electronic properties [9,10]. With the help of the concept of magnetic monopole suggested by Dirac [11], one can interpret the Mead-Berry curvature as a virtual magnetic field generated from a monopole in the parameter space. And the total charge of the monopole is quantized in units of 2π. The multiple of 2π is the so-called Chern number [12], which is essential for understanding various quantization effects such as anomalous Hall effect [13].

Recently, the field of associating ultracold atoms into molecules has received more interest than many other fields [14,15] because of the wide applications ranging from the research on the BCS-BEC crossover physics [16] to the exploration of the quantum phase transition (QPT) [17]. When both the atoms and molecules are bosons, we can describe this kind of system by adopting a mean-field theory in the large particle-number limit. In a mean-field treatment, the atom-molecule systems can be modeled by a nonlinear Schrödinger equation in which the nonlinearity arises from the fact that two or more atoms are needed to form one molecule. These systems do not satisfy the superposition principle due to the presence of nonlinearity [18] and do not have U(1) invariance because of the nonidentical chemical potentials for the atom and molecule [19]. For the U(1)-invariance system, the adiabatic geometric phase was found to be modified by the nonlinearity [20]. In order to overcome the difficulty caused by the lack of U(1) invariance, the geometric phase and related Mead-Berry connection have been generalized to the atom–homonuclear-molecule systems [21,22]. The connection between the geometric phase and QPT in these atom–homonuclear-molecule systems have been investigated as well [23,24]. In contrast to the homonuclear-molecule systems, the atom–heteronuclear-molecule systems [25] are more interesting and calls for further theoretical considerations. This is because the heteronuclear molecules can be either bosons or fermions, thus quantum statistics will play important roles in such systems [26] and a large electric dipole moment will be induced with the prospect of creating a dipolar superfluid [27].

In this paper, we study the monopoles in a three-mode atom-molecule boson system, which depends on three external parameters. In the quantum theory, we calculate the monopole field and its charge for the ground state. In particular, we explore the effects of the particle-number imbalance between two atomic species on the monopole field. We find that the monopole field in this system is not spherically symmetrical and the charge of the monopole strongly depends on the particle-number imbalance. The corresponding mean-field analysis is also given.

The rest of the paper is organized as follows. In Sec. II, we introduce the second-quantized model. In Sec. III, we study the properties of the ground-state monopole in the system. In Sec. IV, we give a mean-field analysis. Section V presents our discussion and conclusion.

II. THREE-MODE ATOM-MOLECULE MODEL

We adopt a three-mode model to describe an atom–heteronuclear-molecule system. The basic assumption here is that the spatial wave functions for three modes are fixed so that

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one can associate each mode with an annihilation operator $\hat{a}_j$ of a particle in atomic mode $j = 1,2$ and in molecular mode $j = m$, respectively. Under this three-mode approximation, the second-quantized Hamiltonian of the system reads ($\hbar = 1$ throughout) [28]

$$\hat{H} = \Delta\hat{N}_m + \sum_{i,j}\frac{\chi_{ij}}{N}\hat{N}_i\hat{N}_j + \left(\frac{\Omega e^{i\varphi}}{\sqrt{N}}\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_m + \text{H.c.}\right), \quad (1)$$

where $\hat{N}_j = \hat{a}_j^\dagger\hat{a}_j$. The detuning Δ represents the energy difference between the molecular and atomic levels, which can be tuned by an external field. The parameter $\Omega e^{i\varphi}$ specifies the atom-molecule coupling. The parameters χ_{ij} describe s-wave scattering, taking into account the intraspecies ($i = j$) and interspecies ($i \neq j$) with $\chi_{ij} = \chi_{ji}$. It should be mentioned that the model (1) can be mapped to a trilinear Hamiltonian describing the nondegenerate parametric down-conversion in quantum optics [29]. In this analogy, the molecular mode plays the role of the pump photon, while the two atomic modes denote the signal and idler photons, respectively. The collisional terms would correspond to the Kerr-type cubic nonlinearity which will be present in the optical system with some nonlinear medium [30]. In experiments, the model (1) may apply to the ^{85}Rb-^{87}Rb system where the ultracold heteronuclear Feshbach molecules were produced starting with a ^{87}Rb Bose-Einstein condensate (BEC) and a cold atomic gas of ^{85}Rb [25]. These ultracold heteronuclear molecules in low-lying vibrational states are of particular interest since they could be a permanent dipole moment due to the unequal distribution of electrons. The Hamiltonian (1) commutes with the total atom number N and thus the N is a conserved quantity of the system. Indeed there exists another conserved quantity, namely, $D = \langle\hat{a}_1^\dagger\hat{a}_1\rangle - \langle\hat{a}_2^\dagger\hat{a}_2\rangle$, which denotes the particle-number imbalance between two atomic species. Using these two conserved constants and neglecting the trivial constant terms which are proportional to D or N, the Hamiltonian (1) can be simplified as follows:

$$\hat{H} = Z\hat{N}_m + \frac{\chi}{N}\hat{N}_m^2 + \left(\frac{X+iY}{\sqrt{N}}\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_m + \text{H.c.}\right), \quad (2)$$

where $X + iY = \Omega e^{i\varphi}$, $Z = \Delta - (D+N)(\chi_{11} - \chi_{1m}) - 2N\chi_{12} + (D-N)(\chi_{22} - \chi_{2m})$, and $\chi = \chi_{11} + \chi_{22} + \chi_{mm} + 2(\chi_{12} - \chi_{1m} - \chi_{2m})$. For our atom-molecule system, the Bloch space is expanded by the following three angular momentum operators: $\hat{L}_X = 2\sqrt{2}\frac{\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_m + \hat{a}_m^\dagger\hat{a}_1\hat{a}_2}{N^{3/2}}$, $\hat{L}_Y = i2\sqrt{2}\frac{\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_m - \hat{a}_m^\dagger\hat{a}_1\hat{a}_2}{N^{3/2}}$, and $\hat{L}_Z = \frac{2\hat{a}_m^\dagger\hat{a}_m - \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2}{N}$. They compose a generalized Bloch representation [31,32] and in this representation the Hamiltonian (2) becomes $\hat{H} = \frac{N}{4}(Z + \frac{\chi}{2})\hat{L}_Z + \frac{N\chi}{16}\hat{L}_Z^2 + \frac{N}{4\sqrt{2}}[(X + iY)(\hat{L}_X - i\hat{L}_Y) + \text{H.c.}]$. The corresponding parameter space is spanned by the vector $\boldsymbol{R} = (X,Y,Z)$, which represent the influences of an external field.

III. MONOPOLES FOR THE GROUND STATES

In the following study, we only focus on the monopoles for the ground states. For convenience, we restrict ourselves to the states with even N and D, where $D = 0,2,\ldots,N-2$. From the Hamiltonian (2), one can compute the ground-state Mead-Berry curvature (i.e., virtual magnetic field) by using the formula [2]

$$\boldsymbol{B}_N(\boldsymbol{R}) = \text{Im}\sum_{m\neq 0}\frac{\langle 0|\nabla_{\boldsymbol{R}}\hat{H}|m\rangle \times \langle m|\nabla_{\boldsymbol{R}}\hat{H}|0\rangle}{(E_m - E_0)^2}, \quad (3)$$

where $|0\rangle$ denotes the quantum ground state. E_m specifies the eigenenergy of the eigenstate $|m\rangle$ and satisfies the eigenequation $\hat{H}|m\rangle = E_m|m\rangle$. The energy denominator in Eq. (3) implies that the virtual magnetic field usually diverges at the degenerate point where the energy levels cross and may have maximum values at avoided level crossings. These level structures are reflected in the geometry of the Hilbert space of the system, which can be captured by the virtual magnetic field of the ground-state monopole.

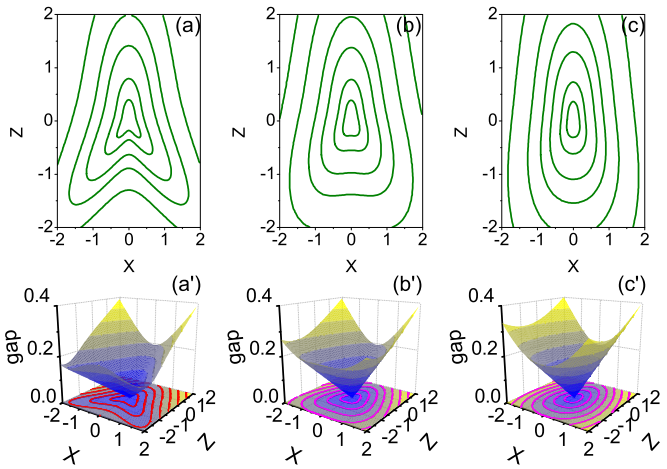

FIG. 1. (Color online) Magnitude of the virtual field for the ground state (upper panels) and the level gap between the ground state and the first exited state (lower panels) with $N = 10$ and $\chi = Y = 0$. (a) and (a′): $D = 0$; (b) and (b′): $D = 2$; (c) and (c′): $D = 4$. In the upper panels, the contours denote the rescaled quantity $\log_{10}|\frac{\boldsymbol{B}_N}{N}|$ and the values are $-1.5,-1.2,-0.9,-0.6,-0.1$, and 0.5 from out to core, respectively.

When the particle interaction is absent, i.e., $\chi = 0$, in our system there only exists a pointlike magnetic monopole, which is located at the origin $\boldsymbol{R} = 0$ in the parameter space. We have numerically computed the virtual magnetic field of the ground-state monopole from Eqs. (2) and (3) by using the exact diagonalization method in double precision arithmetic in the Fock-state representation. The results are shown in Figs. 1(a)–1(c). We find that the virtual magnetic field is symmetric with respect to the Z axis in our atom-molecule system, which is different from the isotropic field generated by a standard pointlike monopole. However, the symmetry of the virtual field will approach spherical symmetry as the atom-number imbalance parameter D increases. When $D = N - 2$, a spherically symmetric field will be recovered. The similarity between the upper panels and the lower panels of Fig. 1 reveals that the properties of the virtual field are mainly determined by the structure of the level gap between the ground state and the first excited state.

In the presence of the particle interaction, i.e., $\chi \neq 0$, the ground state of the system exhibits a monopole chain that includes $\frac{N-D}{2}$ pointlike monopoles on the Z axis, which are located at the points $(X = Y = 0, \frac{Z}{\chi} = -\frac{n}{N})$ with

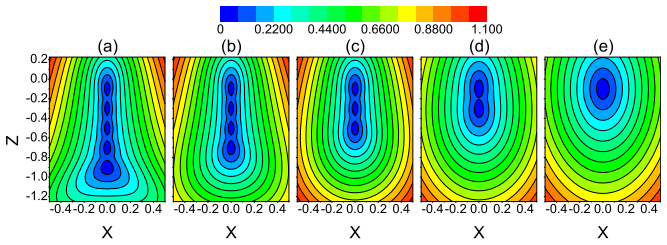

FIG. 2. (Color online) Level gap between the ground state and the first exited state with $N = 10$ and $Y = 0$. From (a) to (e), $D = 0,2,4,6$, and 8, respectively. The interaction parameter $\chi = 1$ has been used.

$n = N - D - 1, N - D - 3, \ldots, 1$. When the particle-number imbalance between two atomic species increases, the number of monopoles will decrease. When $D = N - 2$, there is only one monopole located at the point $(X = Y = 0, \frac{Z}{\chi} = -\frac{1}{N})$. The level gap between the ground state and the first excited state for this case is demonstrated in Fig. 2.

Now we calculate the charge of the monopole numerically. According to "Gauss's law," the charge of the monopole can be defined by

$$\oint_S \boldsymbol{B}_N \cdot d\boldsymbol{S} = 4\pi g_N, \tag{4}$$

where $d\boldsymbol{S}$ is the area element and S denotes any surface boundary enclosing all monopole points. To numerically calculate the monopole charge from Eq. (4), we first compute the monopole field from Eqs. (2) and (3) with the exact diagonalization method, and then choose a sphere as the closed surface enclosing all monopoles. With increasing the radius of the sphere, the numerical tests demonstrate good convergence and accuracy of the results for g_N. The results of the monopole charge for $\chi = 0$ with different total atom numbers are demonstrated in Fig. 3. We see that the monopole charge in

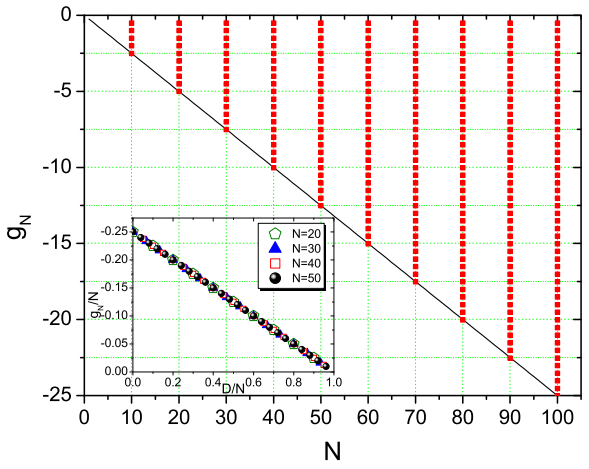

FIG. 3. (Color online) Monopole charge of the ground state in the model (2) as a function of the total particle number N with $\chi = 0$. For each N, the points shown from bottom to top denote the cases $D = 0,2,\ldots$, and $N - 2$, respectively. The inset shows the rescaled charge as a function of the rescaled population imbalance between the two atomic species.

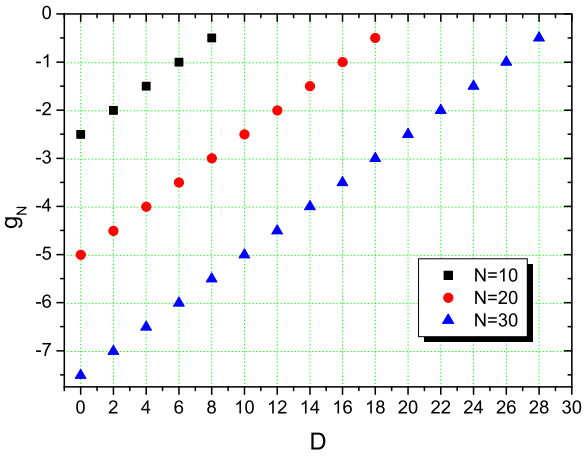

FIG. 4. (Color online) Monopole charge of the ground state in the model (2) as a function of the population imbalance parameter D with $\chi = 1$.

our system is not equal to the elementary charge $g_0 = \frac{1}{2}$. In the special case $D = 0$, the charge $g_N = -\frac{N}{4} = -\frac{N}{2}g_0$. In the general case $D \neq 0$, the charge of the monopole decreases smoothly from $-\frac{N}{2}g_0$ to $-g_0$ as the atom-number imbalance between two atomic species increases. The general formula for the monopole charge of the ground state in our system is given by

$$g_N = -\frac{N-D}{4} = -\frac{N-D}{2}g_0, \tag{5}$$

which is determined by the property of $(\frac{N-D}{2}+1)$-fold degeneracy of the ground states and the degenerate states are $|\frac{N+D-2m}{2}, \frac{N-D-2m}{2}, m\rangle$ with $m = 0,1,\ldots,\frac{N-D}{2}$. This result implies that one can create a magnetic monopole with nonelementary monopole charge in an atom–heteronuclear-molecule system. Notice that the total charge of the monopole for the ground state in the second-quantized model is $Q = 4\pi g_N = -(N-D)\pi$ and the Chern number is $-\frac{N-D}{2}$. We find that the Chern number is also an integer in our system, which is similar to the result shown in an interacting boson system [33].

When $\chi \neq 0$, we choose a closed surface that encloses all $\frac{N-D}{2}$ degenerate points and the monopole charge of the ground state for $\chi = 1$ is obtained as shown in Fig. 4. For $D = 0$, the number of the degenerate points is $\frac{N}{2}$ and the charge is $-\frac{N}{4}$. For $D \neq 0$, the degenerate points reduce to $\frac{N-D}{2}$ and the charge becomes $-\frac{N-D}{4}$. This implies that the result (5) obtained for $\chi = 0$ is also appropriate for the case $\chi \neq 0$. Obviously, the total charge of the monopoles for the ground state is also $-(N-D)\pi$ and the Chern number is also $-\frac{N-D}{2}$. In fact, the quantum ground state is doubly degenerate at $\frac{N-D}{2}$ points in this case. At each degenerate point, the two degenerate eigenstates are $|\frac{N+D-n+1}{2}, \frac{N-D-n+1}{2}, \frac{n-1}{2}\rangle$ and $|\frac{N+D-n-1}{2}, \frac{N-D-n-1}{2}, \frac{n+1}{2}\rangle$.

IV. MEAN-FIELD ANALYSIS

In the mean-field limit, i.e., $N \to \infty$, the atom-molecule system becomes classical [34] and can be well described by

the following semiclassical Hamiltonian [28,35]:

$$H_c = \lim_{N\to\infty} \frac{\langle \hat{H}\rangle}{N} = Z|a_m|^2 + \chi|a_m|^4 + [(X+iY)a_1^* a_2^* a_m + \text{c.c.}], \quad (6)$$

where a_j are complex amplitudes for the system in the three quantum modes and the normalized condition is given by $|a_1|^2 + |a_2|^2 + 2|a_m| = 1$.

It is noted that the mean-field model (6) does not admit the U(1) gauge transformation. This is an especially interesting point about our atom-molecule system. In fact, this model is invariant under the gauge transformation as follows:

$$|\psi\rangle \to U(\phi_1,\phi_2)|\psi\rangle = e^{i\Theta(\phi_1,\phi_2)}|\psi\rangle, \quad (7)$$

where

$$\Theta(\phi_1,\phi_2) = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_1+\phi_2 \end{pmatrix}. \quad (8)$$

This type of gauge transformation includes two nonidentical phase parameters (i.e., ϕ_1 and ϕ_2), which is different from the so-called skewed U(1) gauge transformation introduced in Refs. [21,22] where the transformation only depends on one parameter. However, we notice that the nonlinear model (6) has a classical Hamiltonian structure (i.e., we can introduce three pairs of conjugate variables through $p_j = \sqrt{i}a_j^*$ and $q_j = \sqrt{i}a_j$). Following Refs. [22,36], for our system we can define the Mead-Berry connection $\boldsymbol{A}$ and see its general properties by making a gauge transformation as follows:

$$\begin{aligned} \boldsymbol{A} = i\overline{\langle\psi|\nabla_{\boldsymbol{R}}|\psi\rangle} &\to \boldsymbol{A}' = i\overline{\langle\psi'|\nabla_{\boldsymbol{R}}|\psi'\rangle} \\ &= \boldsymbol{A} - (|a_1|^2 + |a_2|^2 + 2|a_m|^2)\nabla_{\boldsymbol{R}}\phi_1 \\ &\quad + \tfrac{1}{2}[1 - (|a_1|^2 - |a_2|^2)]\nabla_{\boldsymbol{R}}(\phi_1 - \phi_2), \end{aligned} \quad (9)$$

where $|\psi'\rangle = U(\phi_1,\phi_2)|\psi\rangle$ and the overbar indicates an average over all initial angles with the same actions. It is found that the second term and the last term in Eq. (9) are trivial total derivatives due to the conservation of the total particle number and the constant particle-number imbalance between two atomic species (i.e., $d = |a_1|^2 - |a_2|^2$). This result implies that the Mead-Berry connection defined above is gauge invariant under the transformation $U(\phi_1,\phi_2)$. For an instantaneous eigenstate, the unnecessary averaging operation can be safely neglected. This is because the population probabilities of different eigenstates are just the classical actions, which are found to be the adiabatic constants in an adiabatic evolution [37].

From the definition (9), one can calculate the mean-field curvature $\boldsymbol{B}$ and the monopole charge g for the ground state through $\boldsymbol{B} = \nabla_{\boldsymbol{R}} \times \boldsymbol{A}$ and its closed surface integral, respectively. Actually, the mean-field and quantum virtual fields satisfy the relation $\lim_{N\to\infty}(\frac{\boldsymbol{B}_N}{N} - \boldsymbol{B}) = 0$, which can be proved from the fact that H_c is the semiclassical limit of the second-quantized model $\hat{H}$ (for details, see Refs. [22,33]) and by following Berry's argument about the semiclassical connection between the Berry phase and Hannay angle [38,39]. Our numerical simulations have shown that this relation indeed holds everywhere in the parameter space except at the degenerate points. For the monopole charge, we have

$$\lim_{N\to\infty}\left(\frac{g_N}{N} - g\right) = 0, \quad (10)$$

with

$$g = \lim_{N\to\infty}\frac{g_N}{N} = -\frac{1}{4}(1-d) = -\frac{1}{2}(1-d)g_0, \quad (11)$$

where the population-imbalance parameter $d = \lim_{N\to\infty}\frac{D}{N}$. Equation (11) implies that the ground-state monopole in our system in the mean-field limit can carry an arbitrary charge ranging from zero to $-\frac{1}{4}(1-d)$ (i.e., the discrete points in Figs. 3 and 4 will be connected to form continuous lines).

V. DISCUSSION AND CONCLUSION

Before concluding, we present some discussion. When $D = 0$, our system reduces to the homonuclear molecule system. In this case, the system exhibits a QPT from a mixture phase to a pure molecule phase at the critical point $Z_c = -\sqrt{2(X^2+Y^2)}$ [17]. To show the connection of the virtual field to the QPT, we plot the contour lines of the magnitude $(|\boldsymbol{B}_{N+2}| - |\boldsymbol{B}_N|)$ for the ground state in Fig. 5. We find that the contours with the same values for different particle numbers cross at the phase transition point. This result is independent of the system size, which implies that, even though a QPT is only rigorously defined in the thermodynamic limit $N \to \infty$, the virtual field of the monopole does exactly mark the changes in the ground states of the system for a finite particle-number case [23]. However, in the case of $D \neq 0$, there is no QPT even in the semiclassical limit [28].

In summary, we have investigated the ground-state monopoles for a three-level atom-molecule system. Our study mainly focuses on the virtual field and the charge

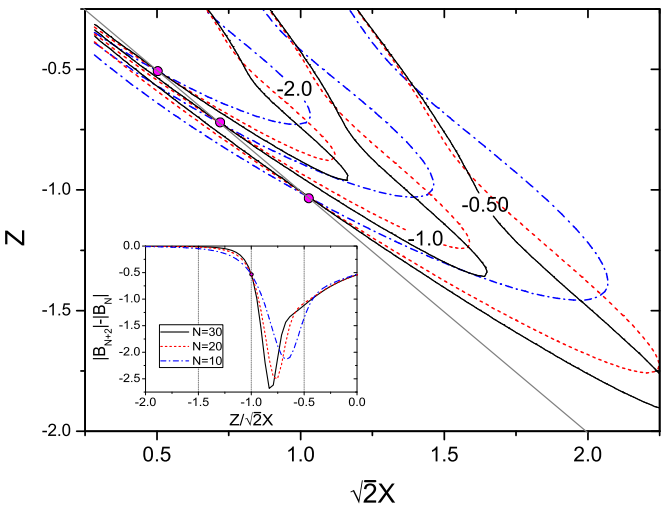

FIG. 5. (Color online) Contour lines of $(|\boldsymbol{B}_{N+2}| - |\boldsymbol{B}_N|)$ for the ground state at $Y = 0$ with $\chi = D = 0$. Each set of contour lines with same value includes the data with $N = 10$ (blue dash-dotted line), $N = 20$ (red dashed line), and $N = 30$ (black solid line). Solid circles indicate the intersection points. The inset shows the dependence of $(|\boldsymbol{B}_{N+2}| - |\boldsymbol{B}_N|)$ on the parameter $Z/\sqrt{2}X$ with different N. The gray straight line is plotted as a guide for the eyes with the slope being -1.

of the monopoles. When the particle interaction is absent, the calculation shows that in our system there only exists a pointlike monopole which is located at the origin in the parameter space. However, the virtual field does not have spherical symmetry and its structure is determined by the atom-number imbalance parameter D. The charge of the monopole is found to be $-\frac{N-D}{4}$, which is not equal to the elementary charge g_0. When we consider the particle interaction, the ground state of the system exhibits a pointlike monopole chain with $\frac{N-D}{2}$ monopoles. The charge for each monopole is $-g_0$ and therefore the charge of the monopole chain is $-\frac{N-D}{2}g_0$. Moreover, we find that the Chern number in our system is also an integer. In the mean-field limit, our analysis illustrates that the monopole charge can be arbitrary ranging from $-\frac{1}{4}(1-d)$ to zero.

In fact, the degeneracies of the spectrum in the parameter space are the singularities of the virtual field, and therefore the monopoles play an important role in connection with the geometric phase. Each degeneracy can be seen as a charge distribution located at the contact point between energy surfaces. Because the eigenstates are smooth and single valued outside the degeneracies, the monopole charge is necessarily an integer multiple of the elementary charge g_0. In the generic case of a diabolical contact [2], the monopole charges are precisely $\pm g_0$. However, higher integer multiples of g_0 may occur [40]. For instance, for light propagating through a twisted anisotropic dielectric medium there are experimental situations [41] where the monopole charges are $\pm 2g_0$. Our present study provides a perfect example for higher integer multiples of g_0 in ultracold atom-molecule systems. We emphasize that even though our results are obtained with a specific three-level boson model. The results are expected to hold in a general interacting atom–heteronuclear-molecule boson system where many heteronuclear molecules in high vibrational states will be included. The reason is that our system is the simplest atom–heteronuclear-molecule system in which we only consider the zero-temperature homogeneous case so that all the bosons are condensed into zero center-of-mass momentum states. Particularly, the monopole as the degeneracy point in the Brillouin zone was found to play a pivotal role in the anomalous Hall effect [13]. This fact implies that it may be possible to observe our predicted virtual monopole fields in experiments by setting up a bosonic Hall system with a molecular BEC in a rotating optical lattice.

ACKNOWLEDGMENTS

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[1] C. A. Mead and D. G. Truhlar, J. Chem. Phys. **70**, 2284 (1979).
[2] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
[3] C. F. Roos, M. Chwalla, K. Kim, M. Riebe, and R. Blatt, Nature (London) **443**, 316 (2006).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[5] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) **403**, 869 (2000).
[6] Q. Niu, X. Wang, L. Kleinman, W.-M. Liu, D. M. C. Nicholson, and G. M. Stocks, Phys. Rev. Lett. **83**, 207 (1999).
[7] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[8] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanziger, *The Geometric Phase in Quantum Systems: Foundations, Mathematical Concepts and Applications in Molecular and Condensed Matter Physics* (Springer-Verlag, New York, 2003).
[9] D. Xiao, M.-C. Zhang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
[10] R. Resta, J. Phys.: Condens. Matter **12**, R107 (2000).
[11] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931).
[12] J. Avron, L. Sadun, J. Segert, and B. Simon, Commun. Math. Phys. **124**, 595 (1989).
[13] Z. Fang, N. Nagaosa, K. S. Takahashi, A. Asamitsu, R. Mathieu, T. Ogasawara, H. Yamada, M. Kawasaki, Y. Tokura, and K. Terakura, Science **302**, 92 (2003).
[14] T. Köhler, K. Góral, and P. S. Julienne, Rev. Mod. Phys. **78**, 1311 (2006).
[15] E. A. Donley, N. R. Claussen, S. T. Thompson, and C. E. Wieman, Nature (London) **417**, 529 (2002).
[16] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).
[17] G. Santos, A. Foerster, J. Links, E. Mattei, and S. R. Dahmen, Phys. Rev. A **81**, 063621 (2010).
[18] Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[19] S.-Y. Meng, L.-B. Fu, and J. Liu, Phys. Rev. A **78**, 053410 (2008).
[20] J. Liu and L. B. Fu, Phys. Rev. A **81**, 052112 (2010).
[21] L. B. Fu and J. Liu, Ann. Phys. **325**, 2425 (2010).
[22] F. Cui and B. Wu, Phys. Rev. A **84**, 024101 (2011).
[23] S. C. Li and L. B. Fu, Phys. Rev. A **84**, 023605 (2011).
[24] S. C. Li, L. B. Fu, and F. L. Li, Phys. Rev. A **88**, 013602 (2013).
[25] S. B. Papp and C. E. Wieman, Phys. Rev. Lett. **97**, 180404 (2006).
[26] A. Nunnenkamp, D. Meiser, and P. Meystre, New J. Phys. **8**, 88 (2006).
[27] B. Damski, L. Santos, E. Tiemann, M. Lewenstein, S. Kotochigova, P. Julienne, and P. Zoller, Phys. Rev. Lett. **90**, 110401 (2003).
[28] L. Zhou, W. Zhang, H. Y. Ling, L. Jiang, and H. Pu, Phys. Rev. A **75**, 043603 (2007).
[29] G. Drobný, I. Jex, and V. Bužek, Phys. Rev. A **48**, 569 (1993).
[30] Y. R. Shen, *The Principle of Nonlinear Optics* (Wiley Interscience, New York, 1984).

[31] A. Vardi, V. A. Yurovsky, and J. R. Anglin, Phys. Rev. A **64**, 063611 (2001).
[32] E. Pazy, I. Tikhonenkov, Y. B. Band, M. Fleischhauer, and A. Vardi, Phys. Rev. Lett. **95**, 170403 (2005); I. Tikhonenkov, E. Pazy, Y. B. Band, M. Fleischhauer, and A. Vardi, Phys. Rev. A **73**, 043605 (2006).
[33] B. Wu, Q. Zhang, and J. Liu, Phys. Lett. A **375**, 545 (2011).
[34] L. G. Yaffe, Rev. Mod. Phys. **54**, 407 (1982).
[35] S. C. Li, J. Phys. B: At. Mol. Opt. Phys. **43**, 205303 (2010).
[36] B. Wu, J. Liu, and Q. Niu, Phys. Rev. Lett. **94**, 140402 (2005).
[37] J. Liu, B. Wu, and Q. Niu, Phys. Rev. Lett. **90**, 170404 (2003).
[38] M. V. Berry, J. Phys. A **18**, 15 (1985).
[39] S. C. Li, J. Liu, and L. B. Fu, Phys. Rev. A **83**, 042107 (2011).
[40] P. Leboeuf and A. Mouchet, J. Phys. A: Math. Gen. **36**, 2847 (2003).
[41] M. V. Berry, in *Fundamental Aspects of Quantum Theory*, edited by V. Gorini and A. Frigerio, NATO ASI Series Vol. 144 (Plenum, New York, 1986), p. 267.