Towards loophole-free Bell inequality test with preselected unsymmetrical singlet states of light

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Can a Bell test with no detection loophole be demonstrated for multiphoton entangled states of light within the current technology? We examine the possibility of a postselection-free Clauser-Horne-Shimony-Holt (CHSH)-Bell inequality test with an unsymmetrical polarization singlet. To that end we employ a preselection procedure which is performed prior to the test. It allows using imperfect (coarse-grained) binary photodetection in the test. We show an example of a preselection scheme which improves violation of the CHSH inequality with the micro-macro polarization singlet produced by the optimal quantum cloning. The preselection is realized by a quantum filter which is believed not to be useful for this purpose.

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I. INTRODUCTION

Quantum-mechanical laws apply to single particles, complex molecules involving tens of atoms, as well as to living organisms [1–4]. Recently, the first optical almost-loophole-free Bell tests for a two-photon singlet, eliminating the famous detection loophole, have been performed by the Zeilinger and Kwiat groups [5,6]. Can this also be demonstrated for systems with higher mean photon numbers using the current technology? An indisputable Bell test [7,8] ultimately rejects the local realistic description of the world in favor of quantum mechanics. It is also of practical importance, allowing for implementation of quantum technology protocols such as device-independent quantum key distribution (QKD) [9], randomness generation [10], and reduction of communication complexity [11].

The detection loophole arises from inefficient (lossy) photodetection. The local realistic models do not necessarily satisfy the fair sampling assumption and they might exploit the postselection, i.e., discarding some of the experimental data, to mimic the violation of a Bell inequality. Closing the detection loophole for a two-photon singlet was possible due to employment of superconducting transition-edge sensors [12,13], quantum detectors with a near-perfect efficiency. However, if we examined states of light involving a large number of photons, elimination of this loophole would be more involved since the imperfect (coarse-grained) measurements come into play [14].

Quantum phenomena on the macroscopic scale have been intriguing and puzzling to physicists since the inception of quantum theory. Recently, macroscopically populated entangled states of light became available experimentally: the micro-macro polarization singlet [15], the entangled bright squeezed vacuum [16,17], and the displaced single-photon path-entangled state [18,19]. An important question regarding the possibility of performing a loophole-free Bell test [20,21] for these states has been posed. The probability of a no-detection event for these states is very low. This property gives hope to close the detection loophole. In Ref. [21] we showed that if the postselection is simply omitted, the micro-macro polarization state fails to pass the Bell test with efficient coarse-grained (binary) analog detection, although the loophole is closed. We also emphasized that preselection can solve this problem (it improves the visibility (distinguishability) of the multiphoton qubit in analog detection [22]), but we did not provide any example to support our claim. However, the considerations in [23] contradict this statement: the authors showed that all preselections tested thus far are not useful for increasing the distinguishability of the micro-macro polarization singlet in analog detection. Additionally, the results in Refs. [24,25] emphasized the significance of the detection loophole for the test of macroscopic entanglement discussed in Ref. [15]: it was demonstrated that in the presence of this loophole, separable states may falsely reveal entanglement. Moreover, in Ref. [14] it was demonstrated that a single-photon resolution is essential in observing the micro-macro entanglement with photon counting measurements.

Here we examine a loophole(postselection)-free Clauser-Horne-Shimony-Holt (CHSH)-Bell inequality test with preselected unsymmetrical polarization singlet states of light of a general form and imperfect binary analog detection. We explicitly show an example of a preselection scheme which improves violation of the CHSH inequality with the micro-macro polarization singlet produced by optimal quantum cloning.

In the unsymmetrical singlets under consideration, one of the modes is occupied by a single photon (the microqubit) whereas the second one contains a pair of mutually orthogonal multiphoton states (the multiphoton qubit). We assume that the average photon number in the multiphoton qubit can be controlled by some external parameter in an experimental setup, and it may vary from a single photon to the macroscopic quantity of thousands of photons. Furthermore, we consider a Bell test based only on linear optical elements. The unsymmetrical singlet is prepared before the test by a special filtering procedure applied to the mode containing the multiphoton state. The filter is described by a positive operator valued measure (POVM). For example, it may be realized by

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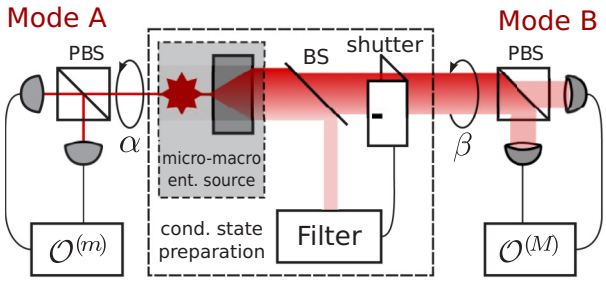

FIG. 1. (Color online) Bell inequality test involving a preselection strategy and un unsymmetrical singlet state of light.

the modulus of intensity difference filter [26,27]. Filtering belongs to the conditional state preparation, not to the test. Only if the state is successfully preselected is the Bell test performed, where every measurement outcome is conclusive and is taken into account. This eliminates the necessity of data postselection and closes the detection loophole.

This paper is organized as follows: In Sec. II we discuss a general scenario of the CHSH-Bell inequality test with preselection strategy for an unsymmetrical singlet. Section III is devoted to a short summary of the experimentally available unsymmetrical polarization singlets of light. We further discuss the CHSH inequality violation for these states preselected by the special case of the modulus of intensity difference filter, namely, the corner filter, in Sec. IV. Finally, we discuss the possible future steps towards genuine loophole-free Bell testing for states of light with a large photon population.

II. CHSH-BELL TEST WITH PRESELECTION

The Bell inequality test with an unsymmetrical singlet state and imperfect intensity measurements has to employ a preselection strategy [21,22]. The role of preselection is to prevent from deterioration the ability to witness quantum correlations in the singlet state, resulting from coarse-graining measurements. We call the singlet unsymmetrical if the dimensions of the Hilbert spaces corresponding to its modes are unequal.

Let us start our discussion with a two-mode unsymmetrical polarization singlet state of a general form

$$|\Psi^-\rangle = 1/\sqrt{2}(|1_\varphi\rangle_A|\bar{\Phi}\rangle_B - |1_{\varphi_\perp}\rangle_A|\Phi\rangle_B) \tag{1}$$

and an arbitrary preselection strategy executed by a POVM $\mathcal{P}$. The states $|1_\varphi\rangle$ and $|1_{\varphi_\perp}\rangle$ denote a microqubit, e.g., a single photon in polarization φ and $\varphi_\perp$, respectively, whereas $|\Phi\rangle$ and $|\bar{\Phi}\rangle$ are multiphoton states which constitute a multiphoton qubit. Of course, the two states of the qubits are pairwise orthogonal $\langle 1_\varphi|1_{\varphi_\perp}\rangle = 0$, $\langle\Phi|\bar{\Phi}\rangle = 0$.

A setup for the Bell test is depicted in Fig. 1. The multiphoton part of the state (mode B) impinges on a beam splitter (BS) with a low reflectivity r, e.g., 10%, which taps only a small fraction of the state, leaving it almost unaffected. Next, the preselection strategy is implemented by the analysis of the reflected part; it is examined by a filter described by a POVM $\mathcal{P}$ and the result is feed forwarded to the transmitted beam. This procedure belongs to a conditional state preparation before the test. After the successful preselection, the Bell test consists of polarization rotations, by the angles α, α' on mode A and β, β' on mode B, and intensity measurements of polarization components of both modes.

In general, the operator $\mathcal{P}$ may suffer from lack of the rotational invariance being an important property of the original singlet and thus, the form of the preselected state may be basis dependent. For example, for the modulus of intensity difference filter it is known that it improves the visibility of a multiphoton state for a measurement in one polarization basis but deteriorates for measurements in the other polarization bases. Thus, such filtering strategies are believed to be useless for preselection [23]. This problem arises if the usual settings for the CHSH-Bell inequality are considered: $\alpha = 0$, $\alpha' = \pi/2$, $\beta = -\pi/4$, $\beta' = \pi/4$. However, it does not need to be the case if nonorthogonal polarization directions in measurements on the multiphoton mode are chosen, at the expense of obtaining a nonmaximal Bell inequality violation. Moreover, the visibility is not the only factor contributing to the CHSH-Bell parameter computed for an unsymmetrical singlet. The other parameter is a quantity which we call antivisibility, and we explain its physical meaning below. In order to maximize the value of the Bell violation, the rotation angles on the multiphoton mode should optimize the two parameters simultaneously.

We consider the CHSH inequality with Bell parameter

$$B = E(\alpha,\beta) + E(\alpha,\beta') + E(\alpha',\beta) - E(\alpha',\beta'), \tag{2}$$

and $E(\alpha,\beta) = \langle\mathcal{O}^{(m)}(\alpha)\otimes\mathcal{O}^{(M)}(\beta)\rangle$ is the correlation function where one observer, Alice, measures the microscopic part (mode A) and the other, Bob, measures the multiphoton part of the singlet (mode B). We assume the ideal measurement operator $\mathcal{O}^{(m)}(\alpha) = |1_\alpha\rangle\langle 1_\alpha| - |1_{\alpha_\perp}\rangle\langle 1_{\alpha_\perp}|$ for the microscopic qubit. Rotating the polarization of the microscopic part by an angle α yields

$$|1_\alpha\rangle = \cos(\alpha/2)|1\rangle + \sin(\alpha/2)|1_\perp\rangle,$$
$$|1_{\alpha_\perp}\rangle = -\sin(\alpha/2)|1\rangle + \cos(\alpha/2)|1_\perp\rangle,$$

which allows expressing the micro-observable in terms of the projectors in the reference basis $\varphi = 0$:

$$\mathcal{O}^{(m)}(\alpha) = \cos\alpha(|1\rangle\langle 1| - |1_\perp\rangle\langle 1_\perp|) + \sin\alpha(|1\rangle\langle 1_\perp| + |1_\perp\rangle\langle 1|). \tag{3}$$

For the multiphoton mode we take the binary threshold detection operator $\mathcal{O}^{(M)}(\beta)$ adapted to the preselection strategy $\mathcal{P}$. The value $+1$ (-1) is assigned to this observable when the state $|\Phi\rangle$ $(|\bar{\Phi}\rangle)$ is identified. We assume it belongs to a class of diagonal observables such that $\mathrm{Tr}\{\mathcal{O}^{(M)}(\beta)\} = 0$. The general form of such an observable reads

$$\mathcal{O}^{(M)}(\beta) = \sum_{\substack{k,l=0\\ C(k,l)}}^{\infty} |k_\beta,l_{\beta_\perp}\rangle\langle k_\beta,l_{\beta_\perp}|, \tag{4}$$

where the condition $C(k,l)$ is such that it ensures the observable to be traceless.

After a short algebra, we obtain the correlation function for the state in Eq. (1),

$$E(\alpha,\beta) = -\cos\alpha\, V^\theta(\beta) - \sin\alpha\, A^\theta(\beta), \tag{5}$$

where $V^\theta(\beta) = \langle\Phi^\theta|\mathcal{O}^{(M)}(\beta)|\Phi^\theta\rangle$ is the visibility and $A^\theta(\beta) = \langle\bar{\Phi}^\theta|\mathcal{O}^{(M)}(\beta)|\Phi^\theta\rangle$ is called the antivisibility of the state Φ preselected in the polarization basis θ and observed in the polarization basis β. The antivisibility quantifies the ability of the observable $\mathcal{O}^{(M)}(\beta)$ to erase the information on which state of $|\bar{\Phi}_\theta\rangle$ or $|\Phi_\theta\rangle$ entered the detector. In this derivation, due to the condition $\mathrm{Tr}\{\mathcal{O}^{(M)}(\beta)\} = 0$, we noticed that $V_\perp^\theta(\beta) = \langle\bar{\Phi}^\theta|\mathcal{O}^{(M)}(\beta)|\bar{\Phi}^\theta\rangle = -V^\theta(\beta)$. Without a loss of generality, we also took that $A^\theta(\beta)$ is real valued. After inserting Eq. (5) into Eq. (2) we obtain

$$\begin{aligned} B = &-\cos\alpha\,[V^\theta(\beta) + V^\theta(\beta')] - \sin\alpha\,[A^\theta(\beta) + A^\theta(\beta')] \\ &- \cos\alpha'\,[V^\theta(\beta) - V^\theta(\beta')] - \sin\alpha'\,[A^\theta(\beta) - A^\theta(\beta')]. \end{aligned} \tag{6}$$

We will first consider the following rotation angles $\theta = 0$ for preselection, and $\alpha = 0$, $\alpha' = \pi/2$, $\beta' = -\beta$ for the Bell test. This choice is quite natural, because in the limit of a small photon population in mode B, i.e., for a two-photon singlet, these are the optimal angles maximizing the value of the CHSH-Bell parameter with $\beta = -\pi/4$. Later, we will show that for the specific examples we examined it is also the optimal set of angles even for amplified mode B; however, the optimal value of β (β_{opt}) changes with the mean number of photons in the multiphoton qubit in the presence of preselection. In this case, the Bell parameter reads

$$B_{\mathrm{opt}} = -[V(\beta_{\mathrm{opt}}) + V(-\beta_{\mathrm{opt}}) + A(\beta_{\mathrm{opt}}) - A(-\beta_{\mathrm{opt}})].$$

Due to the diagonal form of the multiphoton observable, the visibility and antivisibility can be expressed as a convex sum of contributions resulting from various photon-number sectors

$$V(\beta) = \sum_{k=0}^{\infty} \beta_k^2\, V_k(\beta), \qquad A(\beta) = \sum_{k=0}^{\infty} \beta_k^2\, A_k(\beta), \tag{7}$$

where $V_k(\beta)$ and $A_k(\beta)$ are computed for the kth photon-number sector of a multiphoton qubit. (We can always decompose multiphoton states in the Fock basis as follows: $|\Phi\rangle = \sum_{k=0}^{\infty} \beta_k\,|\Phi_k\rangle$ with $|\Phi_k\rangle = \sum_{j=0}^{k} \xi_{k,j}|k-j, j_\perp\rangle$, where β_k and $\xi_{k,j}$ are certain probability amplitudes.) Thus, a similar decomposition holds true for the Bell parameter:

$$\begin{aligned} B &= \sum_{k=0}^{\infty} \beta_k^2\, B_k, \\ B_k &= -[V_k(\beta_{\mathrm{opt}}) + V_k(-\beta_{\mathrm{opt}}) + A_k(\beta_{\mathrm{opt}}) - A_k(-\beta_{\mathrm{opt}})]. \end{aligned} \tag{8}$$

Decompositions in Eqs. (7) and (8) give insight into the contribution of each sector separately by taking into account the structure of the multiphoton qubit. Due to this, we know which photon numbers most often lead to the Bell violation and which deteriorate it.

The above formula may be further simplified by noticing that when $\xi_{k,j}$ and $\bar{\xi}_{k,j}$ fulfill additional conditions, e.g., $\xi_{k,j} = 0$ and $\bar{\xi}_{k,j} \neq 0$ for odd j but $\xi_{k,j} \neq 0$ and $\bar{\xi}_{k,j} = 0$ for even j, then $V_k(\beta) = V_k(-\beta)$ and $A_k(\beta) = -A_k(-\beta)$ (see Appendix A). In this case it is possible to write Eq. (8) as $B_k = -2[V_k(\beta_{\mathrm{opt}}) + A_k(\beta_{\mathrm{opt}})]$.

III. EXAMPLE: UNSYMMETRICAL POLARIZATION SINGLET STATES OF LIGHT EMERGING FROM PHASE-COVARIANT QUANTUM CLONER

In this section we will discuss a specific example of the experimentally available unsymmetrical polarization singlet states of light. They are produced in the process of the phase-covariant optimal quantum cloning. It is based on phase-sensitive parametric amplification [15,24,28] and requires a pair of linearly polarized photons in a standard singlet state, obtained through parametric down conversion, as an input. The single-photon seeding is coherently amplified to produce a multiphoton state by an intensely pumped high-gain g nonlinear medium (the cloner). The equatorial states of the Poincaré sphere of all polarization states, parametrized by the polar angle $\varphi \in \langle 0, 2\pi)$, are privileged for the phase-covariant cloners, only for this subspace their Hamiltonian $H = \frac{i\chi}{2}(a_\varphi^{\dagger 2} + a_{\varphi_\perp}^{\dagger\ 2}) + \mathrm{H.c.}$ is rotationally invariant and they work equally well for all the equatorial states. The operators $a_\varphi^\dagger$ and $a_{\varphi_\perp}^\dagger$ denote the creation operators for the equatorial polarization modes φ and $\varphi_\perp$, respectively, and χ is the coupling strength, proportional to the pumping power. We restrict ourselves to the equatorial polarization state subspace for the seeding photon. The subspace basis is set by two states, $|1_\varphi\rangle = 1/\sqrt{2}(|1_H\rangle + e^{i\varphi}|1_V\rangle)$ and its orthogonal counterpart $|1_{\varphi_\perp}\rangle$, where $|k_H\rangle$ ($|l_V\rangle$) denote k (l) photons polarized horizontally (vertically) and $\varphi_\perp = \varphi + \pi$. Due to its rotational invariance, we express the initial singlet in this basis: $|\psi^-\rangle = 1/\sqrt{2}(|1_\varphi\rangle_A|1_{\varphi_\perp}\rangle_B - |1_{\varphi_\perp}\rangle_A|1_\varphi\rangle_B)$. Cloning is a unitary process, and the original two-photon entanglement is transferred to the unsymmetrical singlet with

$$\begin{aligned} |\Phi\rangle &= \sum_{i,j=0}^{\infty} \gamma_{ij}|(2i+1)_\varphi, (2j)_{\varphi_\perp}\rangle, \\ |\bar{\Phi}\rangle &= \sum_{i,j=0}^{\infty} \gamma_{ij}|(2j)_\varphi, (2i+1)_{\varphi_\perp}\rangle. \end{aligned} \tag{9}$$

$|\Phi\rangle$ and $|\bar{\Phi}\rangle$ are the amplified single photons, with the real-valued probability amplitude $\gamma_{ij} = \cosh^{-2} g(\tanh g/2)^{i+j}\sqrt{(1+2i)!(2j)!}/i!/j!$, where g is the parametric gain. In the experiment, their average population equals $4\sinh^2 g + 1$, varied from less than one up to 10^4 of photons. Due to different parity of the occupation number in the Fock-state basis, $|\Phi\rangle$ and $|\bar{\Phi}\rangle$ are orthogonal. However, in the high-photon-number regime, photon detectors are not single-photon resolving [28] and visibility of the multiphoton states is quite low [29,30], making them inapplicable for quantum protocols and Bell inequality testing.

A. Quantum filtering

Visibility of the multiphoton qubit can be improved by quantum-state filtering performed by certain POVM filters. They modify the state but preserve quantum superpositions. Recently, such a filter has been proposed [26]: the modulus of intensity difference filter selects two-mode states of light whose mode populations differ by more than a certain threshold δ_{th}. It estimates the absolute value of difference

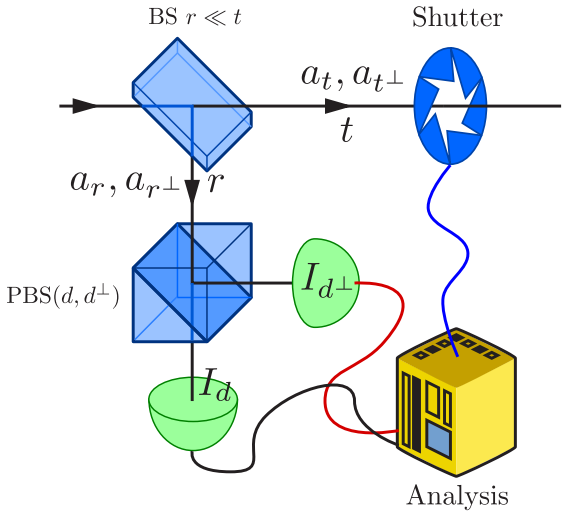

FIG. 2. (Color online) Realization and application of the modulus of intensity difference and other quantum filters.

instead of the difference. It does not provide any information on which polarization mode was more populated, and thus it is not able to distinguish the multiphoton states and preserves superpositions. It performs an almost nondestructive measurement implemented by the tapping and feed-forward loop, i.e., the filtered output state is almost pure. Qualitatively, it approximates the following operation:

$$\mathcal{P}_{\mathrm{MDF}} = \sum_{\substack{k,l=0 \\ |k-l| \geqslant \delta_{\mathrm{th}}}}^{\infty} |k,l\rangle\langle k,l|, \tag{10}$$

where $|k,l\rangle$ is a polarization two-mode Fock state. Below, we will briefly discuss its physical implementation and the principle of operation. More detailed discussion, including action of the filter on multiphoton states from Eq. (9), is given in [26], and a description of the first attempt of its experimental realization can be found in [27].

The approximate realization and application of the quantum filter is presented in Fig. 2. The state to be filtered enters a feed-forward loop. It impinges on a highly unbalanced beam splitter with a very small reflectivity r which splits the state into the reflected a_r ($a_{r\perp}$) and transmitted modes a_t ($a_{t\perp}$). The reflected mode is examined by the quantum filter. Since the reflected and transmitted beams are correlated, estimating the modulus of the population difference for the former gives an estimate for the latter. Depending on the result of this analysis, the transmitted mode is either passed or blocked by the shutter. In this way, it is possible to block light of unwanted properties. The quantum filter consists of a polarization beam splitter (PBS), which works in a basis unbiased with respect to the polarization basis of the incoming field [$a_{d(\perp)} = 1/\sqrt{2}(a_r \pm a_{r\perp})$], and the photon counting detectors. These may be the superconducting transition-edge sensors (TESs) with a very high quantum efficiency of ca. 95% [12,13]. In the up-to-date experiments, they achieved an overall efficiency of 75% [5,6]. This result can be improved by using the integrated optics setups. These detectors posses a single-photon resolution in the range of ca. 0–23 photons. This makes quantum filtering of the macroscopically populated superpositions of light experimentally feasible, taking into account that the population of the reflected mode may constitute, say, 1–10% of the total population of the incoming field. In the higher photon number range the detectors may also work quite well for filtering purposes, since the relative error of the measurement (the uncertainty in photon counting compared to the incoming population) is pretty small: for 1000 incoming photons it is ca. 30 photons.

The key property of the filter is that the more unequally populated a two-mode Fock state entering the PBS, the more equally populated are the output modes (and vice versa). This effect is especially pronounced for highly populated states. It allows one to estimate the population difference in the reflected mode with high probability, and consequently, in the transmitted mode as well.

IV. CHSH-BELL TEST FOR THE UNSYMMETRICAL POLARIZATION SINGLET EMERGING FROM PHASE-COVARIANT QUANTUM CLONER

In order to investigate the Bell inequality violation for the unsymmetrical singlet described in Sec. III, we rewrite the multiphoton states given in Eq. (9) as superpositions of states of a fixed photon number $2k+1$ distributed over the two polarization modes:

$$|\Phi\rangle = \sum_{k=0}^{\infty} \beta_k\, |\Phi_k\rangle, |\Phi_k\rangle = \frac{1}{\sqrt{\mathcal{N}_k}} (a^{\dagger 2} + a_\perp^{\dagger\, 2})^k a^\dagger\, |0\rangle,$$
$$|\bar{\Phi}\rangle = \sum_{k=0}^{\infty} \beta_k |\bar{\Phi}_k\rangle, |\bar{\Phi}_k\rangle = \frac{1}{\sqrt{\mathcal{N}_k}} (a^{\dagger 2} + a_\perp^{\dagger\, 2})^k\, a_\perp^\dagger\, |0\rangle, \tag{11}$$

where we took $\varphi = 0$, $\mathcal{N}_k = 4^k\, k!^2\, (1+k)$,

$$\beta_k = \cosh^{-2} g\, (\tanh g)^k \sqrt{1+k}, \quad \sum_{k=0}^{\infty} \beta_k^2 = 1. \tag{12}$$

We note that the visibility and antivisibility have the following symmetry properties for states from Eq. (11): $V(\beta) = V(-\beta)$, $A(\beta) = -A(-\beta)$, see Eqs. (16) and (17), and Appendix A. They allow us to simplify the Bell parameter to the following form:

$$B = -2[V(\beta_{\mathrm{opt}}) + A(\beta_{\mathrm{opt}})]. \tag{13}$$

It depends on two factors: the visibility $V(\beta_{\mathrm{opt}})$ and the antivisibility $A(\beta_{\mathrm{opt}})$ measured in the same basis, rotated by β_{opt} with respect to the reference. A violation of Bell's inequality is obtained if $|V(\beta_{\mathrm{opt}}) + A(\beta_{\mathrm{opt}})| > 1$. We emphasize that this result holds true for any preselection strategy applied to states in Eq. (11):

$$B^P = -2[V^P(\beta_{\mathrm{opt}}) + A^P(\beta_{\mathrm{opt}})], \tag{14}$$

where the index P denotes the quantities evaluated for the preselected multiphoton states.

A. Bell test without preselection

In order to analyze the difficulties with Bell inequality violation we adapt the following observable:

$$\mathcal{O}^{(M)}(\beta) = \left(\sum_{\substack{k,l=0 \\ k-l \geqslant 0}}^{\infty} - \sum_{\substack{k,l=0 \\ k-l<0}}^{\infty} \right) |k_\beta, l_{\beta_\perp}\rangle\langle k_\beta, l_{\beta_\perp}|. \tag{15}$$

This is a diagonal, traceless operator. It is well suited to the photon number distribution of the multiphoton qubit in Eq. (11): $|\Phi\rangle$ and $|\bar{\Phi}\rangle$ have unequal average population in the two polarization modes. The mean photon number in polarization φ is three times larger than the mean number of photons in polarization $\varphi_\perp$ in $|\Phi\rangle$ (the opposite relation holds true for $|\bar{\Phi}\rangle$) [31]. This property allows us to distinguish these states in analog detection. Possible implementation of the measurement of the observable $\mathcal{O}^{(M)}(\beta)$ would require splitting the two-mode polarization multiphoton beam by a polarization beam splitter followed by TES detection. We would assign $+1$ to the measurement outcome if the signal difference between polarizations β and $\beta_\perp$ was positive and -1 if it was negative. The detectors composed of PIN diodes followed by charge-sensitive amplifiers, such as those used in [32] for measuring uncertainties of the Stokes variables for a macroscopically populated squeezed vacuum, should also be sufficient for adequate implementation of this measurement.

The visibility and antivisibility evaluated for this observable equal

$$V_k(\beta) = \frac{1}{\mathcal{N}_k}\left[\sum_{u-w\geqslant 0} - \sum_{u-w<0}\right]\frac{\delta_{u+w,2k+1}}{u!\,w!}\left\{\sum_{j=0}^{k}\binom{k}{j}(2j+1)!\,(2k-2j)!\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\right.$$
$$\left.\times\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k-2j,m+n}\right\}^2, \tag{16}$$

$$A_k(\beta) = \frac{1}{\mathcal{N}_k}\left[\sum_{u-w>0} - \sum_{u-w<0}\right]\frac{\delta_{u+w,2k+1}}{u!\,w!}\left\{\sum_{j=0}^{k}\binom{k}{j}(2j+1)!\,(2k-2j)!\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\right.$$
$$\left.\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k-2j,m+n}\right\}\cdot\left\{\sum_{j=0}^{k}\binom{k}{j}(2j)!\,(2k+1-2j)!\right.$$
$$\left.\times\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k+1-2j,m+n}\right\}, \tag{17}$$

where $\delta_{i,j}$ denotes Kronecker's δ, equal to 1 when $i = j$ and 0 otherwise.

We now look for β_{opt}, which maximizes the value of the CHSH-Bell parameter in Eq. (13). We numerically checked that regardless of the mean number of photons in the multiphoton state, $\beta_{\text{opt}} = -\pi/4$.

In Fig. 3 we depicted the visibility, the antivisibility, and the Bell parameter for each $2k+1$-photon-number sector separately, computed for the optimal set of angles. We note that V_k and A_k both quickly tend to the value less than $1/2$ for increasing k and the Bell parameter B_k drops below 2 for $k \geqslant 3$. The probability distribution β_k^2 depends on the amplification gain g and gives a weight to each B_k contributing to the Bell parameter B. Since the greatest weight is given to the small values of k (see Fig. 4), it is possible to weakly violate the CHSH-Bell inequality for a very small average photon population, so for very small g. Note that the considered state approaches the Bell singlet state in the limit $g \to 0$. For $g = 0.8$ (total mean number of photons equals 4.15), $B = 2.06$, but for $g = 1.1$ (total mean number of photons equals 8.13), we found $B = 2.01$. Thus, a Bell inequality violation for the multiphoton unsymmetrical singlet with the multiphoton qubit given in Eq. (11) and the population of a few photons already, on average, is difficult to detect.

B. Bell test with a preselection strategy followed by imperfect binary detection

A preselection strategy described by a POVM of the form

$$\mathcal{P}_{\text{C}} = \sum_{\substack{k,l=0 \\ \text{C}(\sigma=k+l,\Delta=l-k)}}^{\infty} |k,l\rangle\langle k,l|, \tag{18}$$

with a general preselection condition C on σ and Δ, insignificantly influences the form of the convex sum in which the Bell parameter is expressed in Eq. (7). The detailed calculations are presented in Appendix B. Assuming that the condition $\text{C}(\sigma,\Delta)$ is symmetric, i.e., $\text{C}(\sigma,\Delta) = \text{C}(\sigma,-\Delta)$, we show that the preselection modifies both multiphoton states from Eq. (11) in the same way:

$$|\Phi_k^P\rangle = \frac{1}{\sqrt{\mathcal{N}_k^P}}\mathcal{P}_{\text{C}}|\Phi_k\rangle,\ |\bar{\Phi}_k^P\rangle = \frac{1}{\sqrt{\mathcal{N}_k^P}}\mathcal{P}_{\text{C}}|\bar{\Phi}_k\rangle,$$
$$|\Phi^P\rangle = \sum_{k=0}^{\infty}\beta_k^P|\Phi_k^P\rangle,\ |\bar{\Phi}^P\rangle = \sum_{k=0}^{\infty}\beta_k^P|\bar{\Phi}_k^P\rangle, \tag{19}$$

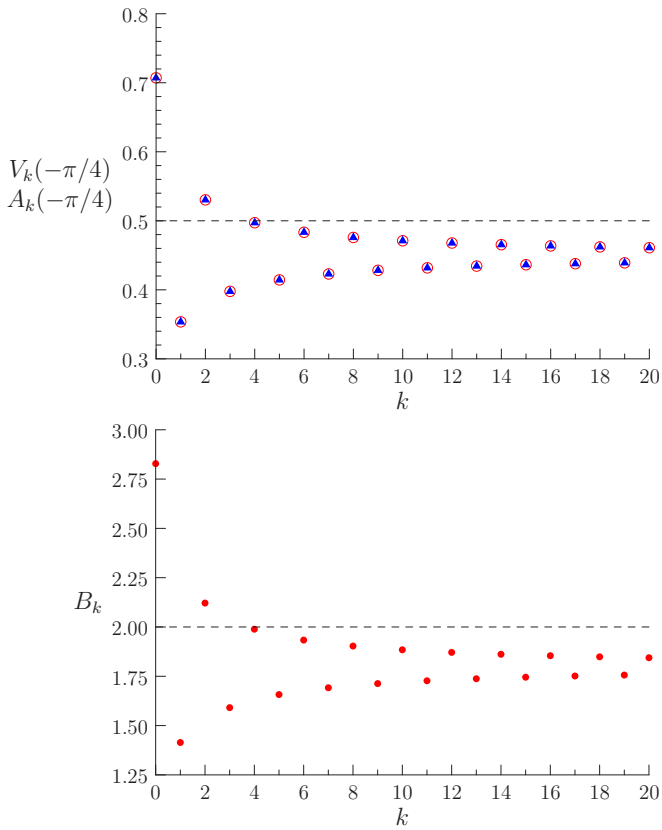

FIG. 3. (Color online) Plot of the visibility V_k from Eq. (16) (top figure, red circles), antivisibility A_k from Eq. (17) (top figure, blue triangles), and the Bell parameter B_k from Eq. (8) (bottom figure) computed for $\beta_{\rm opt} = -\pi/4$ as a function of k corresponding to the $(2k+1)$-photon-number sector of the multiphoton states $|\Phi_k\rangle$ and $|\bar{\Phi}_k\rangle$ given in Eq. (11).

where $\mathcal{N}_k^P$ [Eq. (B3)] is the new normalization constant and β_k^P [Eq. (B6)] is the new probability amplitude in the decomposition of the states into the photon-number sectors.

We would like to mention that although preselection may enable application of coarse-graining measurements in observing quantum effects in the multiphoton superpositions by increasing their visibility, e.g., in the Bell test, it will not be a direct remedy to the deteriorating effect of losses in an experimental setup. Preselection helps in conditional state generation. Thus, the robustness of the new (preselected) state will determine the robustness of the whole Bell test against losses.

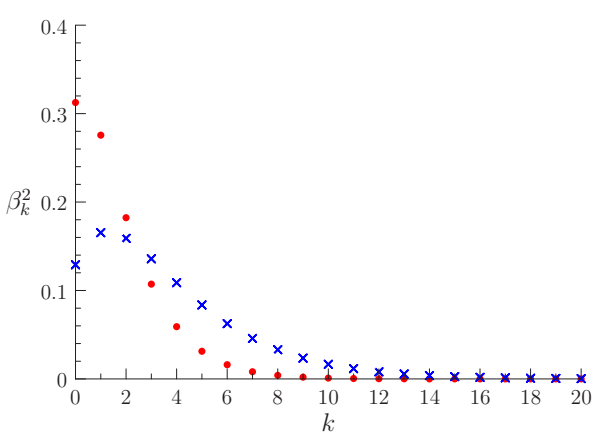

FIG. 4. (Color online) Plot of β_k^2 defined in Eq. (12) for $g = 0.8$ (red dots) and $g = 1.1$ (blue crosses).

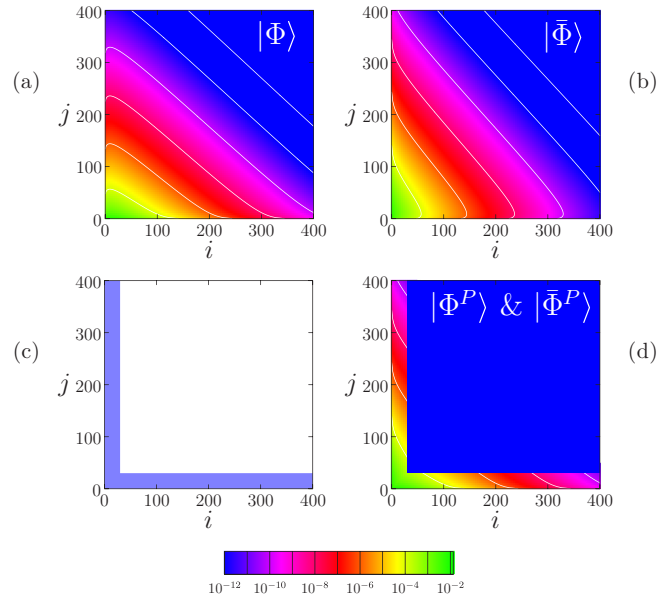

FIG. 5. (Color online) Photon number distribution of multiphoton states (a) $|\Phi\rangle$, (b) $|\bar{\Phi}\rangle$ before and after preselection (d) $|\Phi^P\rangle$ and $|\bar{\Phi}^P\rangle$ with the corner filter. The distributions are evaluated for $g = 1.1$ and $\delta_{\rm th} = 30$. Action of the filter in the photon number space is shown in (c); only the components of filtered superposition which belong to the blue area are preserved.

C. Example: The corner filter

We will now examine the preselection procedure $\mathcal{P}_C$ for the following preselection condition $\mathrm{C}(\sigma,\Delta)$: $\sigma - \delta_{\rm th} \leqslant |\Delta|$, where $\delta_{\rm th}$ is a threshold. This condition means that those components of the multiphoton superpositions are selected whose polarization mode's population difference is higher than the population sum reduced by $\delta_{\rm th}$. We call this kind of filtering the corner filter, since the analysis of the modification of a state in terms of its photon number distribution shows that the filter preserves these components of a superposition which belong to the region in the shape of a corner. Photon number distributions for the original multiphoton states and for those preselected with the corner filter are depicted in Fig. 5. The filtering is most restrictive if $\delta_{\rm th} = 0$. Here $|\Delta| = \sigma$ and only the N00N-like components are left from the initial polarization singlet. The case of $\delta_{\rm th} \to \infty$ corresponds to no filtering, since $0 \leqslant |\Delta|$ is always fulfilled.

We will now focus on two cases: $\delta_{\rm th} = 0$ and $\delta_{\rm th} = 2$. Based on the considerations presented at the beginning of Sec. IV, we compute the visibility $V^P(\beta)$, antivisibility $A^P(\beta)$, and the CHSH-Bell parameter B^P for the preselected states in Eq. (19). The formulas are lengthy and we display them in Appendix B, Eqs. (B13)–(B18).

The optimal rotation angle $\beta_{\rm opt}$ for the preselected states depends on $\delta_{\rm th}$ and varies with amplification gain g. Figure 6

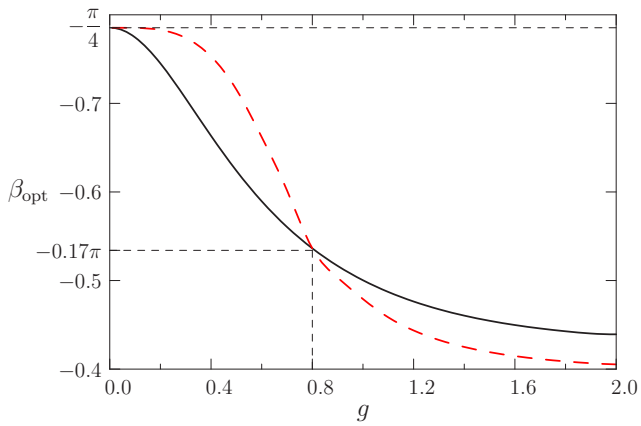

FIG. 6. (Color online) The optimal rotation angle $\beta_{\rm opt}$ for the multiphoton mode of an unsymmetrical singlet in the CHSH-Bell inequality test, computed for the corner filter with $\delta_{\rm th} = 0$ (black solid line) and $\delta_{\rm th} = 2$ (red dashed line) as a function of amplification gain g.

depicts $\beta_{\rm opt}$ as a function of g for $\delta_{\rm th} = 0$ and $\delta_{\rm th} = 2$. It can be found numerically by solving $d/d\beta[V(\beta) + A(\beta)] = 0$. The nonorthogonal choice for the measurement settings on the multiphoton mode indicates that the unsymmetrical singlet looses the phase-covariant symmetry after preselection.

Figure 7 illustrates how the Bell parameter B^P for the states $|\Phi^P\rangle$ and $|\bar{\Phi}^P\rangle$ preselected by the corner filter is constructed. It simultaneously presents the values of the visibility V_k^P from Eq. (B14), antivisibility A_k^P from Eq. (B16), the Bell parameter B_k^P from Eq. (B18), and $(\beta_k^P)^2$ from Eq. (B6), computed for the first $k = 0,\ldots,20$ sectors (each containing $2k+1$ photons), $g = 0.8$, and $\delta_{\rm th} \in \{0,2\}$. The particular choice of the amplification gain g allows us to choose $\beta_{\rm opt} = -0.17\pi$ for both thresholds (see Fig. 6), making the comparison easier.

In the case of $\delta_{\rm th} = 0$ one may notice that $V_k^P = 1$ and $A_k^P = 0$ for all $k \geqslant 4$, which results in $B_k^P = 2$ for sectors of nine or more photons. Moreover, for $k \in \{0,3\}$ the sum $B_k^P = 2(V_k^P + A_k^P)$ is greater than 2 and, together with $(\beta_k^P)^2$ equal to 0.42 and 0.15, gives the maximal contribution to the result. In case of $k = 1$ the value of B_k is below 2, but due to a smaller value of $(\beta_2^P)^2 = 0.27$, its negative influence does not completely destroy the total Bell parameter. Finally, $B^P = 2.26 > B = 2.06$ for $g = 0.8$ (total mean number of photons equals 4.15).

Similar behavior could be observed for other values of g and $\delta_{\rm th} = 0$. Of course, $\beta_{\rm opt}$ changes with g, but it is always possible to find k for which $(\beta_k^P)^2$ is relatively large and $B_k^P > 2$ at the same time, thus resulting in the total Bell parameter which exceeds the classical limit.

For $\delta_{\rm th} = 2$ in turn, setting the rotation angle at Bob's side to $\beta_{\rm opt}$ gives a sawtooth shape of V_k^P, A_k^P and thus B_k^P. The values of visibility and antivisibility for the sectors lie between 0 and 1, but nevertheless, B_k^P exceeds 2 for a few k, for which $(\beta_k^P)^2$ is the greatest. The obtained Bell parameter equals $B^P = 2.08$, which is less than for $\delta_{\rm th} = 0$ but still a bit more than for not-preselected states and the same amplification gain.

The reason behind the shape of V_k^P and A_k^P and thus B_k^P for a given filter threshold is the structure of the preselected

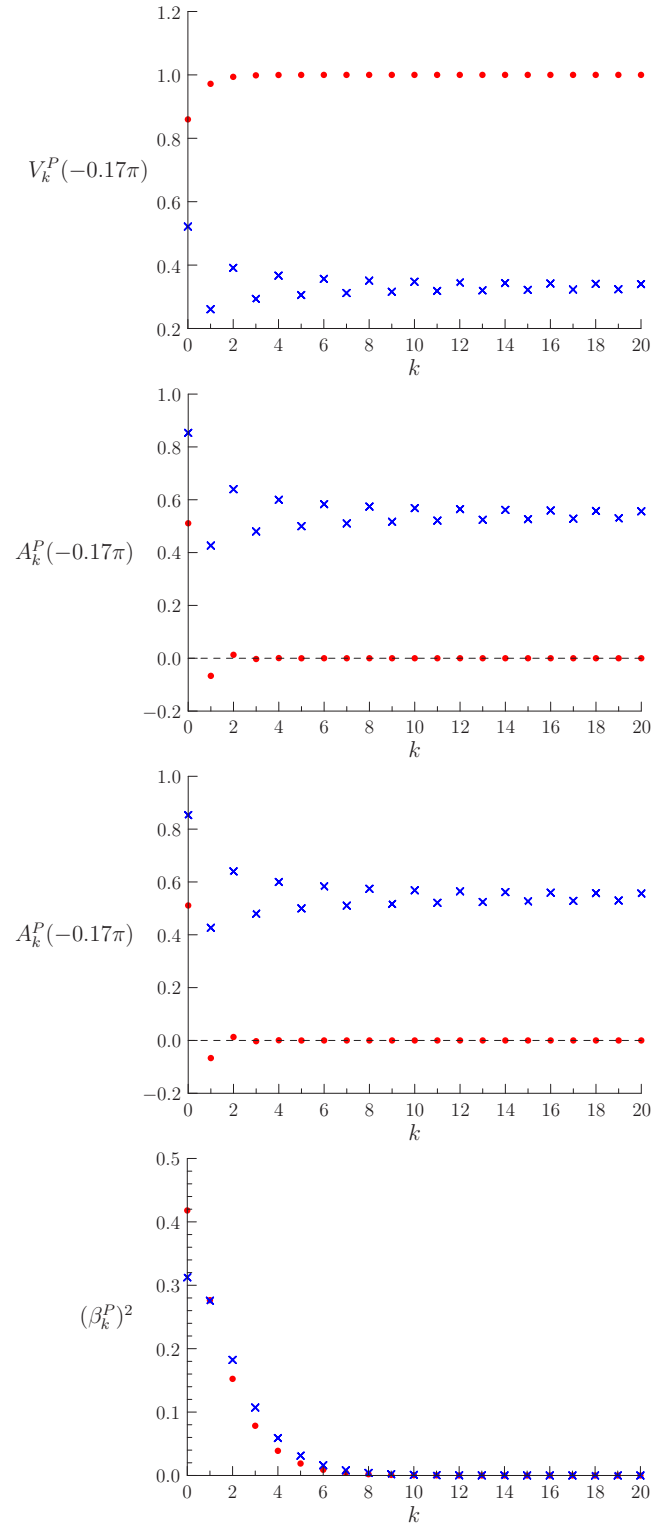

FIG. 7. (Color online) Plot of the visibility V_k^P from Eq. (B14) (top figure), antivisibility A_k^P from Eq. (B16) (upper middle figure), the Bell parameter B_k^P from Eq. (B18) (bottom middle figure), and $(\beta_k^P)^2$ from Eq. (B6), computed for the multiphoton states $|\Phi_k^P\rangle$ and $|\bar{\Phi}_k^P\rangle$ given in Eqs. (B1) and (B2), preselected by the corner filter with threshold values $\delta_{\rm th} = 0$ (red dots) and $\delta_{\rm th} = 2$ (blue crosses) as a function of k corresponding to the $(2k+1)$-photon-number sector of the multiphoton states.

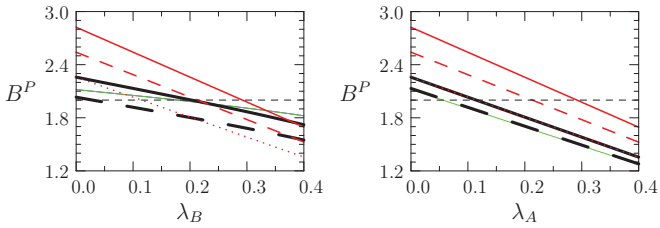

FIG. 8. (Color online) Bell parameter B^P evaluated for the state $|\Psi_P^-\rangle$ from Eq. (20) as a function of losses in the micromode λ_A and the macromode λ_B for three values of amplification gain: $g = 1.1$ (8.13 photons on average, thin green curves), $g = 0.8$ (4.15 photons on average, thick black curves), and $g = 0.05$ (1.01 photons on average, medium red curves). The left figure shows $B^P(\lambda_B)$ for fixed values of $\lambda_A = 0\%$ (solid line), $\lambda_A = 10\%$ (the dashed line), and $\lambda_A = 20\%$ (dotted line). The right figure depicts $B^P(\lambda_A)$ for $\lambda_B = 0\%$ (solid line), $\lambda_B = 10\%$ (the dashed line), $\lambda_B = 20\%$ (dotted line).

states obtained with the filter. Setting $\delta_{\text{th}} = 0$ allows only the $N00N$-like components of the original polarization singlet to pass through. $|\Phi^P\rangle$ and $|\bar{\Phi}^P\rangle$ become superpositions of $|2i+1,0_\perp\rangle$ and $|0,(2i+1)_\perp\rangle$, respectively, with varying i. Then the obtained preselected polarization singlet from Eq. (1) takes the new form of a superposition of polarization $N11N$ states with odd N,

$$|\Psi_P^-\rangle = \sum_{i=0}^{\infty} \tilde{\gamma}_{i0}(|1\rangle_A|(2i+1)_\perp\rangle_B - |1_\perp\rangle_A|2i+1\rangle_B), \quad (20)$$

where $N = 2i+1$ and $\tilde{\gamma}_{i0} = \sqrt{\cosh g/2}\gamma_{i0}$. One can notice that $B^P \geqslant 2$ regardless of the gain g for $\delta_{\text{th}} = 0$, because for $\beta = 0$, which is a suboptimal choice of angle, we have $V_k^P = 1$ and $A_k^P = 0$.

For $\delta_{\text{th}} = 2$, the states $|\Phi^P\rangle$ and $|\bar{\Phi}^P\rangle$ are superpositions not only of $|2i+1,0_\perp\rangle$ and $|0,(2i+1)_\perp\rangle$, so $N00N$s with odd N, but they also include terms like $|1,(2j)_\perp\rangle$ and $|2j,1_\perp\rangle$, i.e., superpositions of $N11N$ states with even N. Depending on the parity of k, the probability amplitudes are summed up with different signs, resulting in the sawtooth shape of the plot.

At the end of this paragraph we would like to comment on losses for the Bell test with the corner filter as a preselection strategy. As expected, since the filter preselects the state in the $N00N$-like form, the Bell test is quite fragile to losses in the setup. Figure 8 depicts the Bell parameter computed for the state $|\Psi_P^-\rangle$ in Eq. (20) for $g = 0.05$ (1.01 photons on average), $g = 0.8$ (4.15 photons on average), and $g = 0.1$ (8.13 photons on average) as a function of losses in the multi-λ_B and the single-photon mode λ_A for $\delta_{\text{th}} = 0$. The violation is much more robust to losses on the amplified-qubit side than on the single-photon side. For example, if $\lambda_A = 0$, then for gain $g = 1.1$ losses on the multiphoton mode up to 20% can be tolerated and $B^P > 2$. As expected, the more the state is populated (gain g increases), the less loss tolerant the Bell test becomes.

V. DISCUSSION AND CONCLUSIONS

We have discussed a possibility of performing a postselection-free Bell test with the use of multiphoton quantum states of light and coarse-grained measurements. For this purpose, we examined the CHSH inequality and exemplary unsymmetrical polarization singlet state with the multiphoton states produced in optimal-phase-covariant quantum cloning. Our work is a proof of principle: we show that it is possible to apply a feasible quantum-state engineering to multiphoton states and in this way to overcome the problem of imperfect analog detection and violate the classical bound. For the states we discussed, the corner filter is a good choice. It filters out the $N00N$-like components from the initial superpositions.

We do not claim that the amplified single photons, the CHSH-Bell inequality, and the modulus intensity filter are the best strategy for obtaining the loophole-free violation for multiphoton entangled states of light. The CHSH inequality itself imposes the need for the coarse-grained measurements in the case of the unsymmetrical singlets. Perhaps an inequality with much less coarse graining, i.e., with nonbinary measurement outcomes, would be required. Also, the analysis of losses in the present model shows that amplification of a two-photon singlet decreases the robustness of the state against losses. Nevertheless, we think that our analysis is an important result because, at least in the near future, it will be difficult to increase arbitrarily the resolution of the measuring devices with increasing population of the states; thus to some extent the coarse graining is unavoidable. Generally, finding a feasible preselection which both enables using coarse-grained detection and creates a state robust against losses is a difficult task. We conjecture that employing an amplified symmetrical singlet state of light instead of the unsymmetrical one for preselection and Bell testing will increase the robustness against losses.

It is also worth noting that so far there is no proposal allowing for a "genuine macroscopic" violation of a Bell inequality. From Figs. 3 and 7 it is clear that the photon number sectors which contribute most to the violation come from the small photon numbers. The Bell parameter decreases with increasing photon number $2k+1$. This is a general tendency one observes also for the bright squeezed vacuum state and other Bell inequalities, with observables which are dichotomic or not. Indeed, the preselection helps to increase the values of B_k for all k, but the question of what is the observable which will reverse the decreasing trend presented in the figures, so that the high photon numbers contribute to violation most, remains open.

At the end it is interesting to note that in Ref. [33] it was shown that a Bell inequality violation may be achieved with extremely coarse-grained measurement in the presence of a nonlinear interaction. Our results seem to follow this statement: quantum engineering may be viewed as a highly nonlinear operation performed on multiphoton states.

We conclude that by taking into account the achievements presented in [5,6], it is possible to demonstrate a loophole-free Bell inequality violation for multiphoton singlet states of light within the current technology in the near future.

ACKNOWLEDGMENTS

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Appendix A

In a general case, one may decompose the state $|\Psi\rangle$ into the k-photon sectors of the form $|\Phi_k\rangle = \sum_{j=0}^{k} \xi_{k,j}|k-j,j_\perp\rangle$ ($|\bar{\Phi}_k\rangle = \sum_{j=0}^{k} \bar{\xi}_{k,j}|k-j,j_\perp\rangle$ for $|\bar{\Psi}\rangle$), where $\xi_{k,j}$ ($\bar{\xi}_{k,j}$) are certain probability amplitudes. A general form of observable $\mathcal{O}^{(M)}(\beta)$ is given by Eq. (4) and can be expressed as

$$\begin{aligned}\mathcal{O}^{(M)}(\beta) = &\sum_{u,w=0\,C(u,w)}^{\infty} \sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n \sum_{m'=0}^{u}\sum_{n'=0}^{w}(-1)^{n'} \frac{\sqrt{u!\,(m+n)!\,w!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\\ &\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\frac{\sqrt{u!\,(m'+n')!\,w!\,(u+w-m'-n')!}}{m'!\,(u-m')!\,n'!\,(w-n')!}\cos^{u-m'}\\ &\times\left(\frac{\beta}{2}\right)\sin^{m'}\left(\frac{\beta}{2}\right)\sin^{w-n'}\left(\frac{\beta}{2}\right)\cos^{n'}\left(\frac{\beta}{2}\right)|u+w-m-n,(m+n)_\perp\rangle\langle u+w-m'-n',(m'+n')_\perp|,\end{aligned} \tag{A1}$$

where $C(u,w)$ represents a condition which ensures the observable to be traceless. The visibility evaluated for this observable equals

$$\begin{aligned}V_k(\beta) = \langle\Phi_k|\mathcal{O}^{(M)}(\beta)|\Phi_k\rangle = &\sum_{u,w=0\,C(u,w)}^{\infty}\left\{\sum_{j=0}^{k}\xi_{k,j}\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\right.\\ &\left.\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\}^2.\end{aligned} \tag{A2}$$

Substituting $-\beta$ as a rotation angle in Eq. (A2) gives a formula of a similar form, which differs only with the coefficient $(-1)^j$ in the probability amplitude:

$$\begin{aligned}V_k(-\beta) = &\sum_{\substack{u,w=0\\C(u,w)}}^{\infty}\left\{\sum_{j=0}^{k}\xi_{k,j}(-1)^j\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\right.\\ &\left.\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\}^2.\end{aligned} \tag{A3}$$

Equations (A2) and (A3) are equivalent $[V_k(-\beta) = V_k(\beta)]$ when $\xi_{k,j} = \xi_{k,j}(-1)^j$ for all k and j. This is fulfilled when $\xi_{k,j} = 0$ for odd j.

Similarly, the antivisibility is computed as follows:

$$\begin{aligned}A_k(\beta) = \langle\Phi_k|\mathcal{O}^{(M)}(\beta)|\Phi_k\rangle = &\sum_{\substack{u,w=0\\C(u,w)}}^{\infty}\left\{\sum_{j=0}^{k}\xi_{k,j}\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\right.\\ &\left.\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\}\\ &\cdot\left\{\sum_{j=0}^{k}\bar{\xi}_{k,j}\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\},\end{aligned} \tag{A4}$$

$$\begin{aligned}A_k(-\beta) = &\sum_{\substack{u,w=0\\C(u,w)}}^{\infty}\left\{\sum_{j=0}^{k}\xi_{k,j}(-1)^j\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\right.\\ &\left.\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\}\cdot\left\{\sum_{j=0}^{k}\bar{\xi}_{k,j}(-1)^j\sum_{m=0}^{u}\sum_{n=0}^{w}(-1)^n\right.\\ &\left.\times\frac{\sqrt{u!\,w!\,(m+n)!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{j,m+n}\right\}.\end{aligned} \tag{A5}$$

Equations (A4) and (A5) are equivalent, i.e., $A_k(-\beta) = A_k(\beta)$, when for all k,j either $\xi_{k,j} = \xi_{k,j}(-1)^j$ and $\bar{\xi}_{k,j} = \bar{\xi}_{k,j}(-1)^j$ or $\xi_{k,j} = \pm\bar{\xi}_{k,j}(-1)^j$. Similarly, $A_k(-\beta) = -A_k(\beta)$ when for all k,j $\xi_{k,j} = \pm\xi_{k,j}(-1)^j$ and $\bar{\xi}_{k,j} = \mp\bar{\xi}_{k,j}(-1)^j$. The last condition is fulfilled, e.g., when $\xi_{k,j} = 0$ and $\bar{\xi}_{k,j} \neq 0$ for odd j but $\xi_{k,j} \neq 0$ and $\bar{\xi}_{k,j} = 0$ for even j.

Appendix B

The preselection modifies the sectors of the fixed photon number states in the following way:

$$\left|\Phi_k^P\right\rangle = \frac{1}{\sqrt{\mathcal{N}_k^P}}\mathcal{P}_{\mathrm{C}}\left|\Phi_k\right\rangle = \frac{1}{\sqrt{\mathcal{N}_k^P}}\sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j+1-2k)}}^{k}\binom{k}{j}\sqrt{(2j+1)!\,(2k-2j)!}\,|2j+1,(2k-2j)_\perp\rangle, \tag{B1}$$

$$\left|\bar{\Phi}_k^P\right\rangle = \mathcal{P}_{\mathrm{C}}\left|\bar{\Phi}_k\right\rangle = \frac{1}{\sqrt{\bar{\mathcal{N}}_k^P}}\sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j-1-2k)}}^{k}\binom{k}{j}\sqrt{(2j)!\,(2k+1-2j)!}\,|2j,(2k+1-2j)_\perp\rangle, \tag{B2}$$

with normalization constants equal to

$$\mathcal{N}_k^P = \sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j+1-2k)}}^{k}\binom{k}{j}^2(2j+1)!\,(2k-2j)!, \tag{B3}$$

$$\bar{\mathcal{N}}_k^P = \sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j-1-2k)}}^{k}\binom{k}{j}^2(2j)!\,(2k+1-2j)!. \tag{B4}$$

The multiphoton states equal

$$|\Phi^P\rangle = \sum_{k=0}^{\infty}\beta_k^P\left|\Phi_k^P\right\rangle, \tag{B5}$$

$$\beta_k^P = C_g^{-2}\left(\frac{T_g}{2}\right)^k\frac{1}{k!}\sqrt{\frac{\mathcal{N}_k^P}{\mathcal{N}^P}},\quad \sum_{k=0}^{\infty}\left(\beta_k^P\right)^2 = 1, \tag{B6}$$

$$\mathcal{N}^P = C_g^{-4}\sum_{k=0}^{\infty}\left(\frac{T_g}{2}\right)^{2k}\frac{1}{k!^2}\mathcal{N}_k^P, \tag{B7}$$

$$|\bar{\Phi}^P\rangle = \sum_{k=0}^{\infty}\bar{\beta}_k^P\left|\bar{\Phi}_k^P\right\rangle, \tag{B8}$$

$$\bar{\beta}_k^P = C_g^{-2}\left(\frac{T_g}{2}\right)^k\frac{1}{k!}\sqrt{\frac{\bar{\mathcal{N}}_k^P}{\bar{\mathcal{N}}^P}},\quad \sum_{k=0}^{\infty}\left(\bar{\beta}_k^P\right)^2 = 1, \tag{B9}$$

$$\bar{\mathcal{N}}^P = C_g^{-4}\sum_{k=0}^{\infty}\left(\frac{T_g}{2}\right)^{2k}\frac{1}{k!^2}\bar{\mathcal{N}}_k^P. \tag{B10}$$

We now change the variable j to j' in $\bar{\mathcal{N}}_k^P$ given in Eq. (B4), so $j' = k - j$ ($j = k - j'$). The sum over j' remains from 0 to k:

$$\begin{aligned}\bar{\mathcal{N}}_k^P &= \sum_{\substack{j'=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4(k-j')-2k-1)}}^{k}\binom{k}{k-j'}^2[2(k-j')]!\,[2k+1-2(k-j')]!\\ &= \sum_{\substack{j'=0\\ \mathrm{C}(\sigma=2k+1,\Delta=-(4j'+1-2k))}}^{k}\binom{k}{j'}^2(2k-2j')!\,(2j'+1)!.\end{aligned} \tag{B11}$$

Assuming that $\mathrm{C}(\sigma,\Delta)$ is symmetric with respect to Δ, we got $\mathrm{C}(\sigma = 2k+1, \Delta = -(4j+1-2k)) = \mathrm{C}(\sigma = 2k+1, \Delta = 4j+1-2k)$, so $\bar{\mathcal{N}}_k^P = \mathcal{N}_k^P$ and therefore $\bar{\mathcal{N}}^P = \mathcal{N}^P$ and $\bar{\beta}_k^P = \beta_k^P$.

The observable $\mathcal{O}^{(M)}(\beta)$ is given by Eq. (15) and can be expressed as

$$\begin{aligned}\mathcal{O}^{(M)}(\beta) = &\left[\sum_{u-w\geqslant 0} - \sum_{u-w<0}\right]\sum_{m=0}^{u}\sum_{n=0}^{w}\frac{\sqrt{u!\,(m+n)!\,w!\,(u+w-m-n)!}}{m!\,(u-m)!\,n!\,(w-n)!}(-1)^n\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\\ &\times\sum_{m'=0}^{u}\sum_{n'=0}^{w}\frac{\sqrt{u!\,(m'+n')!\,w!\,(u+w-m'-n')!}}{m'!\,(u-m')!\,n'!\,(w-n')!}(-1)^{n'}\cos^{u-m'}\left(\frac{\beta}{2}\right)\sin^{m'}\left(\frac{\beta}{2}\right)\sin^{w-n'}\left(\frac{\beta}{2}\right)\cos^{n'}\left(\frac{\beta}{2}\right)\\ &\times|u+w-m-n,(m+n)_\perp\rangle\langle u+w-m'-n',(m'+n')_\perp|. \end{aligned}\tag{B12}$$

Visibility takes the form

$$V^P(\beta) = \sum_{k,k'=0}^{\infty}\beta_k^P\,\beta_{k'}^P\left\langle\Phi_k^P\middle|\mathcal{O}^{(M)}(\beta)\middle|\Phi_{k'}^P\right\rangle = \sum_{k}^{\infty}\left(\beta_k^P\right)^2\left\langle\Phi_k^P\middle|\mathcal{O}^{(M)}(\beta)\middle|\Phi_k^P\right\rangle = \sum_{k=0}^{\infty}\left(\beta_k^P\right)^2 V_k^P(\beta),\tag{B13}$$

where

$$\begin{aligned}V_k^P(\beta) = \frac{1}{\mathcal{N}_k}&\left[\sum_{u-w\geqslant 0} - \sum_{u-w<0}\right]\frac{\delta_{u+w,2k+1}}{u!\,w!}\Bigg\{\sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j+1-2k)}}^{k}\binom{k}{j}(2j+1)!\,(2k-2j)!\\ &\times\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k-2j,m+n}\Bigg\}^2.\end{aligned}\tag{B14}$$

From the form of Eq. (B14) it is possible to derive the property $V_k^P(-\beta) = V_k^P(\beta)$, which implies $V^P(-\beta) = V^P(\beta)$. This is consistent with the condition found in Appendix A, since probability amplitudes of $|\Phi_k^P\rangle$ are nonzero only for even number of photons in one of the polarizations. Similarly, the antivisibility equals

$$A^P(\beta) = \sum_{k,k'=0}^{\infty}\beta_k^P\,\bar{\beta}_{k'}^P\left\langle\Phi_k^P|\mathcal{O}^{(M)}(\beta)|\bar{\Phi}_{k'}^P\right\rangle = \sum_{k}^{\infty}\beta_k^P\,\bar{\beta}_k^P\left\langle\Phi_k^P|\mathcal{O}^{(M)}(\beta)|\bar{\Phi}_k^P\right\rangle = \sum_{k}^{\infty}\left(\beta_k^P\right)^2 A_k^P(\beta),\tag{B15}$$

where

$$\begin{aligned}A_k^P(\beta) = \frac{1}{\mathcal{N}_k}&\left[\sum_{u-w>0} - \sum_{u-w<0}\right]\frac{\delta_{u+w,2k+1}}{u!\,w!}\Bigg\{\sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j+1-2k)}}^{k}\binom{k}{j}(2j+1)!\,(2k-2j)!\\ &\times\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k-2j,m+n}\Bigg\}\\ &\cdot\Bigg\{\sum_{\substack{j=0\\ \mathrm{C}(\sigma=2k+1,\Delta=4j-1-2k)}}^{k}\binom{k}{j}(2j)!\,(2k+1-2j)!\sum_{m=0}^{u}\sum_{n=0}^{w}\binom{u}{m}\binom{w}{n}(-1)^n\cos^{u-m}\\ &\times\left(\frac{\beta}{2}\right)\sin^m\left(\frac{\beta}{2}\right)\sin^{w-n}\left(\frac{\beta}{2}\right)\cos^n\left(\frac{\beta}{2}\right)\delta_{2k+1-2j,m+n}\Bigg\},\end{aligned}\tag{B16}$$

which implies $A_k^P(-\beta) = -A_k^P(\beta)$ and therefore $A^P(-\beta) = -A^P(\beta)$. Again, it is consistent with the condition derived in Appendix A, because for the same number of photons in both polarizations, probability amplitudes in $A_k^P(-\beta)$ and $A_k^P(\beta)$ have the same modules and opposite signs. Finally, the Bell parameter [Eq. (6)] can be simplified for angles $\theta = 0$, $\alpha = 0$, $\alpha' = \frac{\pi}{2}$, and $\beta' = -\beta$ to

$$B^P = 2V^P(\beta) + 2A^P(\beta),\quad B = 2\sum_{k=0}^{\infty}\left(\beta_k^P\right)^2 V_k^P + 2\sum_{k=0}^{\infty}\left(\beta_k^P\right)^2 A_k^P,\quad B^P = \sum_{k=0}^{\infty}\left(\beta_k^P\right)^2 B_k^P,\tag{B17}$$

where

$$B_k^P = 2\left[V_k^P(\beta) + A_k^P(\beta)\right].\tag{B18}$$

The formulas in Eqs. (B1)–(B18) hold true also for the Bell test without preselection. In this case, the condition $C(\sigma,\Delta)$ is always fulfilled and $|\Phi^P\rangle = |\Phi\rangle$, $|\bar{\Phi}^P\rangle = |\bar{\Phi}\rangle$, $\beta_k^P = \beta_k$, $\mathcal{N}_k^P = \mathcal{N}_k$, $\mathcal{N}^P = 1$, and $B_k^P = B_k$.

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